A PICARD-MACLAURIN THEOREM FOR INITIAL VALUE PDES

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In 1988, Parker and Sochacki announced a theorem which proved that the Picard iteration, properly modified, generates the Taylor series solution to any ordinary differential equation (ODE) on $\Re^n$ with a polynomial generator. In this paper, we present an analogous theorem for partial differential equations (PDEs) with polynomial generators and analytic initial conditions. Since the domain of a solution of a PDE is a subset of $\Re^n$, we identify one component of the domain to achieve the analogy with ODEs. The generator for the PDE must be a polynomial and autonomous with respect to this component, and no partial derivative with respect to this component can appear in the domain of the generator. The initial conditions must be given in the designated component at zero and must be analytic in the nondesignated components. The power series solution of such a PDE, whose existence is guaranteed by the Cauchy theorem, can be generated to arbitrary degree by Picard iteration. As in the ODE case these conditions can be met, for a broad class of PDEs, through polynomial projections.

1. Introduction

In [6] the authors developed a completely explicit notation for presenting ordinary differential equations (ODEs) with polynomial generators that allowed them to make a reasonably transparent proof for the following theorem.

If F is a polynomial from $\Re^n$ into $\Re^n$, then the kth Picard iterate for the ODE

$$y' = F \circ y; \qquad y(0) = x, \tag{1.1}$$

defined by

$$p_1(t) = x, \qquad p_k(t) = x + \int_0^t F \circ p_{k-1}, \tag{1.2}$$

is the $(k-1)$st degree Maclaurin polynomial plus a polynomial all of whose terms have degree greater than $k-1$.

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The notation developed translates directly to implementation of the algorithm arising from the proof of this theorem in either a symbolic or numeric computing environment.

To avoid ambiguity in proving a similar theorem for initial value partial differential equations (PDEs) and to afford direct translation of the resulting algorithm to either a symbolic or numeric computing environment we will again introduce a formal notation.

In order to realize the full impact of the theory on applications it is important to have a precise definition for a polynomial functional F on $\Re^n$ ($D_F = \Re^n$) since the components of the generator and the iterates of the solution are polynomial functionals. However, we note that their roles within any computing environment are different. The components of the generator must be stored and accessed; the iterates of the solution must be tracked.

We offer here a definition that translates directly into a (numerical) computing environment.

Definition 1.1. Suppose that $n \in N$. Let $\underline{n}$ denote $\{k \mid k \in N \text{ and } k \leq n\}$.

$$E_n = \big\{h \mid h : \underline{n} \longrightarrow \{0\} \cup N\big\}. \tag{1.3}$$

Elements of E_n will define the exponents for the terms of polynomial functionals on $\Re^n$.

Definition 1.2. Suppose that $n \in N$ and $F : \Re^n \to \Re$.

The statement that "F is a polynomial functional on $\Re^n$" means that there is a finite subset, Λ, of E_n and $A : \{0\} \cup \Lambda \to \Re$, so that if $x \in \Re^n$, then

$$F(x) = A(0) + \sum_{\mu \in \Lambda} A(\mu) \prod_{i=1}^{n} x_i^{\mu(i)}. \tag{1.4}$$

The three main structural differences between the PDE problem and the ODE problem are

(1) the initial conditions are analytic functions;
(2) the PDE need be autonomous only in the designated component; and
(3) the presence of partial derivatives in the domain of the generator for the PDE.

For computational reasons it is important to have a formal notation for these partial derivatives.

Definition 1.3. Let $\Theta = \{h \mid \text{there is } m \in N \text{so that } h : \underline{m} \to N\}$. Suppose $u : \Re^n \to \Re$, $\nu \in \Theta$, and $D_\nu = \underline{m}$ and the range of ν, R_ν, satisfies $\max R_\nu \leq n$. Then

$$d_\nu u = d_{\nu(1)}\big(d_{\nu(2)}\big(\cdots\big(d_{\nu(m)}u\big)\cdots\big)\big). \tag{1.5}$$

To illustrate, consider a function $u : \Re^2 \to \Re$. It is common to denote the first component of a typical element of D_u as x, the second component as y and, for instance, to write one of the third partial derivatives as u_{xxy}. We prefer $d_{112}u$. To translate to the formalism suggested above, let $\nu(1) = 1$, $\nu(2) = 1$, and $\nu(3) = 2$. The partial derivative is then denoted as $d_\nu u$.

To set the stage for the presentation of a PDE as our theorem applies to it, consider the following PDE on $\Re^3$ (from gas dynamics) as it might be typically presented.

Example 1.4.

$$\begin{aligned} w_t &= -(wu)_x - (wv)_y, \\ u_t &= uu_x + vu_y + u_{xx} + u_{yy} - \frac{x^2 y}{w} w_x, \\ v_t &= uv_x + vv_y + v_{xx} + v_{yy} - \frac{x^2 y}{w} w_y. \end{aligned} \tag{1.6}$$

Let $\alpha(1) = 1$, $\beta(1) = 2$, $\chi(1) = 3$, $\phi(1) = 2$, and $\phi(2) = 2$, and $\gamma(1) = 3$ and $\gamma(2) = 3$. For (t, x, y) in the common domains of w, u, and v, this translates into the formalism as

$$\begin{aligned} d_\alpha w(t,x,y) &= -d_\beta(wu)(t,x,y) - d_\chi(wv)(t,x,y), \\ d_\alpha u(t,x,y) &= u(t,x,y) d_\beta u(t,x,y) + v(t,x,y) d_\chi u(t,x,y) + d_\phi u(t,x,y) \\ &\quad + d_\gamma u(t,x,y) - \frac{x^2 y}{w(t,x,y)} d_\beta w(t,x,y), \\ d_\alpha v(t,x,y) &= u(t,x,y) d_\beta v(t,x,y) + v(t,x,y) d_\chi v(t,x,y) + d_\phi v(t,x,y) \\ &\quad + d_\gamma v(t,x,y) - \frac{x^2 y}{w(t,x,y)} d_\chi w(t,x,y). \end{aligned} \tag{1.7}$$

