NONEXISTENCE THEOREMS FOR WEAK SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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New nonexistence results are obtained for entire bounded (either from above or from below) weak solutions of wide classes of quasilinear elliptic equations and inequalities. It should be stressed that these solutions belong only locally to the corresponding Sobolev spaces. Important examples of the situations considered herein are the following: $\sum_{i=1}^{n}(a(x)|\nabla u|^{p-2}u_{x_i})_{x_i} = -|u|^{q-1}u$, $\sum_{i=1}^{n}(a(x)|u_{x_i}|^{p-2}u_{x_i})_{x_i} = -|u|^{q-1}u$, $\sum_{i=1}^{n}(a(x)|\nabla u|^{p-2}u_{x_i}/\sqrt{1+|\nabla u|^2})_{x_i} = -|u|^{q-1}u$, where $n \geq 1$, $p > 1$, $q > 0$ are fixed real numbers, and $a(x)$ is a nonnegative measurable locally bounded function. The methods involve the use of capacity theory in connection with special types of test functions and new integral inequalities. Various results, involving mainly classical solutions, are improved and/or extended to the present cases.

1. Introduction

This work is devoted to the study of nonexistence phenomena for entire (defined on the whole space) bounded (either from above or from below) solutions of elliptic partial differential equations and inequalities. This classical field of analysis, well known as "Liouville-type theorems," is again of interest (cf. [1, 2, 3, 5, 14, 15, 16, 17, 18, 19] and the references therein) due to the nonlinearity of the equations involved.

Our main purpose here is to obtain new nonexistence results for entire bounded (either from above or from below) weak solutions of general classes of quasilinear elliptic equations and inequalities, that may belong only locally to the corresponding Sobolev spaces. We also have succeeded in establishing a precise dependence between the character of degeneracy of ellipticity for differential operators and the nonexistence results for entire bounded (either from above or from below) weak solutions of the corresponding partial differential

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equations and inequalities. Here, we apply and extend the approach developed, initially in [7, 8, 9, 10, 11, 12] and later in [16, 17]. Note that a brief version of the present paper was announced in [6].

Typical examples of the equations considered are the following:

$$\sum_{i=1}^{n} \left(a(x)|\nabla u|^{p-2}u_{x_i}\right)_{x_i} = -|u|^{q-1}u, \tag{1.1}$$

$$\sum_{i=1}^{n} \left(a(x)\left|u_{x_i}\right|^{p-2}u_{x_i}\right)_{x_i} = -|u|^{q-1}u, \tag{1.2}$$

$$\sum_{i=1}^{n} \left(\frac{a(x)|\nabla u|^{p-2}u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right)_{x_i} = -|u|^{q-1}u, \tag{1.3}$$

where $n \geq 1$, $p > 1$, $q > 0$ are fixed real numbers, and $a(x)$ is a measurable nonnegative locally bounded function.

Note that for $a(x) \equiv 1$ the differential operators standing on the left-hand sides of (1.1), (1.2), and (1.3) are the well-known p–Laplacian, its modification (cf. [13]), and the mean curvature operator (for $p = 2$), respectively. In particular, the equation

$$\Delta u = -|u|^{q-1}u \tag{1.4}$$

is a special case of (1.1) and (1.2) with $a(x) \equiv 1$ and $p = 2$.

We consider sufficiently general classes of quasilinear elliptic equations (see the conditions (2.2), (2.3) below in comparison with the well-known ones (2.9), (2.10)). Even in the case $k(x) \equiv$ constant, differential operators satisfying conditions (2.2), (2.3) may possess an arbitrary degeneracy of ellipticity. In particular, in (1.1), (1.2), and (1.3) a function $a(x)$ can be zero on an arbitrary set in $\mathbb{R}^n$. Furthermore, for the typical equations (1.1), (1.2), and (1.3), as well as in more general situations, a function $a(x)$ may approach infinity as $x \to \infty$. What is most interesting here is that we have established a precise dependence between the character of degeneracy of ellipticity near infinity and nonexistence results. For example, there are no entire nonnegative generalized solutions of (1.1), (1.2), and (1.3) for any $p-1 < q \leq (p-1)n/(n+\delta-p)$, where $\delta \in (p-n, p)$ is, so to speak, a certain measure of degeneracy of the function $a(x)$ at infinity (see condition (2.29) and Theorems 2.4, 2.6). Note that the quantity $(p-1)n/(n+\delta-p)$ can become infinitely large as $\delta \to p-n$. Therefore, under special conditions on the nontrivial function $a(x)$, (1.1), (1.2), and (1.3) have no entire nonnegative generalized solutions for any $p-1 < q < \infty$. We have also obtained analogous results for sufficiently general classes of quasilinear elliptic equations (see conditions (2.2), (2.3), and (2.29)).

All the results of the paper are new even for (1.1), (1.2), and (1.3). Similar results to those of Theorem 2.4, for semilinear elliptic equations were obtained in [12]. For $\delta = 0$, $k(x) \equiv$ constant, Theorems 2.4, 2.6, 2.9, 2.10, and 2.15

were obtained in [9, 10, 11, 14], respectively. For $\delta = 0$ and $a(x) \equiv 1$, results close to those of Theorem 2.6 were obtained for entire positive supersolutions of (1.1) and (1.3) (for $p = 2$), provided that $p-1 \le q \le (p-1)n/(n-p)$, in [16]. Similar results to those of Theorems 2.4, 2.6, 2.10, and 2.13 were obtained for a very special case of function spaces in [17] (see the remarks after the corresponding theorems).

It is evident that similar results to those of Theorems 2.4, 2.6, 2.9, 2.10, 2.13, and 2.15 are valid for entire nonpositive (negative) generalized subsolutions of (1.1), (1.2), (1.3), (2.25), and (2.33).

The main result of the paper is Theorem 2.4. The rest of the results are also proved by the method of Theorem 2.4. We have followed this approach because of our future considerations about extending this theory to Riemannian manifolds, higher order equations, and nonlinear parabolic problems.

2. Definitions and main results

Let L be a differential operator defined formally by

$$Lu = \sum_{i=1}^{n} \frac{d}{dx_i} A_i(x, u, \nabla u). \tag{2.1}$$

We assume that the functions $A_i(x, \eta, \xi)$, $i = 1, \dots, n$, $n \ge 1$, satisfy the usual Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$. Namely, they are continuous in η, ξ for a.e. $x \in \mathbb{R}^n$ and measurable in x for any $\eta \in \mathbb{R}^1$ and $\xi \in \mathbb{R}^n$.

Definition 2.1. Let $\alpha \ge 1$ be an arbitrary fixed constant. An operator L, defined by (2.1), belongs to the class $A(\alpha)$, if

$$0 \le \sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi), \tag{2.2}$$

$$\left| \sum_{i=1}^{n} \psi_i A_i(x, \eta, \xi) \right|^{\alpha} \le k(x) |\psi|^{\alpha} \left(\sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi) \right)^{\alpha - 1}, \tag{2.3}$$

for any $\eta \in \mathbb{R}^1$, any $\xi, \psi \in \mathbb{R}^n$, and almost all $x \in \mathbb{R}^n$, where $k(x)$ is a measurable nonnegative locally bounded function.

It is easy to see that condition (2.3) is fulfilled whenever

$$\left(\sum_{i=1}^{n} A_i^2(x, \eta, \xi) \right)^{\alpha/2} \le k(x) \left(\sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi) \right)^{\alpha - 1}, \tag{2.4}$$

because the inequality

$$\left|\sum_{i=1}^{n} \psi_i A_i(x,\eta,\xi)\right|^{\alpha} \le |A|^{\alpha} |\psi|^{\alpha} \tag{2.5}$$

is valid for almost all $x \in \mathbb{R}^n$, all $\eta \in \mathbb{R}^1$, and all $\xi, \psi \in \mathbb{R}^n$.

Note that the restrictions on the behavior of the coefficients of the differential operator L in (2.3) and (2.4), for $k(x) \equiv$ constant, were introduced in [14].

It is not difficult to verify that the differential operators on the left-hand sides of (1.1), (1.2), and (1.3), respectively, belong to the classes $A(p)$, for $p > 1$. We show this, for example, for (1.1). We need to check that its coefficients satisfy the conditions (2.2) and (2.4) for $\alpha = p$, where $p > 1$. In fact, in the case of any measurable nonnegative locally bounded function $a(x)$ the expression

$$\sum_{i=1}^{n} \xi_i A_i(x,\eta,\xi) \tag{2.6}$$

equals

$$a(x)|\xi|^p, \tag{2.7}$$

and is therefore nonnegative for almost all $x \in \mathbb{R}^n$, all $\eta \in \mathbb{R}^1$, and all $\xi, \psi \in \mathbb{R}^n$.