In order to meet the conditions of the hypothesis of our theorem we convert the products to polynomials by differentiating, project $1/w$ into a polynomial by letting $z = 1/w$, and identify the component in which the equation must be autonomous and whose partial derivatives cannot appear in the domain of the generator as the first component. Under these conditions the PDE (suppressing (t, x, y)) is

$$\begin{aligned} d_1 w &= -u d_\beta w - w d_\beta u - v d_\chi w - w d_\chi v, \\ d_1 u &= u d_\beta u + v d_\chi u + d_\phi u + d_\gamma u - x^2 y z d_\beta w, \\ d_1 v &= u d_\beta v + v d_\chi v + d_\phi v + d_\gamma v - x^2 y z d_\chi w, \\ d_1 z &= -z^2 \big(-u d_\beta w - w d_\beta u - v d_\chi w - w d_\chi v \big). \end{aligned} \tag{1.8}$$

In some computing environments it is advantageous to write the last equation of the PDE as

$$d_1 z = z^2 u d_\beta w + z^2 w d_\beta u + z^2 v d_\chi w + z^2 w d_\chi v. \tag{1.9}$$

We note the solution U has four components given by $U_1 = w$, $U_2 = u$, $U_3 = v$, and $U_4 = z$, and each of these components has domain a subset of $\Re^3$. Thus, the PDE can be written as

$$\begin{aligned} d_1 U_1 &= -U_2 d_\beta U_1 - U_1 d_\beta U_2 - U_3 d_\chi U_1 - U_1 d_\chi U_3, \\ d_1 U_2 &= U_2 d_\beta U_2 + U_3 d_\chi U_2 + d_\phi U_2 + d_\gamma U_2 - U_4 x^2 y d_\beta U_1, \\ d_1 U_3 &= U_2 d_\beta U_3 + U_3 d_\chi U_3 + d_\phi U_3 + d_\gamma U_3 - U_4 x^2 y d_\chi U_1, \\ d_1 U_4 &= U_4^2 U_2 d_\beta U_1 + U_4^2 U_1 d_\beta U_2 + U_4^2 U_3 d_\chi U_1 + U_4^2 U_1 d_\chi U_3. \end{aligned} \tag{1.10}$$

Notice that the second and the third components of the generator are evolutionary in x and y, whereas its first and fourth components are autonomous. Thus the polynomial functionals that express the projections of the generator can be given domains $\Re^8$, $\Re^{11}$, $\Re^{11}$, and $\Re^8$, and degrees $2,5,5$, and 4, respectively. (Actually the domains of the projections of the generator into the second and the third components can be taken to be $\Re^{10}$, but in our own programming, we typically allow space for all components of the solution and build in flexibility to accommodate the partial derivatives present.)

Finally, we illustrate the use of Definitions 1.1 and 1.2 by considering the second equation. The domain of the generator in the second component has two components for the second and the third components of an element of D_U, four components for U (although U_1 is not acted upon), and five components for the pertinent partial derivatives. We reserve the first six components, then designate the final five components. This can be formalized by identifying appropriate functions, ν_2 and δ_2, from $\{n : n \in N \text{ and } 6 < n \leq 11\}$ into Θ and from $\{n : n \in N \text{ and } 6 < n \leq 11\}$ into $\underline{4}$. Letting

$$
\begin{aligned}
\rho &= \{(1,0),(2,0),(3,0),(4,1),(5,0),(6,0),(7,0),(8,1),(9,0),(10,0),(11,0)\},\\
\sigma &= \{(1,0),(2,0),(3,0),(4,0),(5,1),(6,0),(7,0),(8,0),(9,0),(10,0),(11,1)\},\\
\tau &= \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,1),(10,0),(11,0)\},\\
\omega &= \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,0),(10,1),(11,0)\},\\
\psi &= \{(1,2),(2,1),(3,0),(4,0),(5,0),(6,1),(7,1),(8,0),(9,0),(10,0),(11,0)\},\\
\Lambda_2 &= \{\rho,\sigma,\tau,\omega,\psi\},\\
\nu_2 &= \{(7,\beta),(8,\beta),(9,\phi),(10,\gamma),(11,\chi)\},\\
\delta_2 &= \{(7,1),(8,2),(9,2),(10,2),(11,2)\},\\
A_2 &= \{(0,0),(\rho,1),(\sigma,1),(\tau,1),(\omega,1),(\psi,-1)\},
\end{aligned}
\tag{1.11}
$$

(note $\Lambda_2 \subset E_{11}$) gives the second equation as

$$
\begin{aligned}
&d_1U_2(t,x_1,x_2)\\
&\quad = A_2(0)+\sum_{\mu\in\Lambda_2}\left(A_2(\mu)\prod_{k=1}^{2}x_k^{\mu(k)}\prod_{k=3}^{6}U_{k-2}(t,x_1,x_2)^{\mu(k)}\prod_{k=7}^{11}d_{\nu_2(k)}U_{\delta_2(k)}(t,x_1,x_2)^{\mu(k)}\right).
\end{aligned}
\tag{1.12}
$$

2. Theorems

Since the presentation of the problem is dependent on the designation of a particular component, we adopt, for an element of $\Re^m$, the notation (t,x). t will denote the designated component and x will consist of the other $m-1$ components; we will denote the components of x as $x_1,x_2,\dots,x_{m-1}$. I will denote the identity function in the designated component. We will write the right-hand side of a PDE as $P(x,U(t,x),d_\nu U(t,x))$. This is done to emphasize that we demarcate the domain of the generator into components accepting the non-designated components of the domain, components of the solution, and partial derivatives of components of the solution.

THEOREM 2.1. *Let $n \in N$ and $m \in N$. Suppose that $k \leq n$, $\Theta_k \subset \{f \mid$ there is $j \in N$ so that $f : \underline{j} \to \{l \mid l \in N$ and $1 < l \leq m\}\}$, $n_k \in N$ so that $n_k \geq m-1+n$, Λ_k is a finite subset of E_{n_k}, and if $n_k > m-1+n$ then $\nu_k : \{j \mid j \in N$ and $m-1+n < j \leq n_k\} \to \Theta_k$ and $\delta_k : \{j \mid j \in N$ and $m-1+n < j \leq n_k\} \to \underline{n}$, and $A_k : \{0\} \cup \Lambda_k \to \Re$. Let $U : \Re^m \to \Re^n$ be so that if $(t, x) \in D_U$,*

$$d_1 U_k(t,x) = A_k(0) + \sum_{\mu\in\Lambda_k}\left(A_k(\mu)\prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} U_{j-m+1}(t,x)^{\mu(j)} \prod_{j=m+n}^{n_k} d_{\nu_k(j)} U_{\delta_k(j)}(t,x)^{\mu(j)}\right), \tag{2.1}$$

and $U_k(0, x)$ is analytic. Let $a_k(x)$ denote $U_k(0, x)$.

Define, for $k \in \underline{n}$, $p_{k,1}$ by if $(t, x) \in D_U$

$$p_{k,1}(t,x) = a_k(x) \tag{2.2}$$

and for $l > 1$

$$p_{k,l}(t,x) = a_k(x) + \int_0^t \left(A_k(0) + \sum_{\mu\in\Lambda_k}\left(A_k(\mu)\prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \left(p_{j-m+1,l-1}\circ(I,x)\right)^{\mu(j)} \times \prod_{j=m+n}^{n_k}\left(d_{\nu_k(j)} p_{\delta_k(j),l-1}\circ(I,x)\right)^{\mu(j)}\right)\right). \tag{2.3}$$

Then if $k \in \underline{n}$ and $r, s \in N$, $(t, x) \in D_U$, $p_{k,r}(t, x) = b_{r,0}(x) + \sum_{i=1}^{q_r} b_{r,i}(x)t^i$, and $p_{k,s}(t, x) = B_{s,0}(x) + \sum_{i=1}^{q_s} B_{s,i}(x)t^i$; then if $Q < \min(r, s)$, $b_{r,Q}(x) = B_{r,Q}(x)$.

Before proving this theorem we note that, by the Cauchy theorem the PDE must have a unique analytic solution. Also, since components of the generator for the PDE and the iterates for the solution are both polynomial functionals, it is important to distinguish between them. We will write the functionals for the generator as indicated in the example. Also note that $b_{r,i}(x)$ and $B_{s,i}(x)$ are power series in x. Using these series as coefficients for powers of t we write the iterates for the solution in powers of the designated component, t.

Proof. The proof will use induction on the powers of the designated component. Let $1 \leq k \leq n$ and for $l > 1$ consider $p_{k,l}$.

If $(t,x) \in \Re^m$ then

$$p_{k,l}(t,x) = a_k(x) + \int_0^t \left(A_k(0) + \sum_{\mu \in \Lambda_k} \left(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \left(p_{j-m+1,l-1} \circ (I,x) \right)^{\mu(j)} \times \prod_{j=m+n}^{n_k} \left(d_{\nu_k(j)} p_{\delta_k(j),l-1} \circ (I,x) \right)^{\mu(j)} \right) \right). \tag{2.4}$$

Every term from the integral, since the integration is done with respect to the identity in the designated component, must have a power of at least 1 on t. Therefore, the zero degree term in t must be the initial condition, a power series in the components of x. Thus any pair of iterates agree in the zero power term.

Let $Q \in \{0\} \cup N$ and suppose that if $r, s > Q$ and $(t,x) \in D_U$, $p_{k,r}(t,x) = a_k(x) + \sum_{i=1}^{q_r} b_{r,i}(x)t^i$, and $p_{k,s}(t,x) = a_k(x) + \sum_{i=1}^{q_s} B_{s,i}(x)t^i$, then $a_k(x) + \sum_{i=1}^{Q} b_{r,i}(x)t^i$, and $a_k(x) + \sum_{i=1}^{Q} B_{s,i}(x)t^i$ have identical terms. Consider $l > Q+1$ and

$$p_{k,l}(t,x) = a_k(x) + \int_0^t \left(A_k(0) + \sum_{\mu \in \Lambda_k} \left(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \left(p_{j-m+1,l-1} \circ (I,x) \right)^{\mu(j)} \times \prod_{j=m+n}^{n_k} \left(d_{\nu_k(j)} p_{\delta_k(j),l-1} \circ (I,x) \right)^{\mu(j)} \right) \right). \tag{2.5}$$

If $j \in \underline{n}$, define $f_{j,l}$ and $g_{j,l}$ by if $(t,x) \in D_U$, $f_{j,l}(t,x) = a_j(x) + \sum_{i=1}^{Q} b_{r,i}(x)t^i$ and $g_{j,l}(t,x) = \sum_{i=Q+1}^{q_l} b_{r,i}(x)t^i$. Then

$$\begin{aligned} p_{k,Q+1}(t,x) &= a_k(x) + \int_0^t \left(A_k(0) + \sum_{\mu \in \Lambda_k} \left(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \left(f_{j-m+1,Q} + g_{j-m+1,Q} \right) \circ (I,x) \right)^{\mu(j)} \right. \\ &\qquad \left. \times \prod_{j=m+n}^{n_k} \left(d_{\nu_k(j)} \left(f_{\delta_k(j),Q} + g_{\delta_k(j),Q} \right) \circ (I,x) \right)^{\mu(j)} \right), \end{aligned} \tag{2.6}$$

$$\begin{aligned} p_{k,l}(t,x) &= a_k(x) + \int_0^t \left(A_k(0) + \sum_{\mu \in \Lambda_k} \left(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \left(f_{j-m+1,l-1} + g_{j-m+1,l-1} \right) \circ (I,x) \right)^{\mu(j)} \right. \\ &\qquad \left. \times \prod_{j=m+n}^{n_k} \left(d_{\nu_k(j)} \left(f_{\delta_k(j),l} + g_{\delta_k(j),l} \right) \circ (I,x) \right)^{\mu(j)} \right). \end{aligned} \tag{2.7}$$