We now verify the validity of condition (2.4). Because of

$$\begin{aligned} \left(\sum_{i=1}^{n} A_i^2(x,\eta,\xi)\right)^{\alpha/2} &= \left(a(x)|\xi|^{p-1}\right)^{\alpha}, \\ \left(\sum_{i=1}^{n} \xi_i A_i(x,\eta,\xi)\right)^{\alpha-1} &= \left(a(x)|\xi|^{p}\right)^{\alpha-1}, \end{aligned} \tag{2.8}$$

it is evident that condition (2.4) is satisfied with $\alpha = p$ and $k(x) = a(x)$ for almost all $x \in \mathbb{R}^n$, all $\eta \in \mathbb{R}^1$, and all $\xi, \psi \in \mathbb{R}^n$.

It is important to note that if the differential operators defined by (2.1) satisfy the well-known conditions

$$\left(\sum_{i=1}^{n} A_i^2(x,\eta,\xi)\right)^{1/2} \le k_1 |\xi|^{\alpha-1}, \tag{2.9}$$

$$k_2 |\xi|^{\alpha} \le \sum_{i=1}^{n} \xi_i A_i(x,\eta,\xi), \tag{2.10}$$

with some fixed positive constants k_1, k_2, then they belong to $A(\alpha)$.

In connection with this fact, we give another example of an operator that belongs to the class $A(\alpha)$, for arbitrary fixed $\alpha > 1$, but does not satisfy the condition (2.10) even if $k(x) \equiv$ constant.

Let $a(x,\eta,\xi)$ be nonnegative, locally bounded, and satisfying the Carathéodory conditions on $\mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n$. It is not difficult to verify, as above, that the differential operator N defined by

$$Nu=\operatorname{div}\big(a(x,u,\nabla u)|\nabla u|^{p-2}\nabla u\big) \tag{2.11}$$

belongs to $A(p)$, for any $p>1$, and does not satisfy condition (2.10) if the function $a(x,\eta,\xi)$ is assumed only nonnegative, but not bounded below away from zero.

It can happen that an operator L defined by (2.1) belongs simultaneously to several different classes $A(\alpha)$. We verify below that for any fixed number $p\ge 2$ the differential operator L from (1.3) is an element of the class $A(\alpha)$ for any $\alpha\in[p-1,p]$. The same is actually true for the well-known mean curvature operator

$$Lu=\sum_{i=1}^{n}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right)_{x_i}. \tag{2.12}$$

In fact, it belongs to the classes $A(\alpha)$ for any $1\le\alpha\le 2$. It should be noted that the coefficients of this operator do not satisfy condition (2.10) for any $1\le\alpha\le 2$.

Now we check that the coefficients of the differential operator defined formally by

$$Lu=\sum_{i=1}^{n}\left(\frac{a(x)|\nabla u|^{p-2}u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right)_{x_i}, \tag{2.13}$$

for $p\ge 2$, satisfy conditions (2.2) and (2.4) for any $\alpha\in[p-1,p]$. Indeed, for any measurable nonnegative locally bounded function $a(x)$ the expression

$$\sum_{i=1}^{n}\xi_i A_i(x,\eta,\xi) \tag{2.14}$$

equals

$$\frac{a(x)|\xi|^p}{\sqrt{1+|\xi|^2}}, \tag{2.15}$$

and is therefore nonnegative for almost all $x\in\mathbb{R}^n$, all $\eta\in\mathbb{R}^1$, and all $\xi,\psi\in\mathbb{R}^n$.

We now verify condition (2.4). Since

$$\begin{gathered}\left(\sum_{i=1}^{n}A_i^2(x,\eta,\xi)\right)^{1/2}=\frac{a(x)|\xi|^{p-1}}{\sqrt{1+|\xi|^2}},\\ \sum_{i=1}^{n}\xi_i A_i(x,\eta,\xi)=\frac{a(x)|\xi|^p}{\sqrt{1+|\xi|^2}},\end{gathered} \tag{2.16}$$

it is evident that condition (2.4) is satisfied with any $\alpha \in [p-1, p]$ and $k(x) = a(x)$ for almost all $x \in \mathbb{R}^n$, all $\eta \in \mathbb{R}^1$, and all $\xi, \psi \in \mathbb{R}^n$.

In connection with class $A(2)$, let L be defined formally by

$$Lu = \sum_{i,j=1}^{n} \left(a_{ij}(x, u, \nabla u) u_{x_i}\right)_{x_j}, \tag{2.17}$$

where the functions $a_{ij}(x, \eta, \xi)$ are locally bounded, satisfy the Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$, and are such that $a_{ij}(x, \eta, \xi) = a_{ji}(x, \eta, \xi)$, $i, j = 1, \dots, n$,

$$\begin{aligned} &\left(\sum_{i,j=1}^{n} a_{ij}^2(x, \eta, \xi)\right)^{1/2} \le k(x), \\ &0 \le \sum_{i,j=1}^{n} a_{ij}(x, \eta, \xi) \psi_i \psi_j, \end{aligned} \tag{2.18}$$

for almost all $x \in \mathbb{R}^n$, all $\eta \in \mathbb{R}^1$, all ξ and ψ from $\mathbb{R}^n$, and a certain measurable locally bounded $k(x)$.

Note that a linear divergent nonuniformly elliptic differential operator of the form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j}\right) \tag{2.19}$$

is a special case of (2.17).

We verify that the operator L defined formally by (2.17) belongs to the class $A(2)$, or, in other words, its coefficients satisfy conditions (2.2) and (2.3). To this end, let

$$A_i(x, \eta, \xi) = \sum_{j=1}^{n} a_{ij}(x, \eta, \xi) \xi_j, \tag{2.20}$$

where $i = 1, \dots, n$. It is trivial to verify condition (2.2) because

$$\sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi) = \sum_{i,j=1}^{n} a_{ij}(x, \eta, \xi) \xi_i \xi_j. \tag{2.21}$$

We check the validity of condition (2.3) for $\alpha = 2$. First, we observe that

$$\sum_{i=1}^{n} \psi_i A_i(x, \eta, \xi) = \sum_{i,j=1}^{n} a_{ij}(x, \eta, \xi) \psi_i \xi_j. \tag{2.22}$$

Estimating the right-hand side of this identity by Cauchy's inequality we get

$$\left(\sum_{i=1}^{n} \psi_i A_i(x,\eta,\xi)\right)^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,\eta,\xi)\psi_i\psi_j \sum_{i,j=1}^{n} a_{ij}(x,\eta,\xi)\xi_i\xi_j. \quad (2.23)$$

Using the condition of local boundedness of the coefficients $a_{ij}(x,\eta,\xi)$, we obtain

$$\left(\sum_{i=1}^{n} \psi_i A_i(x,\eta,\xi)\right)^2 \leq k(x)|\psi|^2 \sum_{i=1}^{n} \xi_i A_i(x,\eta,\xi), \quad (2.24)$$

for almost all $x \in \mathbb{R}^n$, all $\eta \in \mathbb{R}^1$, and all ξ and ψ from $\mathbb{R}^n$.

Hence, the differential operator L defined formally by (2.17) is of class $A(2)$ and does not satisfy, in general, conditions (2.9) and (2.10).

Analogously, the linear divergent elliptic differential operator that does not satisfy a uniform ellipticity condition belongs to the class $A(2)$ and does not satisfy inequalities (2.9) and (2.10).

In this paper we restrict ourselves to the study of the equation

$$Lu = -|u|^{q-1}u, \quad (2.25)$$

with an operator L from the class $A(\alpha)$, for certain fixed $\alpha \geq 1$ and $q \geq 0$, although the results formulated below are easily extendable to equations of the type

$$Lu = -f(x,u,\nabla u), \quad (2.26)$$

where the function $f(x,\eta,\xi)$ satisfies suitable growth and regularity conditions, and, for example, is such that

$$f(x,0,0) = 0, \qquad \eta f(x,\eta,\xi) \geq a|\eta|^{q+1}, \quad (2.27)$$

for certain fixed positive numbers a and q, and almost all $x \in \mathbb{R}^n$, all $\eta \in \mathbb{R}^1$, and all $\xi \in \mathbb{R}^n$.