Differentiating over sums and expanding the resulting integrands using the binomial theorem ($C_a^b = b!/a!(b-a)!$), we get

$$
\begin{aligned}
&p_{k,Q+1}(t,x)\\
&\quad = a_k(x) + \int_0^t \Bigg(A_k(0) + \sum_{\mu\in\Lambda_k} \Bigg(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \big(f_{j-m+1,Q}\circ(I,x)\big)^{\mu(j)} \Bigg)\\
&\qquad + \sum_{i=1}^{\mu(j)} \Bigg(C_i^{\mu(j)} \big(f_{j-m+1,Q}\circ(I,x)\big)^{\mu(j)-i} \big(g_{j-m+1,Q}\circ(I,x)\big)^i\\
&\qquad\qquad \times \prod_{j=m+n}^{n_k} \Big(d_{\nu_k(j)} \big(f_{\delta_k(j),Q}\circ(I,x)\big)^{\mu(j)} \Big)\\
&\qquad\qquad + \sum_{i=1}^{\mu(j)} \Big(C_i^{\mu(j)} d_{\nu_k(j)} f_{\delta_k(j),Q}\circ(I,x) \Big)^{\mu(j)-i}\\
&\qquad\qquad \times \big(d_{\nu_k(j)} g_{\delta_k(j),Q}\circ(I,x)\big)^i \Bigg)\Bigg),
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
&p_{k,l}(t,x)\\
&\quad = a_k(x) + \int_0^t \Bigg(A_k(0) + \sum_{\mu\in\Lambda_k} \Bigg(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \big(f_{j-m+1,l-1}\circ(I,x)\big)^{\mu(j)} \Bigg)\\
&\qquad + \sum_{i=1}^{\mu(j)} \Bigg(C_i^{\mu(j)} \big(f_{j,l-1}\circ(I,x)\big)^{\mu(j)-i} \big(g_{j,l-1}\circ(I,x)\big)^i\\
&\qquad\qquad \times \prod_{j=m+n+2}^{n_k} \Big(d_{\nu_k(j)} \big(f_{\delta_k(j),l-1}\circ(I,x)\big)^{\mu(j)} \Big)\\
&\qquad\qquad + \sum_{i=1}^{\mu(j)} \Big(C_i^{\mu(j)} d_{\nu_k(j)} f_{\delta_k(j),l-1}\circ(I,x) \Big)^{\mu(j)-i}\\
&\qquad\qquad \times \ \big(d_{\nu_k(j)} g_{\delta_k(j),l-1}\big)\circ(I,x))^i \Bigg)\Bigg).
\end{aligned}
\tag{2.9}
$$

From the induction hypothesis, for any j and i, $f_{j,l-1}\circ(I,x) = f_{j,Q}\circ(I,x)$, and therefore, $d_{\nu(i)} f_{j,l-1}\circ(I,x) = d_{\nu(i)} f_{j,Q}\circ(I,x)$. The power of the identity in the designated component in any term of $g_{j,l-1}\circ(I,x)$ or $g_{j,Q}\circ(I,x)$ is at least $Q+1$. Since the differential operators reduce only powers of components of x, the same is true for $d_{\nu(j)}g_{j,l-1}\circ(I,x)$ and $d_{\nu(j)}g_{j,Q}\circ(I,x)$. Thus, the term having factor t^{Q+1} must come from integrating powers of $f_{j,l-1}\circ(I,x)$ and powers of $d_{\nu(j)}f_{j,l-1}\circ(I,x)$ which we have just shown to be identical to the corresponding powers of $f_{j,Q}\circ(I,x)$ and $d_{\nu(j)}f_{j,Q}\circ(I,x)$. Hence the induction is continued.

Define V by if $1\le k\le n$ and $(t,x)\in D_U$ then

$$V_k(t,x)=a_k(x)+\sum_{j\in N}\left(b_j+\sum_{\sigma\in\Lambda_{k,j}}b_{j,\sigma}\prod_{i=1}^{m-1}x_i^{\sigma(i)}\right)t^j, \tag{2.10}$$

that is, if $l\in N$ then

$$f_{k,l}(t,x)=a_k(x)+\sum_{j}^{l-1}\left(b_j+\sum_{\sigma\in\Lambda_{k,j}}b_{j,\sigma}\prod_{i=1}^{m-1}x_i^{\sigma(i)}\right)t^j. \tag{2.11}$$

(Note that $\Lambda_{k,j}\subset E_{m-1}$.) From the induction just established, V is well defined.

Consider $1\le k\le N$, $l\in N$, and d_1V_k. If $(t,x)\in D_U$,

$$d_1V_k=d_1f_{k,l}+d_1\big(V_k-f_{k,l}\big). \tag{2.12}$$

From the fundamental theorem of calculus

$$\begin{aligned}d_1p_{k,l}=A_k(0)+\sum_{\mu\in\Lambda_k}\Bigg(&A_k(\mu)\prod_{j=1}^{m-1}x_j^{\mu(j)}\prod_{j=m}^{m-1+n}\big(p_{j-m+1,l-1}\circ(I,x)\big)^{\mu(j)}\\&\times\prod_{j=m+n}^{n_k}\big(d_{\nu_k(j)}p_{\delta_k(j),l-1}\circ(I,x)\big)^{\mu(j)}\Bigg).\end{aligned} \tag{2.13}$$

If $1\le j\le n$ then $p_{j,l}=f_{j,l}+g_{j,l}$; thus

$$\begin{aligned}&d_1f_{k,l}+d_1g_{k,l}\\&\quad=\left(A_k(0)+\sum_{\mu\in\Lambda_k}\left(A_k(\mu)\prod_{j=1}^{m-1}x_j^{\mu(j)}\prod_{j=m}^{m-1+n}\big(f_{j-m+1,l-1}+g_{j-m+1,l-1}\big)\circ(I,x)\right)^{\mu(j)}\right.\\&\qquad\left.\times\prod_{j=m+n}^{n_k}\big(d_{\nu_k(j)}\big(f_{\delta_k(j),l-1}+g_{\delta_k(j),l-1}\big)\circ(I,x)\big)^{\mu(j)}\right).\end{aligned} \tag{2.14}$$

From the argument in the induction, all terms having degree no greater than $l-1$ in t are terms of $d_1f_{k,l}(t,x)$. All terms of $d_1(V_k-f_{k,l})$ have degree at least l and the terms of $V_j(t,x)$ of degree no greater than $l-1$ are identical to those of $f_{j,l-1}(t,x)$. Therefore, since arbitrarily high powers of $d_1V_k(t,x)$ satisfy the PDE, V_k solves the same PDE as U_k, and $V=U$ by the uniqueness in the premise. $\square$

If the generator is a polynomial on the appropriate space and the initial conditions are polynomials then, as in the ODE case, the computational implementation follows directly from the proof of the theorem since only polynomial algebra and the power rule are involved. On the other hand, even if the initial conditions are polynomial, if the generator is analytic but not a polynomial, and a polynomial projection is necessary to achieve a polynomial generator, then the initial conditions for the corresponding components of the solution are analytic, but need not be polynomials. For example, consider w in Example 1.4 and suppose that $w(0,x,y)=x^2+y^2$. Then $z(0,x,y)=1/(x^2+y^2)$ which is not polynomial. In a symbolic computing environment this problem is easily addressed. However, in a numeric computing environment an analytic function cannot be entered.

Even in a numeric computing environment Theorem 2.1 solves PDEs with polynomial generators and polynomial initial conditions, an interesting and extensive class of problems. However, the theorem would carry even greater impact if analytic initial conditions and/or generators that can be projected as polynomials could also be included. Theorem 2.2 addresses this issue. We introduce Δ as the "exponent finder" for the presentation of the solution in order to distinguish it from Λ, the "exponent finder" for the generator.

THEOREM 2.2. *Suppose U is as in the premise to Theorem 2.1 and if $(t,x)\in D_U$ and $1\le k\le n$ then*

$$U_k(t,x)=\left(b_0+\sum_{\sigma\in\Delta_{k,0}} b_{0,\sigma}\prod_{i=1}^{m-1}x_i^{\sigma(i)}\right)+\sum_{j\in N}\left(b_j+\sum_{\sigma\in\Delta_{k,j}} b_{j,\sigma}\prod_{i=1}^{m-1}x_i^{\sigma(i)}\right)t^j. \tag{2.15}$$

For $r\in N$, $1\le k\le n$, $j\in N\cup\{0\}$, $M=\max\{\dim D_{\nu_s}\mid 1\le s\le n\}$, and $\Gamma_{k,j,r}=\{\alpha\mid \alpha\in\Delta_{k,j}$ and $\sum_{i=1}^{m-1}\alpha(i)\le r+M\}$ and if $(t,x)\in D_U$, then

$$T_{k,r}(t,x)=\left(b_0+\sum_{\sigma\in\Gamma_{k,0,r}} b_{0,\sigma}\prod_{i=1}^{m-1}x_i^{\sigma(i)}\right)+\sum_{j\in\underline{r}}\left(b_0+\sum_{\sigma\in\Gamma_{k,j,r}} b_{j,\sigma}\prod_{i=1}^{m-1}x_i^{\sigma(i)}\right)t^j \tag{2.16}$$

and

$$\begin{aligned}W_{k,r}(t,x)=&\left(b_0+\sum_{\sigma\in\Gamma_{k,0,r}} b_{0,\sigma}\prod_{i=1}^{m-1}x_i^{\sigma(i)}\right)\\&+\int_0^t\left(A_k(0)+\sum_{\mu\in\Lambda_k}\left(A_k(\mu)\prod_{j=1}^{m-1}x_j^{\mu(j)}\prod_{j=m}^{m-1+n}\left(T_{j-m+1,r-1}\circ(I,x)\right)^{\mu(j)}\right.\right.\\&\left.\left.\times\prod_{j=m+n}^{n_k}\left(d_{\nu_k(j)}T_{\delta_k(j),r-1}\circ(I,x)\right)^{\mu(j)}\right)\right).\end{aligned} \tag{2.17}$$

Then $W_{k,r+1}=T_{k,r+1}$ or the degree of $W_{k,r+1}-T_{k,r+1}$ is greater than $r+1$.

Proof. Let U be as in the premise to Theorem 2.1 and define for $r \in N$ and $k \in \underline{n}$, $R_{k,r}$ so that $U_k = T_{k,r} + R_{k,r}$. Then for $(t,x) \in \Re^m$, each term of $R_{k,r}(t,x)$ has degree at least $r+M+1$.

$$U_k(t,x) = a_k(x) + \int_0^t \left(A_k(0) + \sum_{\mu \in \Lambda_k} \left(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \left(U_{j-m+1} \circ (I,x)\right)^{\mu(j)} \right.\right.$$
$$\left.\left. \times \prod_{j=m+n}^{n_k} \left(d_{\nu_k(j)} U_{\delta_k(j)} \circ (I,x)\right)^{\mu(j)} \right)\right), \tag{2.18}$$

then we have

$$U_k(t,x)$$
$$= a_k(x) + \int_0^t \left(A_k(0) + \sum_{\mu \in \Lambda_k} \left(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \left(\left(T_{j-m+1,r} + R_{j-m+1,r}\right) \circ (I,x)\right)^{\mu(j)} \right.\right.$$
$$\left.\left. \times \prod_{j=m+n}^{n_k} \left(d_{\nu_k(j)} \left(\left(T_{\delta_k(j),r} + R_{\delta_k(j),r}\right) \circ (I,x)\right)^{\mu(j)} \right) \right)\right). \tag{2.19}$$