We define below the concept of an entire positive (nonnegative) generalized supersolution of (2.25).

Definition 2.2. A function $u \in L_{1,\text{loc}}(\mathbb{R}^n)$ is said to be positive (nonnegative) in $\mathbb{R}^n$, if ess-inf $u(x)$, taken over any ball in $\mathbb{R}^n$, is finite and positive (nonnegative).

Definition 2.3. Let $q > 0$ and $\alpha \geq 1$ be fixed real numbers, and let the operator L belong to the class $A(\alpha)$. A function $u(x)$ is said to be an entire generalized supersolution ($Lu \le -|u|^{q-1}u$) of (2.25), if it belongs to the space $W^1_{\alpha,\text{loc}}(\mathbb{R}^n) \cap L_{q,\text{loc}}(\mathbb{R}^n)$ and satisfies the integral inequality

$$\int_{\mathbb{R}^n} \left[\sum_{i=1}^{n} \varphi_{x_i} A_i(x,u,\nabla u) - |u|^{q-1}u\varphi \right] dx \ge 0 \tag{2.28}$$

for every nonnegative function $\varphi \in \overset{\circ}{C}{}^{\infty}(\mathbb{R}^n)$.

In what follows, we let δ be a real number less than α, $B(R)$ the open ball in $\mathbb{R}^n$ with center at the origin and radius R, and assume that $k(x)$ in the condition (2.3) is such that

$$K(R) := \sup_{B(R)\setminus B(R/2)} k(x) \le c\left(1+R^2\right)^{\delta/2}, \tag{2.29}$$

for a fixed constant $c > 0$ and any $R > 0$.

THEOREM 2.4. *Let $1 < \alpha < n+\delta$, let $u(x)$ be an entire nonnegative generalized supersolution of (2.25), and let the operator L satisfy conditions (2.2), (2.3), and (2.29). Then $u(x) = 0$ a.e. in $\mathbb{R}^n$, for any $\alpha - 1 < q \le (\alpha-1)n/(n+\delta-\alpha)$.*

Remark 2.5. For $\delta = 0$, $k(x) \equiv$ constant, and $\alpha - 1 < q < (\alpha-1)n/(n-\alpha)$, Theorem 2.4 was obtained in [9, 10, 11].

The following result is a special case of Theorem 2.4.

THEOREM 2.6. *Let $1 < p < n+\delta$, $\delta < p$, let the function $a(x)$ satisfy condition (2.29), and let $u(x)$ be an entire nonnegative generalized supersolution of (1.1), (1.2), or (1.3). Then $u(x) = 0$ a.e. in $\mathbb{R}^n$, for any $p-1 < q \le (p-1)n/(n+\delta-p)$.*

Remark 2.7. Similar results to those of Theorem 2.6 for entire positive supersolutions of (1.1) and (1.3) (for $p = 2$), with $\delta = 0$, $a(x) \equiv 1$, and $p-1 \le q \le (p-1)n/(n-p)$, were announced in [16].

It is important to note that for a suitable constant $c > 0$, $n+\delta > p > 1$, $p > \delta$, and $q > n(p-1)/(n+\delta-p)$, the radially symmetric function

$$u(x) = c\left(1+|x|^{p/(p-1)}\right)^{(1-p)(p-\delta)/p(q-p+1)} \tag{2.30}$$

is an entire nonnegative supersolution of (1.1), (1.2), and (1.3) with the measurable nonnegative locally bounded function

$$a(x) \equiv \left(1+|x|^{p/(p-1)}\right)^{\delta(p-1)/p}. \tag{2.31}$$

However, if an entire generalized supersolution of (2.25) is bounded from below by any positive constant, then the following result is valid.

Definition 2.8. A function $u \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$ is said to be bounded from below by a certain positive constant in $\mathbb{R}^n$, if $\operatorname{ess-inf} u(x)$, taken over any ball in $\mathbb{R}^n$, is finite and not less than that constant.

THEOREM 2.9. *Let $1 < \alpha < n+\delta$, $\alpha - 1 < q$, and let the operator L satisfy conditions (2.2), (2.3), and (2.29). Then there exists no entire generalized supersolution of (2.25) bounded from below by a positive constant.*

The following result, as well as Theorem 2.9, provides more clarity to the understanding of Theorem 2.4.

THEOREM 2.10. *Let $1 < \alpha < n+\delta$, $0 < q < \alpha - 1$, and let the operator L satisfy conditions (2.2), (2.3), and (2.29). Then there exists no entire positive generalized supersolution of (2.25).*

Remark 2.11. Similar results to those of Theorem 2.10 for $\delta = 0$ and $k(x) \equiv$ constant were obtained in [17] in very special function spaces. Note that for $\delta = 0$ and $k(x) \equiv$ constant, Theorems 2.9 and 2.10 were obtained in [9, 10, 11].

Analogous results to those of Theorems 2.4, 2.6, 2.9, and 2.10 are also valid for $\alpha \geq n+\delta$ and are simple corollaries of the fact that in this special case all entire nonnegative solutions of the inequality $Lu \leq 0$, with an operator L from the class $A(\alpha)$, are identically constant under the following condition: if

$$\sum_{i=1}^{n} \xi_i A_i(x,\eta,\xi) = 0, \tag{2.32}$$

then $\xi = 0$.

We now define the concept of a supersolution of the equation

$$Lu = 0. \tag{2.33}$$

Definition 2.12. Let $\alpha \geq 1$ be a fixed real number and let the operator L belong to the class $A(\alpha)$. A function $u(x)$ is said to be an entire generalized supersolution ($Lu \leq 0$) of (2.33), if it belongs to the space $W^1_{\alpha,\mathrm{loc}}(\mathbb{R}^n)$ and satisfies the integral inequality

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{x_i} A_i(x,u,\nabla u)\,dx \geq 0 \tag{2.34}$$

for every nonnegative function $\varphi \in \overset{\circ}{W}{}^1_\alpha(\mathbb{R}^n)$.

THEOREM 2.13. *Let $\alpha > 1$, $\alpha \geq n+\delta$, $q > 0$, and let the operator L satisfy conditions (2.2), (2.3), and (2.29). If $u(x)$ is an entire nonnegative generalized supersolution of (2.25), then $u(x) = 0$ a.e. in $\mathbb{R}^n$.*

Remark 2.14. In the case $\delta = 0$ and $k(x) \equiv$ constant, similar results to those of Theorem 2.13 were announced for supersolutions of (1.1) and (1.3) (for $p = 2$), under the assumption that $a(x) \equiv 1$, in [16]. However, it is not hard to see that these results from [16] are very special cases of similar results from [14].

THEOREM 2.15. *Let $\alpha > 1$, $\alpha \geq n+\delta$, and let the operator L satisfy conditions (2.2), (2.3), (2.29), and (2.32). Let $u(x)$ be an entire nonnegative generalized supersolution of (2.33). Then $u(x) =$ constant a.e. in $\mathbb{R}^n$.*

Remark 2.16. In the case $\delta = 0$ and $k(x) \equiv$ constant, results very close to those of Theorem 2.15 were obtained in [14].

In our proofs of Theorems 2.4, 2.6, 2.9, 2.10, 2.13, and 2.15, we make use of the well-known variational capacity concept. As we mentioned above, our approach (using the concept of the variational capacity) can be directly applied to the study of analogous problems for partial differential equations on Riemannian manifolds.

Definition 2.17. Let G be a domain in $\mathbb{R}^n$ and let P, Q be subsets of G which are disjoint and closed in G (in the relative topology). We call any such triple $(P, Q; G)$ a condenser.

Fix $\gamma \geq 1$. The quantity

$$\operatorname{cap}_\gamma(P, Q; G) = \inf \int_G |\nabla \zeta|^\gamma \, dx \tag{2.35}$$

is called the γ-capacity of the condenser $(P, Q; G)$. Here, the infimum is taken over all nonnegative functions ζ of the space $C^\infty(G)$ which equal 1 on P and 0 on Q.

3. Proofs of the main results

Proof of Theorem 2.4. Let $q > \alpha - 1$, $n+\delta > \alpha > 1$, let $u(x)$ be an entire nonnegative generalized supersolution of (2.25), and let the operator L satisfy conditions (2.2), (2.3), and (2.29). Let r and ε be arbitrary positive numbers, $R = 2r$, and $\zeta(x)$ an arbitrary function from the space $\overset{\circ}{C}{}^\infty(B(R))$ which equals 1 on $B(r)$ and is such that $0 \leq \zeta(x) \leq 1$. Without loss of generality, we substitute $\varphi(x) = (u(x)+\varepsilon)^{-t}\zeta^s(x)$ as a test function in inequality (2.28), where the positive constants $s \geq \alpha$ and $q > t > 0$ will be chosen below. Integrating by parts we obtain

$$
\begin{aligned}
-t\int_{B(R)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{s}\,dx& \\
+s\int_{B(R)}\sum_{i=1}^{n}\zeta_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t}\zeta^{s-1}\,dx& \\
\equiv I_1+I_2\geq a\int_{B(R)}u^{q}(u+\varepsilon)^{-t}\zeta^{s}\,dx.&
\end{aligned}
\tag{3.1}
$$

Using condition (2.3) on the coefficients of the operator L, we easily obtain

$$
\begin{aligned}
|I_2|&=\left|s\int_{B(R)}\sum_{i=1}^{n}\zeta_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t}\zeta^{s-1}\,dx\right| \\
&\leq\int_{B(R)}s\big(k(x)\big)^{1/\alpha}\left(\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha}|\nabla\zeta|(u+\varepsilon)^{-t}\zeta^{s-1}\,dx.
\end{aligned}
\tag{3.2}
$$

Estimating, further, the integrand on the right-hand side of (3.2) by using Young's inequality

$$
AB\leq\rho A^{\beta/(\beta-1)}+\rho^{1-\beta}B^{\beta},
\tag{3.3}
$$

where $\rho=t/2$, $\beta=\alpha$,

$$
A=(u+\varepsilon)^{(1+t)(1-\alpha)/\alpha}\zeta^{s(\alpha-1)/\alpha}\left(\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha},
\tag{3.4}
$$

and $B=s(k(x))^{1/\alpha}|\nabla\zeta|\zeta^{s/\alpha-1}(u+\varepsilon)^{(\alpha-1-t)/\alpha}$, we arrive at

$$
\begin{aligned}
|I_2|\leq\frac{t}{2}\int_{B(R)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{s}\,dx& \\
+\int_{B(R)}s^{\alpha}\left(\frac{t}{2}\right)^{1-\alpha}k(x)|\nabla\zeta|^{\alpha}(u+\varepsilon)^{-t+\alpha-1}\zeta^{s-\alpha}\,dx.&
\end{aligned}
\tag{3.5}
$$

It follows from (3.1), (3.2), and (3.5) that

$$
\begin{aligned}
\int_{B(R)}s^{\alpha}\left(\frac{t}{2}\right)^{1-\alpha}k(x)|\nabla\zeta|^{\alpha}(u+\varepsilon)^{-t+\alpha-1}\zeta^{s-\alpha}\,dx& \\
\geq a\int_{B(R)}u^{q}(u+\varepsilon)^{-t}\zeta^{s}\,dx& \\
+\frac{t}{2}\int_{B(R)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{s}\,dx.&
\end{aligned}
\tag{3.6}
$$

Estimating now the integrand on the left-hand side of (3.6) by Young's inequality (3.3), with $\rho = a/2$, $A = (u+\varepsilon)^{\alpha-t-1}\zeta^{s(\alpha-1-t)/(q-t)}$, $B = k(x)s^{\alpha}(t/2)^{1-\alpha}|\nabla\zeta|^{\alpha}\zeta^{s(q-\alpha+1)/(q-t)-\alpha}$, and $\beta = (q-t)/(q-\alpha+1)$, we obtain

$$\begin{aligned}\frac{1}{2}\int_{B(R)\setminus B(r)}&(u+\varepsilon)^{q-t}\zeta^{s}\,dx+\frac{1}{2}\big(2^{\alpha}s^{\alpha}t^{1-\alpha}a^{-1}K(R)\big)^{(q-t)/(q-\alpha+1)}\\ &\times\int_{B(R)}|\nabla\zeta|^{\alpha(q-t)/(q-\alpha+1)}\zeta^{s-\alpha(q-t)/(q-\alpha+1)}\,dx\\ &\geq\int_{B(R)}u^{q}(u+\varepsilon)^{-t}\zeta^{s}\,dx+\frac{t}{2a}\int_{B(R)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{s}\,dx.\end{aligned}\tag{3.7}$$

We now estimate the integral $\int_{B(R)}u^{q}\zeta^{s}\,dx$ using inequality (3.7). To this end, we substitute $\varphi(x)=\zeta^{s}(x)$ in inequality (2.28). After integration by parts, we have

$$s\int_{B(R)}\sum_{i=1}^{n}\zeta_{x_i}A_i(x,u,\nabla u)\zeta^{s-1}\,dx\geq a\int_{B(R)}u^{q}\zeta^{s}\,dx.\tag{3.8}$$

Since by condition (2.3)

$$\sum_{i=1}^{n}\zeta_{x_i}A_i(x,u,\nabla u)\leq\big(k(x)\big)^{1/\alpha}|\nabla\zeta|\left(\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha},\tag{3.9}$$

we have

$$a\int_{B(R)}u^{q}\zeta^{s}\,dx\leq s\big(K(R)\big)^{1/\alpha}\int_{B(R)}\left(\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha}|\nabla\zeta|\zeta^{s-1}\,dx.\tag{3.10}$$

Estimating the right-hand side of (3.10) by Hölder's inequality, it is easy to see that the inequality

$$\begin{aligned}a\int_{B(R)}u^{q}\zeta^{s}\,dx\leq s\big(K(R)\big)^{1/\alpha}&\left(\int_{B(R)}|\nabla\zeta|^{\alpha}(u+\varepsilon)^{(\alpha-1)(t+1)}\zeta^{s-\alpha}\,dx\right)^{1/\alpha}\\ &\times\left(\int_{B(R)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{s}\,dx\right)^{(\alpha-1)/\alpha}\end{aligned}\tag{3.11}$$

is valid for any $\varepsilon > 0$. Since, for any $d > 1$,

$$\int_{B(R)} |\nabla\zeta|^{\alpha}(u+\varepsilon)^{(\alpha-1)(t+1)}\zeta^{s-\alpha}\,dx \leq \left(\int_{B(R)\setminus B(r)} (u+\varepsilon)^{d(\alpha-1)(1+t)}\zeta^{s}\,dx\right)^{1/d} \times \left(\int_{B(R)} |\nabla\zeta|^{\alpha d/(d-1)}\zeta^{s-\alpha d/(d-1)}\,dx\right)^{(d-1)/d}, \tag{3.12}$$

by choosing, for any fixed and sufficiently small t from the interval $(0,q)\cap(0,(q-\alpha+1)/(\alpha-1))$, the parameter $d = q/(\alpha-1)(1+t)$ such that $q = d(\alpha-1)(1+t)$, it follows from inequalities (3.11) and (3.12) that

$$a\int_{B(R)} u^{q}\zeta^{s}\,dx \leq s\big(K(R)\big)^{1/\alpha}\left(\int_{B(R)} |\nabla\zeta|^{\alpha d/(d-1)}\zeta^{s-\alpha d/(d-1)}\,dx\right)^{(d-1)/\alpha d} \times \left(\int_{B(R)\setminus B(r)} (u+\varepsilon)^{q}\zeta^{s}\,dx\right)^{1/\alpha d} \times \left(\int_{B(R)} \sum_{i=1}^{n} u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{s}dx\right)^{(\alpha-1)/\alpha}. \tag{3.13}$$

Estimating the last term on the right-hand side of inequality (3.13) by formula (3.7), we have

$$\begin{aligned} a\int_{B(R)} u^{q}\zeta^{s}\,dx &\leq s\big(K(R)\big)^{1/\alpha}\left(\int_{B(R)} |\nabla\zeta|^{\alpha d/(d-1)}\zeta^{s-\alpha d/(d-1)}\,dx\right)^{(d-1)/\alpha d} \\ &\quad\times \left(\int_{B(R)\setminus B(r)} (u+\varepsilon)^{q}\zeta^{s}\,dx\right)^{1/\alpha d} \\ &\quad\times \Bigg(at^{-1}\big(2^{\alpha}s^{\alpha}t^{1-\alpha}a^{-1}K(R)\big)^{(q-t)/(q-\alpha+1)} \\ &\qquad\times \int_{B(R)} |\nabla\zeta|^{\alpha(q-t)/(q-\alpha+1)}\zeta^{s-\alpha(q-t)/(q-\alpha+1)}\,dx \\ &\qquad+\frac{a}{t}\int_{B(R)\setminus B(r)} (u+\varepsilon)^{q-t}\zeta^{s}\,dx - \frac{2a}{t}\int_{B(R)} u^{q}(u+\varepsilon)^{-t}\zeta^{s}\,dx\Bigg)^{(\alpha-1)/\alpha}. \end{aligned} \tag{3.14}$$