Expanding as in the proof of Theorem 2.1 using the binomial theorem,

$$U_k(t,x)$$
$$= a_k(x) + \int_0^t \left(A_k(0) + \sum_{\mu \in \Lambda_k} \left(A_k(\mu) \prod_{j=1}^{m-1} x_j^{\mu(j)} \prod_{j=m}^{m-1+n} \left(T_{j-m+1,r} \circ (I,x)\right)^{\mu(j)} \right) \right.$$
$$+ \sum_{i=1}^{\mu(j)} \left(C_i^{\mu(j)} \left(T_{j-m+1,r} \circ (I,x)\right)^{\mu(j)-i} \left(R_{j-m+1,r} \circ (I,x)\right)^i \right.$$
$$\times \prod_{j=m+n}^{n_k} \left(d_{\nu_k(j)} \left(T_{\delta_k(j),r} \circ (I,x)\right)^{\mu(j)} \right)$$
$$+ \sum_{i=1}^{\mu(j)} \left(C_i^{\mu(j)} d_{\nu_k(j)} T_{\delta_k(j),r} \circ (I,x) \right)^{\mu(j)-i}$$
$$\left.\left. \times \left(d_{\nu_k(j)} R_{\delta_k(j),r} \circ (I,x)\right)^i \right)\right). \tag{2.20}$$

Each term of the binomial expansion which has $R_{\delta.(\cdot),r}$ as a factor has degree greater than $r+M$. Any term containing a partial derivative of $R_{\delta.(\cdot),r}$ as a factor, since the order of the equation is M, has degree greater than r. Thus any term from the integral arising from these terms has degree greater than $r+1$ because of the additional power of t from the power rule. Let $a_{k,r}(x)$ denote the terms of $a_k(x)$ of degree less than or equal to $r+1$.

Thus

$$U_k(t,x)=a_{k,r}(x)+\int_0^t\left(A_k(0)\sum_{\mu\in\Lambda_k}\left(A_k(\mu)\prod_{j=1}^{m-1}x_j^{\mu(j)}\prod_{j=m}^{m-1+n}\left(T_{j-m+1,r}\circ(I,x)\right)^{\mu(j)}\right)\right). \tag{2.21}$$

plus terms of degree greater than $r+1$. Since

$$a_{k,r}(x)+\int_0^t\left(A_k(0)+\sum_{\mu\in\Lambda_k}\left(A_k(\mu)\prod_{j=1}^{m-1}x_j^{\mu(j)}\prod_{j=m}^{m-1+n}\left(T_{j-m+1,r}\circ(I,x)\right)^{\mu(j)}\right)\right)=W_{k,r+1}(t,x), \tag{2.22}$$

and $T_{k,r+1}(t,x)$ is the $(r+1)$st degree truncation of $U_k(t,x)$, the conclusion is established. □

Theorem 2.2 shows that if iterates are truncated properly, then succeeding iterates continue to generate the Maclaurin series. This makes it possible to implement the iteration in a numeric computing environment.

3. Examples

In this section, we consider four well-studied PDEs and illustrate the usefulness of the theorems and ideas presented in the last section. Our main goal is to show how the results of the last section provide a method for obtaining polynomial solutions in t with coefficients that are functions of x. We use the software package Maple to obtain symbolic results. We stress that all the presented results could have been obtained in a numerical environment using Theorem 2.2.

Example 3.1 (Burger's equation). Consider the equation

$$d_1u=-ud_2u;\qquad u(0,x)=f(x). \tag{3.1}$$

It is well known that Burger's equation can produce solutions with discontinuities and that many numerical techniques exhibit a Gibbs phenomenon for sharp fronts. If f is twice differentiable then the fourth modified Picard iterate (the modification is to truncate to all powers of the first component less than the number of the iterate; Theorem 2.2 guarantees that these are terms of the power series for the solution and Theorem 2.1 that they will be carried forward in the iteration) is given by

$$\begin{aligned}&\left(\frac{-1}{3f(x)^2f'(x)f''(x)}-\frac{1}{3f(x)f'(x)^3}\right)t^3\\&\qquad+\left(f(x)f'(x)^2+\frac{1}{2f(x)^2f''(x)}\right)t^2-f(x)f'(x)t+f(x).\end{aligned} \tag{3.2}$$

Higher degree Maclaurin polynomials are easily generated and do have high computational utility, but because of the space occupied by the printout, are not presented here.

To model a sharp front that will propagate over time, the initial condition used was $f(x) = 2 + (2/\pi)\arctan(x)$. The results from using only the third degree truncation of the fourth Picard iterate with a time step of 0.25 are presented in Figure 3.1. (Note that the initial condition is updated after each time step to what the iterate from the previous time step pairs $(0.25, x)$ with.) We note the smooth movement of the front in Figure 3.1.

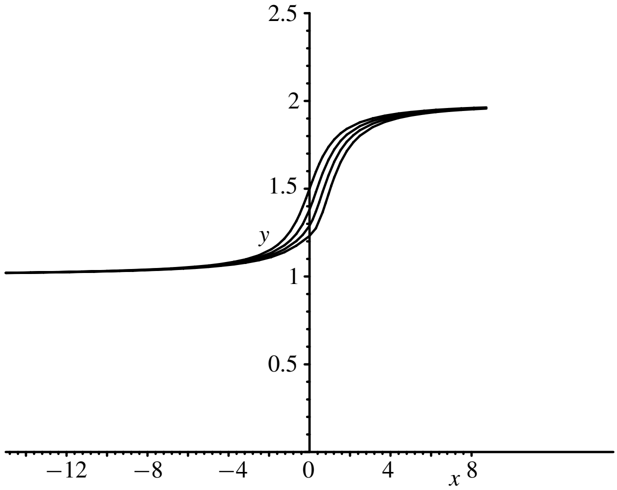


FIGURE 3.1. Graph of third degree Maclaurin polynomials for Burger's equation using modified Picard iteration at $t = 0.25$, 0.5, 0.75, and 1.0.