Passing to the limit as $\varepsilon \to 0$ by Lebesgue's theorem, we get

$$
\begin{aligned}
a & \int_{B(R)} u^q \zeta^s \, dx \\
& \le s\big(K(R)\big)^{1/\alpha} \bigg(\int_{B(R)} |\nabla \zeta|^{\alpha d/(d-1)} \zeta^{s-\alpha d/(d-1)} \, dx \bigg)^{(d-1)/\alpha d} \\
& \quad \times \bigg(\int_{B(R)\setminus B(r)} u^q \zeta^s \, dx \bigg)^{1/\alpha d} \\
& \quad \times \bigg(a t^{-1} \big(2^\alpha s^\alpha t^{1-\alpha} a^{-1} K(R)\big)^{(q-t)/(q-\alpha+1)} \\
& \qquad \times \int_{B(R)} |\nabla \zeta|^{\alpha(q-t)/(q-\alpha+1)} \zeta^{s-\alpha(q-t)/(q-\alpha+1)} \, dx \bigg)^{(\alpha-1)/\alpha},
\end{aligned}
\tag{3.15}
$$

and therefore, for sufficiently large s,

$$
\begin{aligned}
a \bigg(\int_{B(r)} u^q \, dx \bigg)^{(\alpha d-1)/\alpha d} & \le s\big(K(R)\big)^{1/\alpha} \bigg(\int_{B(R)} |\nabla \zeta|^{\alpha d/(d-1)} dx \bigg)^{(d-1)/\alpha d} \\
& \quad \times \bigg(a t^{-1} \big(2^\alpha s^\alpha t^{1-\alpha} a^{-1} K(R)\big)^{(q-t)/(q-\alpha+1)} \\
& \qquad \times \int_{B(R)} |\nabla \zeta|^{\alpha(q-t)/(q-\alpha+1)} dx \bigg)^{(\alpha-1)/\alpha}.
\end{aligned}
\tag{3.16}
$$

Minimizing the right-hand side of the inequality obtained over all admissible functions $\zeta(x)$ of the type indicated above (which is equivalent to the calculation of the γ_1- and γ_2-capacities of the condenser $(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n)$ with $\gamma_1 = \alpha d/(d-1)$ and $\gamma_2 = \alpha(q-t)/(q-\alpha+1)$, (cf. [4])), we obtain

$$
\begin{aligned}
a^{1/\alpha} & \bigg(\int_{B(r)} u^q \, dx \bigg)^{(\alpha d-1)/\alpha d} \\
& \le \Big(t^{-1} \big(2^\alpha t^{1-\alpha} a^{-1}\big)^{(q-t)/(q-\alpha+1)} \Big)^{(\alpha-1)/\alpha} \big(s^\alpha K(R)\big)^{(\alpha q-(\alpha-1)(1+t))/\alpha(q-\alpha+1)} \\
& \quad \times \big(\operatorname{cap}_{\gamma_1}\big(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n\big)\big)^{1/\gamma_1} \big(\operatorname{cap}_{\gamma_2}\big(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n\big)\big)^{(\alpha-1)/\alpha}.
\end{aligned}
\tag{3.17}
$$

Since, for any $\gamma \ge 1$ and $R = 2r$, it is well known that the γ-capacity of the condenser $(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n)$ is $O(R^{n-\gamma})$ as $R \to \infty$, it follows from (2.29) and (3.17) that

$$
\bigg(\int_{B(r)} u^q \, dx \bigg)^{(\alpha d-1)/\alpha d} = O\big(R^{\gamma_3}\big) \tag{3.18}
$$

as $R \to \infty$, where

$$\gamma_3 = \frac{n-\gamma_1}{\gamma_1} + \frac{(\alpha-1)\big(n-\gamma_2\big)}{\alpha} + \delta\frac{\alpha q-(\alpha-1)(1+t)}{\alpha(q-\alpha+1)}, \tag{3.19}$$

or, equivalently,

$$\gamma_3 = \frac{(n+\delta-\alpha)\big(\alpha q-\alpha+1-t(\alpha-1)\big)}{\alpha q(q-\alpha+1)}\left(q-\frac{n(\alpha-1)}{n+\delta-\alpha}\right). \tag{3.20}$$

Now, since, for any $t \in (0,q)$, the quantity

$$\frac{(n+\delta-\alpha)\big(\alpha q-\alpha+1-t(\alpha-1)\big)}{\alpha q(q-\alpha+1)} \tag{3.21}$$

is positive, it follows easily from above that if $\alpha-1<q<n(\alpha-1)/(n+\delta-\alpha)$, then $\int_{\mathbb{R}^n} u^q\,dx=0$. Also, if $q=n(\alpha-1)/(n+\delta-\alpha)$, then $\int_{\mathbb{R}^n} u^q\,dx$ is bounded. Therefore, due to monotonicity, the integral sequence

$$\int_{B(2r_k)\setminus B(r_k)} u^q\,dx \longrightarrow 0 \tag{3.22}$$

for any sequence $r_k \to \infty$. On the other hand, for sufficiently large s, it follows from (3.15) that

$$\begin{aligned} a\int_{B(r)} u^q\,dx \le{}& s\big(K(R)\big)^{1/\alpha}\left(\int_{B(R)} |\nabla\zeta|^{\alpha d/(d-1)}dx\right)^{(d-1)/\alpha d} \\ &\times\left(\int_{B(R)\setminus B(r)} u^q\,dx\right)^{1/\alpha d} \\ &\times\Bigg(at^{-1}\big(2^\alpha s^\alpha t^{1-\alpha}a^{-1}K(R)\big)^{(q-t)/(q-\alpha+1)} \\ &\quad\times\int_{B(R)} |\nabla\zeta|^{\alpha(q-t)/(q-\alpha+1)}dx\Bigg)^{(\alpha-1)/\alpha}. \end{aligned} \tag{3.23}$$

Minimizing again the right-hand side of this inequality over all admissible functions $\zeta(x)$ of the type indicated above, we obtain

$$\begin{aligned} a^{1/\alpha}&\int_{B(r)} u^q\,dx \\ &\le \Big(t^{-1}\big(2^\alpha t^{1-\alpha}a^{-1}\big)^{(q-t)/(q-\alpha+1)}\Big)^{(\alpha-1)/\alpha} \\ &\quad\times\big(s^\alpha K(R)\big)^{(\alpha q-(\alpha-1)(1+t))/\alpha(q-\alpha+1)}\left(\int_{B(R)\setminus B(r)} u^q\,dx\right)^{1/\alpha d} \\ &\quad\times\Big(\mathrm{cap}_{\gamma_1}\big(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n\big)\Big)^{1/\gamma_1} \\ &\quad\times\Big(\mathrm{cap}_{\gamma_2}\big(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n\big)\Big)^{(\alpha-1)/\alpha}. \end{aligned} \tag{3.24}$$

By capacity theory and condition (2.29) we have

$$
\begin{aligned}
&\big(K(R)\big)^{(\alpha q-(\alpha-1)(1+t))/\alpha(q-\alpha+1)} \\
&\qquad \times \big(\operatorname{cap}_{\gamma_1}\big(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n\big)\big)^{1/\gamma_1} \\
&\qquad \times \big(\operatorname{cap}_{\gamma_2}\big(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n\big)\big)^{(\alpha-1)/\alpha} = O\big(R^{\gamma_3}\big)
\end{aligned}
\tag{3.25}
$$

as $R \to \infty$. Thus, (3.22) and (3.24) imply directly, for $q = n(\alpha-1)/(n+\delta-\alpha)$ (i.e., for $\gamma_3 = 0$), that the integral sequence

$$
\int_{B(r_k)} u^q \, dx \longrightarrow 0 \tag{3.26}
$$

as $r_k \to \infty$. This implies again that $\int_{\mathbb{R}^n} u^q \, dx = 0$. □

Proof of Theorem 2.9. Let $n+\delta > \alpha > 1$, $q > \alpha - 1$, and let the operator L belong to the class $A(\alpha)$. Suppose that there exists an entire generalized supersolution $u(x)$ of (2.25) bounded from below by a fixed positive constant. To prove our assertion by contradiction, let r be a positive constant, $R = 2r$, and $\zeta(x)$ an arbitrary function from the space $\overset{\circ}{C}{}^{\infty}(B(R))$ which equals 1 on $B(r)$ and is such that $0 \le \zeta(x) \le 1$. Substituting, without loss of generality, $\varphi(x) = (u(x))^{-t}\zeta^s(x)$ as a test function in the inequality (2.28), where the positive constants $s \ge \alpha$ and $\alpha - 1 > t > 0$ will be suitably chosen below, and integrating by parts, we obtain

$$
\begin{aligned}
&-t \int_{B(R)} \sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u) u^{-t-1} \zeta^s \, dx \\
&\qquad + s \int_{B(R)} \sum_{i=1}^n \zeta_{x_i} A_i(x,u,\nabla u) u^{-t} \zeta^{s-1} \, dx \\
&\quad \equiv I_1 + I_2 \ge a \int_{B(R)} u^{q-t} \zeta^s \, dx.
\end{aligned}
\tag{3.27}
$$

Using condition (2.3) on the coefficients of the operator L, we easily obtain

$$
\begin{aligned}
|I_2| &= \left| s \int_{B(R)} \sum_{i=1}^n \zeta_{x_i} A_i(x,u,\nabla u) u^{-t} \zeta^{s-1} \, dx \right| \\
&\le \int_{B(R)} s(k(x))^{1/\alpha} \left(\sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u) \right)^{(\alpha-1)/\alpha} |\nabla\zeta| u^{-t} \zeta^{s-1} \, dx.
\end{aligned}
\tag{3.28}
$$

Estimating, further, the integrand on the right-hand side of the relation (3.28) by using Young's inequality (3.3), for $\rho = t$, $\beta = \alpha$,

$$A = (u+\varepsilon)^{(1+t)(1-\alpha)/\alpha}\zeta^{s(\alpha-1)/\alpha}\left(\sum_{i=1}^{n} u_{x_i}A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha}, \tag{3.29}$$

and $B = s(k(x))^{1/\alpha}|\nabla\zeta|\zeta^{s/\alpha-1}(u+\varepsilon)^{(\alpha-1-t)/\alpha}$, we get

$$\begin{aligned}\left|I_2\right| &\le t\int_{B(R)}\sum_{i=1}^{n} u_{x_i}A_i(x,u,\nabla u)u^{-t-1}\zeta^s\,dx \\ &\quad + s^\alpha t^{1-\alpha}K(R)\int_{B(R)}|\nabla\zeta|^\alpha u^{-t+\alpha-1}\zeta^{s-\alpha}\,dx.\end{aligned} \tag{3.30}$$

Because of (2.29), it follows from (3.27) and (3.30) that

$$cs^\alpha t^{1-\alpha}\left(1+R^2\right)^{\delta/2}\int_{B(R)}|\nabla\zeta|^\alpha u^{-t+\alpha-1}\zeta^{s-\alpha}\,dx \ge a\int_{B(R)}u^{q-t}\zeta^s\,dx. \tag{3.31}$$

Choosing $s=\alpha(q-t)/(q-\alpha+1)$ in (3.31), so that $(s-\alpha)(q-t)/(\alpha-1-t) = s$, and then estimating the left-hand side of (3.31) by Hölder's inequality, we get

$$\begin{aligned}&cs^\alpha t^{1-\alpha}\left(1+R^2\right)^{\delta/2}\left(\int_{B(R)}|\nabla\zeta|^{\alpha(q-t)/(q-\alpha+1)}dx\right)^{(q-\alpha+1)/(q-t)} \\ &\qquad \times\left(\int_{B(R)}u^{q-t}\zeta^s\,dx\right)^{(\alpha-1-t)/(q-t)} \\ &\quad \ge a\int_{B(R)}u^{q-t}\zeta^s\,dx.\end{aligned} \tag{3.32}$$

Therefore,

$$\left(cs^\alpha t^{1-\alpha}a^{-1}\left(1+R^2\right)^{\delta/2}\right)^{s/\alpha}\int_{B(R)}|\nabla\zeta|^{\alpha(q-t)/(q-\alpha+1)}dx \ge \int_{B(R)}u^{q-t}\zeta^s\,dx. \tag{3.33}$$

Minimizing the left-hand side of the inequality obtained over all admissible functions $\zeta(x)$ of the type indicated above (which is equivalent to the calculation of s-capacity of a certain condenser, (cf. [4])) we get

$$\operatorname{cap}_s\left(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n\right)\left(ca^{-1}\left(1+R^2\right)^{\delta/2}s^\alpha t^{1-\alpha}\right)^{s/\alpha} \ge \int_{B(r)}u^{q-t}\,dx, \tag{3.34}$$

where $\operatorname{cap}_s(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n)$ is the s-capacity of the condenser $(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n)$. From elementary capacity theory we have that the s-capacity of the condenser $(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n)$ is $O(R^{n-s})$ for $R = 2r$ as $R \to \infty$. Therefore, it follows from (3.34) that

$$\int_{B(r)} u^{q-t}\, dx = O\big(R^{n-s+s\delta/\alpha}\big) \tag{3.35}$$

for $R = 2r$ as $R \to \infty$. As long as

$$n - s + \frac{s\delta}{\alpha} = n - \frac{(\alpha-\delta)(q-t)}{q-\alpha+1}, \tag{3.36}$$

the exponent $n - s + s\delta/\alpha$ is strictly less than n for any fixed constant t from the interval $(0, \alpha-1)$. This is impossible because $u(x)$ is bounded below by a fixed positive constant, and we have a contradiction to our assumption. □

Proof of Theorem 2.10. Let $n+\delta > \alpha > 1, \alpha - 1 > q > 0$, and let the operator L belong to the class $A(\alpha)$. Suppose that there exists an entire positive generalized supersolution $u(x)$ of (2.25). Let r be a positive number, $R = 2r$, $\zeta(x)$ be a function from the space $\overset{\circ}{C}{}^{\infty}(B(R))$ which equals 1 on $B(r)$ and is such that $0 \le \zeta(x) \le 1$. Without loss of generality, substitute $\varphi(x) = (u(x))^{-t}\zeta^s(x)$ as a test function in the inequality (2.28), where the positive constants $s \ge \alpha$ and $t > \alpha - 1$ will be chosen below. Integrating by parts we obtain

$$\begin{aligned} -t \int_{B(R)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u) u^{-t-1} \zeta^s\, dx \\ + s \int_{B(R)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u) u^{-t} \zeta^{s-1}\, dx \\ \equiv I_1 + I_2 \ge a \int_{B(R)} u^{q-t} \zeta^s\, dx. \end{aligned} \tag{3.37}$$

Using condition (2.3) on the coefficients of the operator L, we easily get

$$\begin{aligned} |I_2| &= \left| s \int_{B(R)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u) u^{-t} \zeta^{s-1}\, dx \right| \\ &\le \int_{B(R)} s\big(k(x)\big)^{1/\alpha} \left(\sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u) \right)^{(\alpha-1)/\alpha} |\nabla\zeta| u^{-t} \zeta^{s-1}\, dx. \end{aligned} \tag{3.38}$$

Estimating further the integrand on the right-hand side of the relation (3.38) by using of Young's inequality, as well as in the proof of Theorem 2.9, we have

$$\begin{aligned} |I_2| \le t \int_{B(R)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u) u^{-t-1} \zeta^s \, dx \\ + s^\alpha t^{1-\alpha} K(R) \int_{B(R)} |\nabla \zeta|^\alpha u^{-t+\alpha-1} \zeta^{s-\alpha} \, dx. \end{aligned} \tag{3.39}$$

It follows from (2.29), (3.37), (3.38), and (3.39) that

$$cs^\alpha t^{1-\alpha} (1+R^2)^{\delta/2} \int_{B(R)} |\nabla \zeta|^\alpha u^{-t+\alpha-1} \zeta^{s-\alpha} \, dx \ge a \int_{B(R)} u^{q-t} \zeta^s \, dx. \tag{3.40}$$