Of course, there need not be a unique polynomial projection for a given PDE. In [6] the authors show that for ODEs one may be able to choose a polynomial projection that reduces the number of components of the generator or a polynomial projection that reduces the degree of the generator. In that paper it was shown by example that using a polynomial projection that reduces the degree may give more accurate numerical results. However, using a polynomial projection that reduces the degree of the generator usually increases the number of components of the generator.

The following examples show that one can reduce the number of partial derivatives needed by choosing an appropriate polynomial generator. When working in a symbolic environment, at least for these examples, this reduces the computations and computational time needed to generate the modified Picard iterates.

Example 3.2 (The wave equation). Consider the wave equation

$$d_{11}u(t,x) = a\big(d_2\big(bd_2u\big)\big)(t,x); \qquad u(0,x) = p(x), \qquad d_1u(0,x) = q(x), \tag{3.3}$$

where a and b are functions of x only. This can be converted to a form to which our theorem applies in at least two ways. One is by using the polynomial projection $v = d_1 u$. This projection gives the system

$$\begin{aligned} d_1 u = v; \qquad d_1 v = a d_2 b d_2 u + a b d_{22} u, \\ u(0,x) = p(x); \qquad v(0,x) = q(x). \end{aligned} \tag{3.4}$$

If one makes the additional polynomial projection $w = d_2 u$, the system that results is

$$\begin{aligned} d_1 u = v; \qquad d_1 v = a d_2 b w + a b d_2 w; \qquad d_1 w = d_2 v, \\ u(0,x) = p(x); \qquad v(0,x) = q(x); \qquad w(0,x) = p'(x). \end{aligned} \tag{3.5}$$

This second system has more components, but the second system is first order and thus less computationally demanding than the first system. The second system generated the modified Picard iterates significantly faster than the first system. Of course, the modified Picard iterates, being the Maclaurin polynomials for u, are the same for both systems.

The third degree Maclaurin polynomial for $u(t,x)$ in this system is

$$\begin{aligned} p(x) + q(x)t + \frac{1}{2}\left(a(x)b'(x)p'(x) + \frac{1}{2a(x)b(x)p''(x)} \right) t^2 \\ + \left(\frac{1}{6} a(x)b'(x)q'(x) + \frac{1}{6a(x)b(x)q''(x)} \right) t^3. \end{aligned} \tag{3.6}$$

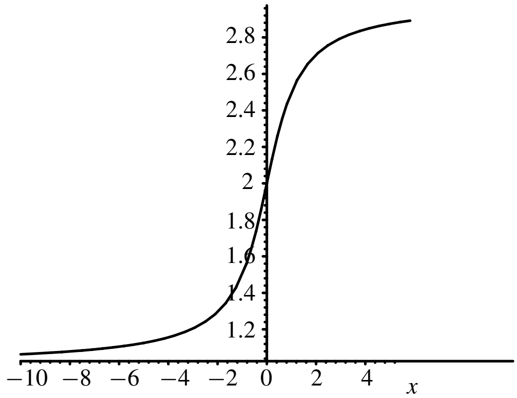
Another projection of (3.3) is obtained by letting $w = b d_2 u$. This gives the polynomial system

$$\begin{aligned} d_1 u = v; \qquad d_1 v = a d_2 w; \qquad d_1 w = b d_2 v, \\ u(0,x) = p(x); \qquad v(0,x) = q(x); \qquad w(0,x) = b p'(x). \end{aligned} \tag{3.7}$$

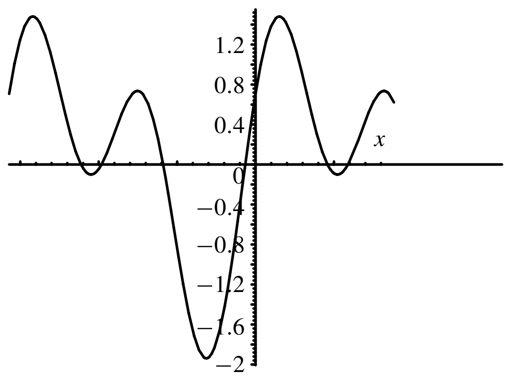
To model a sharp front that will cause reflections, $b(x)$ was set to $2 + (2/\pi)\arctan(x)$ and $a(x)$ was set to 1. A strength of Picard iteration is that one does not have to introduce boundary conditions in wave propagation to obtain numerical results. Therefore, there are no spurious reflections from boundaries. To show this $p(x)$ is set to $\cos(x) + \sin(2x - \pi/4)$ and $q(x)$ is set to 0. The graphical results for the data obtained from using only the second degree Maclaurin polynomials in t and updating the coefficients by setting $p(x)$ to $u(5/32, x)$, $q(x)$ to $d_1 u(5/32, x)$ and $w(0,x)$ to the update of $p'(x)$ are presented in Figure 3.2.

Example 3.3 (The Sine-Gordon equation). The Sine-Gordon equation has the form

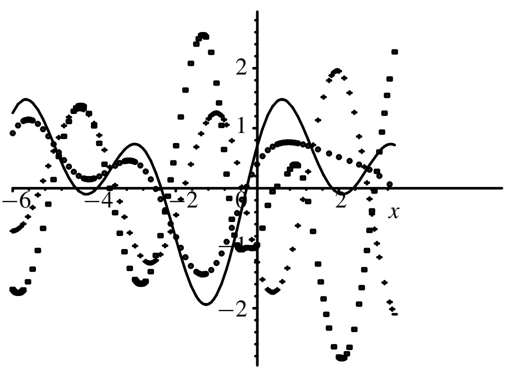
$$d_{11} u = d_{22} u + \sin \circ u; \qquad u(0,x) = p(x), \qquad d_1 u(0,x) = q(x). \tag{3.8}$$



(a) The graph of the parameter b for the wave equation runs.