Choose $s = \alpha(t-q)/(\alpha-1-q)$ in (3.40), so that $(s-\alpha)(t-q)/(t-\alpha+1) = s$, and estimate the left-hand side of (3.40) by Hölder's inequality. We get

$$\begin{aligned} cs^\alpha t^{1-\alpha} (1+R^2)^{\delta/2} \left(\int_{B(R)} |\nabla \zeta|^{\alpha(t-q)/(\alpha-1-q)} dx \right)^{(\alpha-1-q)/(t-q)} \\ \times \left(\int_{B(R)} u^{q-t} \zeta^s \, dx \right)^{(t-\alpha+1)/(t-q)} \ge a \int_{B(R)} u^{q-t} \zeta^s \, dx, \end{aligned} \tag{3.41}$$

and, therefore,

$$\left(cs^\alpha t^{1-\alpha} a^{-1} (1+R^2)^{\delta/2} \right)^{s/\alpha} \int_{B(R)} |\nabla \zeta|^{\alpha(t-q)/(\alpha-1-q)} dx \ge \int_{B(r)} u^{q-t} \, dx. \tag{3.42}$$

Proceeding exactly as in the proof of Theorem 2.9, we now obtain by minimization that

$$\operatorname{cap}_s \left(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n \right) \left(ca^{-1} (1+R^2)^{\delta/2} s^\alpha t^{1-\alpha} \right)^{s/\alpha} \ge \int_{B(r)} u^{q-t} \, dx, \tag{3.43}$$

and, finally,

$$\int_{B(r)} u^{q-t} \, dx = O\left(R^{n-s+s\delta/\alpha} \right) \tag{3.44}$$

for $R = 2r$ as $R \to \infty$. Choose now a parameter t from the interval $(\alpha - 1, \infty)$ so that

$$n - \frac{(\alpha-\delta)(t-q)}{\alpha-1-q} < 0. \tag{3.45}$$

As long as

$$n - s + \frac{s\delta}{\alpha} = n - \frac{(\alpha-\delta)(t-q)}{\alpha-1-q}, \tag{3.46}$$

condition (3.45) implies that the exponent $n - s + s\delta/\alpha$ is negative. Therefore, it follows from (3.44) that $\int_{\mathbb{R}^n} u^{q-t}\,dx = 0$, but this is impossible because $u(x)$ is positive in the whole space. We have thus arrived at the desired contradiction. $\square$

Proof of Theorem 2.13. Let $q > 0$, $\alpha > 1$, $\alpha \geq n+\delta$, and let $u(x)$ be an entire nonnegative generalized supersolution of (2.25). Let the operator L satisfy conditions (2.2), (2.3), and (2.29). Let r and ε be positive numbers, $R = 2r$, and $\zeta(x)$ a function from the space $\overset{\circ}{C}{}^{\infty}(B(R))$ which equals 1 on $B(r)$ and is such that $0 \leq \zeta(x) \leq 1$. Substitute without loss of generality $\varphi(x) = (u(x)+\varepsilon)^{-t}\zeta^{\alpha}(x)$ as a test function in the inequality (2.28), where $t > \alpha - 1$ will be chosen below. Integrating by parts we obtain

$$\begin{aligned}
-t\int_{B(R)} \sum_{i=1}^{n} & u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{\alpha}\,dx \\
&+\alpha\int_{B(R)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t}\zeta^{\alpha-1}\,dx \\
&\equiv I_1 + I_2 \geq a\int_{B(R)} u^q(u+\varepsilon)^{-t}\zeta^{\alpha}\,dx.
\end{aligned} \tag{3.47}$$

Using condition (2.3) on the coefficients of the operator L, we obtain

$$\begin{aligned}
|I_2| &= \left|\alpha\int_{B(R)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t}\zeta^{\alpha-1}\,dx\right| \\
&\leq \int_{B(R)} \alpha\big(k(x)\big)^{1/\alpha}\left(\sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha} |\nabla\zeta|(u+\varepsilon)^{-t}\zeta^{\alpha-1}\,dx.
\end{aligned} \tag{3.48}$$

Estimating further the integrand on the right-hand side of relation (3.48) by use of Young's inequality (3.3) (as well as in the proof of Theorem 2.4), where $\rho = t/2$,

$$\begin{aligned}
A &= (u+\varepsilon)^{(1+t)(1-\alpha)/\alpha}\zeta^{\alpha-1}\left(\sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha}, \\
B &= \alpha\big(k(x)\big)^{1/\alpha}|\nabla\zeta|(u+\varepsilon)^{(\alpha-1-t)/\alpha},
\end{aligned} \tag{3.49}$$

and $\beta = \alpha$, we get

$$\begin{aligned}
|I_2| \leq \frac{t}{2}\int_{B(R)} & \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{\alpha}\,dx \\
&+\int_{B(R)} \alpha^{\alpha}\left(\frac{t}{2}\right)^{1-\alpha} k(x)|\nabla\zeta|^{\alpha}(u+\varepsilon)^{-t+\alpha-1}\,dx.
\end{aligned} \tag{3.50}$$

It follows from (3.47) and (3.50) that

$$\begin{aligned}\int_{B(R)} \alpha^{\alpha}\left(\frac{t}{2}\right)^{1-\alpha} & k(x)|\nabla\zeta|^{\alpha}(u+\varepsilon)^{-t+\alpha-1}dx \\ & \geq a\int_{B(R)} u^{q}(u+\varepsilon)^{-t}\zeta^{\alpha}\,dx \\ & \quad + \frac{t}{2}\int_{B(R)}\sum_{i=1}^{n} u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{\alpha}\,dx.\end{aligned} \tag{3.51}$$

Since $t > \alpha - 1$, it follows from (3.51) that

$$\begin{aligned}\alpha^{\alpha}\left(\frac{t}{2}\right)^{1-\alpha} & \varepsilon^{-t+\alpha-1}\int_{B(R)} k(x)|\nabla\zeta|^{\alpha}dx \\ & \geq a\int_{B(R)} u^{q}(u+\varepsilon)^{-t}\zeta^{\alpha}\,dx \\ & \quad + \frac{t}{2}\int_{B(R)}\sum_{i=1}^{n} u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{\alpha}\,dx.\end{aligned} \tag{3.52}$$

Minimizing the left-hand side of the inequality obtained over all admissible functions $\zeta(x)$ of the type indicated above (which is equivalent to the calculation of α-capacity of the condenser $(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n)$; cf. [4]), we have

$$\begin{aligned}\alpha^{\alpha}\left(\frac{t}{2}\right)^{1-\alpha} & \varepsilon^{-t+\alpha-1}K(R)\,\mathrm{cap}_{\alpha}\left(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n\right) \\ & \geq a\int_{B(r)} u^{q}(u+\varepsilon)^{-t}dx + \frac{t}{2}\int_{B(r)}\sum_{i=1}^{n} u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx.\end{aligned} \tag{3.53}$$

Since for any $\alpha \geq 1$ and $R = 2r$ it is well known that the α-capacity of the condenser $(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n)$ is $O(R^{n-\alpha})$ as $R\to\infty$, it follows from (2.29) and (3.53) that

$$\int_{B(r)} u^{q}(u+\varepsilon)^{-t}dx = O\left(R^{n+\delta-\alpha}\right) \tag{3.54}$$

as $R\to\infty$. It is easy to see that if $\alpha > n+\delta$, then

$$\int_{B(r)} u^{q}(u+\varepsilon)^{-t}dx \longrightarrow 0 \tag{3.55}$$

as $r\to\infty$. This implies that $u(x) = 0$ for a.e. $x\in\mathbb{R}^n$. If $\alpha = n+\delta$ we can see from (3.53) that the integral

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1} dx \tag{3.56}$$

is bounded. Therefore, due to monotonicity, the integral sequence

$$\int_{B(2r_k)\setminus B(r_k)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1} dx \longrightarrow 0 \tag{3.57}$$

for any sequence $r_k \to \infty$. On the other hand, it follows from (3.47) and (3.48) that

$$\begin{aligned}\int_{B(R)} \alpha\big(k(x)\big)^{1/\alpha} \left(\sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha} & |\nabla\zeta|(u+\varepsilon)^{-t}\zeta^{\alpha-1}\,dx \\ & \geq a \int_{B(r)} u^q (u+\varepsilon)^{-t} dx.\end{aligned} \tag{3.58}$$

Estimating, further, the integrand on the left-hand side of the relation (3.58) by using of Hölder's inequality, we get

$$\begin{aligned}&\left(\int_{B(R)\setminus B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1} dx\right)^{(\alpha-1)/\alpha} \\ &\qquad \times \left(\int_{B(R)} \alpha^\alpha k(x)|\nabla\zeta|^\alpha (u+\varepsilon)^{-t+\alpha-1} dx\right)^{1/\alpha} \\ &\geq a \int_{B(r)} u^q (u+\varepsilon)^{-t} dx.\end{aligned} \tag{3.59}$$

Minimizing the left-hand side of inequality (3.59) over all admissible functions $\zeta(x)$ of the type indicated above (which is equivalent to the calculation of α-capacity of the condenser $(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n)$; cf. [4]), we obtain

$$\begin{aligned}&\left(\int_{B(R)\setminus B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1} dx\right)^{(\alpha-1)/\alpha} \\ &\qquad \times \left(\alpha^\alpha \varepsilon^{-t+\alpha-1} K(R)\,\mathrm{cap}_\alpha\left(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n\right)\right)^{1/\alpha} \\ &\geq a \int_{B(r)} u^q (u+\varepsilon)^{-t} dx.\end{aligned} \tag{3.60}$$

From (2.29) and capacity properties, we have

$$\alpha^\alpha \varepsilon^{-t+\alpha-1} K(R)\,\mathrm{cap}_\alpha\left(B(r), \mathbb{R}^n \setminus B(R); \mathbb{R}^n\right) \text{ is } O\left(R^{n+\delta-\alpha}\right) \tag{3.61}$$

for $R = 2r$ as $R \to \infty$. It then follows directly from (3.57) and (3.60) under $\alpha = n+\delta$ that the integral sequence

$$\int_{B(r_k)} u^q (u+\varepsilon)^{-t} dx \longrightarrow 0 \tag{3.62}$$

as $r_k \to \infty$. This implies in turn again that $u(x) = 0$ for a.e. $x \in \mathbb{R}^n$. □

Proof of Theorem 2.15. Let $\alpha > 1$, $\alpha \geq n+\delta$, and let $u(x)$ be an entire nonnegative generalized supersolution of (2.33). Let the operator L satisfy conditions (2.2), (2.3), (2.29), and (2.32). Let r and ε be positive constants, $R = 2r$, and $\zeta(x)$ a function from the space $\overset{\circ}{C}{}^{\infty}(B(R))$ which equals 1 on $B(r)$ and is such that $0 \leq \zeta(x) \leq 1$. Substituting, without loss of generality, $\varphi(x) = (u(x)+\varepsilon)^{-t}\zeta^{\alpha}(x)$ in the inequality (2.34) as a test function, where $t > \alpha - 1$, and integrating by parts, we obtain

$$\begin{aligned} &\alpha \int_{B(R)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t}\zeta^{\alpha-1}\,dx \\ &\qquad \geq t \int_{B(R)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{\alpha}\,dx. \end{aligned} \tag{3.63}$$

Estimating the left-hand side of (3.63) by using condition (2.3) on the coefficients of the operator L, we easily get

$$\begin{aligned} &\left| \alpha \int_{B(R)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t}\zeta^{\alpha-1}\,dx \right| \\ &\qquad \leq \int_{B(R)} \alpha\big(k(x)\big)^{1/\alpha} \left(\sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u) \right)^{(\alpha-1)/\alpha} |\nabla\zeta|(u+\varepsilon)^{-t}\zeta^{\alpha-1}\,dx. \end{aligned} \tag{3.64}$$

Estimating further the integrand on the right-hand side of the relation (3.64) by Hölder's inequality, we have

$$\begin{aligned} &\left| \alpha \int_{B(R)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t}\zeta^{\alpha-1}\,dx \right| \\ &\qquad \leq \alpha \left(\int_{B(R)} k(x)|\nabla\zeta|^{\alpha}(u+\varepsilon)^{-t+\alpha-1}dx \right)^{1/\alpha} \\ &\qquad\quad \times \left(\int_{B(R)\setminus B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{\alpha}\,dx \right)^{\alpha/(\alpha-1)}. \end{aligned} \tag{3.65}$$

It follows immediately from (3.63) and (3.65) that

$$\begin{aligned}\alpha\varepsilon^{(\alpha-1-t)/\alpha}&\left(\int_{B(R)}k(x)|\nabla\zeta|^{\alpha}\,dx\right)^{1/\alpha}\\&\times\left(\int_{B(R)\setminus B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{\alpha}\,dx\right)^{\alpha/(\alpha-1)}\\&\geq t\int_{B(R)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\zeta^{\alpha}\,dx.\end{aligned}\tag{3.66}$$

Therefore,

$$\begin{aligned}\alpha\varepsilon^{(\alpha-1-t)/\alpha}&\left(\int_{B(R)}k(x)|\nabla\zeta|^{\alpha}dx\right)^{1/\alpha}\\&\geq t\left(\int_{B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx\right)^{1/\alpha}.\end{aligned}\tag{3.67}$$

Minimizing the left-hand side of (3.67) over all admissible functions $\zeta(x)$ of the type indicated above (which is equivalent to the calculation of α-capacity of the condenser $(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n)$; cf. [4]), we obtain

$$\begin{aligned}\alpha^{\alpha}\varepsilon^{\alpha-t-1}K(R)\,&\mathrm{cap}_{\alpha}\big(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n\big)\\&\geq t^{\alpha}\int_{B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}\,dx.\end{aligned}\tag{3.68}$$

Since for any $\alpha\geq 1$ and $R=2r$ it is well known that the α-capacity of the condenser $(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n)$ is $O(R^{n-\alpha})$ as $R\to\infty$, it follows from (2.29) and (3.68) that the integral

$$\int_{B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx\tag{3.69}$$

is $O(R^{n+\delta-\alpha})$ as $R\to\infty$. Now, if $\alpha>n+\delta$, it is evident that

$$\int_{\mathbb{R}^n}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx=0.\tag{3.70}$$

Therefore, because of the condition (2.32), $u(x)=$ constant for a.e. $x\in\mathbb{R}^n$. If $\alpha=n+\delta$, then it follows directly from above that the integral

$$\int_{\mathbb{R}^n}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx\tag{3.71}$$

is bounded. Therefore, due to monotonicity, the integral sequence

$$\int_{B(2r_k)\setminus B(r_k)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx \longrightarrow 0 \tag{3.72}$$

as an arbitrary sequence $r_k \to \infty$. On the other hand, it follows from (3.66) that

$$\begin{aligned} &\alpha\varepsilon^{(\alpha-1-t)/\alpha}\left(\int_{B(R)} k(x)|\nabla\zeta|^{\alpha}dx\right)^{1/\alpha} \\ &\qquad \times\left(\int_{B(R)\setminus B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx\right)^{\alpha/(\alpha-1)} \\ &\quad \geq t\int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx. \end{aligned} \tag{3.73}$$

Minimizing again the integral $\int_{B(R)} |\nabla\zeta|^{\alpha}dx$ over all admissible functions $\zeta(x)$ of the type indicated above, we obtain

$$\begin{aligned} &\alpha\varepsilon^{(\alpha-1-t)/\alpha}\big(K(R)\operatorname{cap}_{\alpha}\big(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n\big)\big)^{1/\alpha} \\ &\qquad \times\left(\int_{B(R)\setminus B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx\right)^{\alpha/(\alpha-1)} \\ &\quad \geq t\int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx. \end{aligned} \tag{3.74}$$

Because of the above

$$\alpha\varepsilon^{(\alpha-1-t)/\alpha}\big(K(R)\operatorname{cap}_{\alpha}\big(B(r),\mathbb{R}^n\setminus B(R);\mathbb{R}^n\big)\big)^{1/\alpha} = O\big(R^{(n+\delta-\alpha)/\alpha}\big) \tag{3.75}$$

as $R = 2r$ and $R \to \infty$, then it follows directly from (3.72) and (3.74) under $\alpha = n+\delta$ that the integral sequence

$$\int_{B(r_k)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx \longrightarrow 0 \tag{3.76}$$

as $r_k \to \infty$. This implies in turn

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u+\varepsilon)^{-t-1}dx = 0. \tag{3.77}$$

Therefore, because of condition (2.32), $u(x) =$ constant for a.e. $x \in \mathbb{R}^n$. □

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