(b) The graph of the initial condition for the wave equation runs.



(c) Graph of second degree Maclaurin polynomials for the wave equation using modified Picard iteration at $t = 0(\cdot)$, $4h(\circ)$, $8h(+)$ and $11h(\square)$ for $h = 5/32$.

FIGURE 3.2.

In order to use the theorems from Section 2 the polynomial projection we use is $v = d_1 u$, $y = \sin \circ u$, and $z = \cos \circ u$. Again, in order to reduce the amount of differentiation, we also let $w = d_2 u$. The resulting system is

$$\begin{aligned} d_1 u &= v; & u(0,x) &= p(x), \\ d_1 v &= d_2 w + y; & v(0,x) &= q(x), \\ d_1 w &= d_2 v; & w(0,x) &= p'(x), \\ d_1 y &= zv; & y(0,x) &= \sin\big(p(x)\big), \\ d_1 z &= -yv; & z(0,x) &= \cos\big(p(x)\big). \end{aligned} \tag{3.9}$$

This system is only a second degree polynomial system. Thus it is straightforward to program and requires little computing time. The second degree Maclaurin polynomial for $u(t,x)$ is

$$\left(\frac{1}{2} p''(x) + \frac{1}{2} \sin\big(p(x)\big)\right) t^2 + q(x)t + p(x). \tag{3.10}$$

Again boundary conditions do not have to be introduced to obtain numerical results. The graphical results in Figure 3.3 are from updating p and q as explained in Example 3.2 with $p(x)$ set to $\sin(x)$ and $q(x)$ set to 0, but using a time step of 1.25.

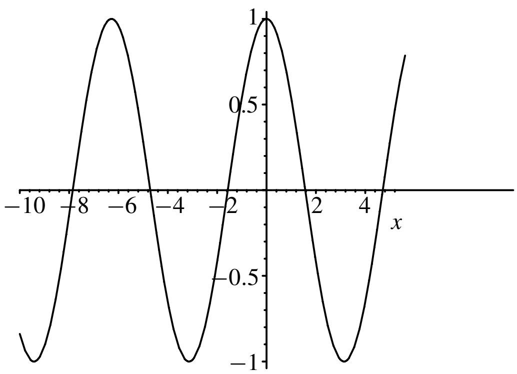
Example 3.4 (Euler's inviscid gas equations). The one dimensional system of equations is

$$\begin{aligned} d_1 \rho + d_2(\rho v) &= 0; & \rho(0,x) &= p(x), \\ d_1(\rho v) + d_2\big(\rho v^2\big) + c^2 d_2 \rho &= 0; & v(0,x) &= q(x), \end{aligned} \tag{3.11}$$

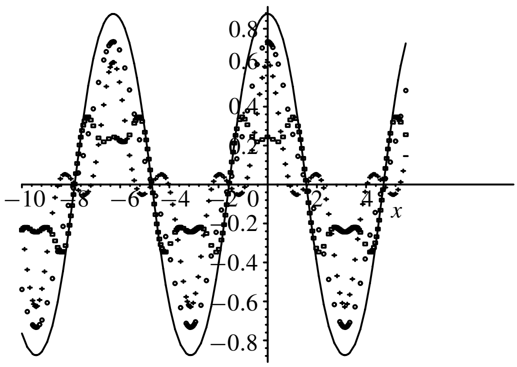
where ρ represents the density, v represents the velocity and c represents the speed of sound for the medium. To use the theorems, we make the polynomial projection $w = 1/\rho$ and the products are differentiated giving the polynomial system

$$\begin{aligned} d_1 \rho &= -v d_2 \rho - \rho d_2 v; & \rho(0,x) &= p(x), \\ d_1 v &= -v d_2 v - c^2 w d_2 \rho; & v(0,x) &= q(x), \\ d_1 w &= w^2 v d_2 \rho + w^2 \rho d_2 v; & w(0,x) &= \frac{1}{p(x)}. \end{aligned} \tag{3.12}$$

Of course other polynomial projections are possible. For example, one could let $z = \rho v$. It is well known that "shocks" can develop in the above equations. The modified Picard process presented here does not converge at the shocks, thereby giving numerical evidence of the development of the shock.



(a) The graph of the initial condition for the Sine-Gordon equation runs.



(b) Graph of second degree Maclaurin polynomials for the Sine-Gordon equation using modified Picard iteration at $t = h(\cdot)$, $2h(\circ)$, $3h(+)$, and $3.25h(\square)$ for $h = 1.25$.

FIGURE 3.3.

The third degree Maclaurin polynomial for $v(t,x)$ is

$$\begin{aligned}
&p(x) - \big(p(x)q'(x) + p'(x)q(x)p(x)\big)t \\
&\quad + \big(6\,p(x)q(x)q''(x) + 6\,p(x)q'(x)^2 + 3\,c^2 p''(x) \\
&\qquad + 12\,p(x)^2 p'(x)q(x)q'(x) + 3\,p(x)^2 q(x)^2 p''(x)\big)t^2 \\
&\quad + \big(\big(2\,p'(x)^3 q(x)c^2 - 2\,p(x)^2 p''(x)q(x)^2 q'(x) - 2\,p(x)^3 q'(x)^3 \\
&\qquad - 4\,p(x)^3 q'(x)q(x)q''(x) - 2\,p(x)q'(x)c^2 p'(x)^2 \\
&\qquad - 6\,p'(x)q(x)q'(x)p(x)^2 - 4\,p'(x)q(x)p(x)c^2 p''(x) \\
&\qquad - 2\,p'(x)q(x)^2 p(x)^2 q''(x) - 2\,p(x)^2 q'(x)c^2 p''(x) \\
&\qquad - 2\,p(x)^2 q''(x)c^2 p'(x)\big)/6p(x)^2\big)t^3.
\end{aligned} \tag{3.13}$$

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