PERIODIC SOLUTIONS OF A CLASS OF NON-AUTONOMOUS SECOND-ORDER DIFFERENTIAL INCLUSIONS SYSTEMS

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Using an abstract framework due to Clarke (1999), we prove the existence of periodic solutions for second-order differential inclusions systems.

1. Introduction

Consider the second-order system

$$\begin{aligned} \ddot{u}(t) &= \nabla F\big(t, u(t)\big) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{1.1}$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for each $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$\begin{aligned} \big|F(t, x)\big| &\le a\big(\|x\|\big) b(t), \\ \big\|\nabla F(t, x)\big\| &\le a\big(\|x\|\big) b(t), \end{aligned} \tag{1.2}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$.

Wu and Tang in [4] proved the existence of solutions for problem (1.1) when $F = F_1 + F_2$ and F_1, F_2 satisfy some assumptions. Now we will consider problem (1.1) in a more general sense. More precisely, our results represent the extensions to systems with discontinuity (we consider the generalized gradients unlike continuously gradient in classical results).

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2. Main results

Consider the second-order differential inclusions systems

$$\begin{aligned} \ddot{u}(t) &\in \partial F\big(t, u(t)\big) \quad \text{a.e. } t \in [0,T], \\ u(0)-u(T) &= \dot{u}(0)-\dot{u}(T) = 0, \end{aligned} \tag{2.1}$$

where $T > 0$, $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ and ∂ denotes the Clarke subdifferential.

We suppose that $F = F_1 + F_2$ and F_1, F_2 satisfy the following assumption:

(A′) F_1, F_2 are measurable in t for each $x \in \mathbb{R}^n$, at least F_1 or F_2 are strictly differentiable in x and there exist $k_1 \in L^2(0,T;\mathbb{R})$ and $k_2 \in L^2(0,T;\mathbb{R})$ such that

$$\begin{aligned} \big|F_1\big(t,x_1\big) - F_1\big(t,x_2\big)\big| &\le k_1(t)\big\|x_1 - x_2\big\|, \\ \big|F_2\big(t,x_1\big) - F_2\big(t,x_2\big)\big| &\le k_2(t)\big\|x_1 - x_2\big\|, \end{aligned} \tag{2.2}$$

for all $x_1, x_2 \in \mathbb{R}^n$ and all $t \in [0,T]$.

THEOREM 2.1. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A′) and the following conditions:*

(i) *$F_1(t,\cdot)$ is (λ,μ)-subconvex with $\lambda > 1/2$ and $\mu < 2\lambda^2$ for a.e. $t \in [0,T]$;*

(ii) *there exist $c_1, c_2 > 0$ and $\alpha \in [0,1)$ such that*

$$\zeta \in \partial F_2(t,x) \Longrightarrow \|\zeta\| \le c_1\|x\|^\alpha + c_2, \tag{2.3}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$;

(iii)

$$\frac{1}{\|x\|^{2\alpha}}\left[\frac{1}{\mu}\int_0^T F_1(t,\lambda x)dt + \int_0^T F_2(t,x)dt\right] \longrightarrow \infty, \quad \text{as } \|x\| \longrightarrow \infty. \tag{2.4}$$

Then problem (2.1) has at least one solution which minimizes φ on H_T^1.

Remark 2.2. Theorem 2.1 generalizes [3, Theorem 1]. In fact, [3, Theorem 1] follows from Theorem 2.1 letting $F_1 = 0$.

THEOREM 2.3. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A′) and the following conditions:*

(iv) *$F_1(t,\cdot)$ is (λ,μ)-subconvex for a.e. $t \in [0,T]$, and there exists $\gamma \in L^1(0,T;\mathbb{R})$, $h \in L^1(0,T;\mathbb{R}^n)$ with $\int_0^T h(t)dt = 0$ such that*

$$F_1(t,x) \ge \big\langle h(t), x\big\rangle + \gamma(t), \tag{2.5}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$;

(v) *there exist $c_1 > 0$, $c_0 \in \mathbb{R}$ such that*

$$\zeta \in \partial F_2(t,x) \Longrightarrow \|\zeta\| \le c_1, \tag{2.6}$$

for all $x \in \mathbb{R}^n$ *and all* $t \in [0,T]$*, and*

$$\int_0^T F_2(t,x)dt \geq c_0, \tag{2.7}$$

for all $x \in \mathbb{R}^n$*;*

(vi)

$$\frac{1}{\mu}\int_0^T F_1(t,\lambda x)dt + \int_0^T F_2(t,x)dt \longrightarrow \infty, \quad \text{as } \|x\| \longrightarrow \infty. \tag{2.8}$$

Then problem (2.1) has at least one solution which minimizes φ *on* H_T^1*.*

THEOREM 2.4. *Assume that* $F = F_1 + F_2$*, where* F_1*,* F_2 *satisfy assumption (A′) and the following conditions:*

(vii) $F_1(t,\cdot)$ *is* (λ,μ)*-subconvex for a.e.* $t \in [0,T]$*, and there exists* $\gamma \in L^1(0,T;\mathbb{R})$*,* $h \in L^1(0,T;\mathbb{R}^n)$ *with* $\int_0^T h(t)dt = 0$ *such that*

$$F_1(t,x) \geq \langle h(t), x\rangle + \gamma(t), \tag{2.9}$$

for all $x \in \mathbb{R}^n$ *and a.e.* $t \in [0,T]$*;*

(viii) *there exist* $c_1, c_2 > 0$ *and* $\alpha \in [0,1)$ *such that*

$$\zeta \in \partial F_2(t,x) \Longrightarrow \|\zeta\| \leq c_1\|x\|^{\alpha} + c_2, \tag{2.10}$$

for all $x \in \mathbb{R}^n$ *and a.e.* $t \in [0,T]$*;*

(ix)

$$\frac{1}{\|x\|^{2\alpha}}\int_0^T F_2(t,x)dt \longrightarrow \infty, \quad \text{as } \|x\| \longrightarrow \infty. \tag{2.11}$$

Then problem (2.1) has at least one solution which minimizes φ *on* H_T^1*.*

3. Preliminary results

We introduce some functional spaces. Let $[0,T]$ be a fixed real interval ($0 < T < \infty$) and $1 < p < \infty$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0,T;\mathbb{R}^n)$ having a weak derivative $\dot{u} \in L^p(0,T;\mathbb{R}^n)$. The norm over $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \left(\int_0^T \|u(t)\|^p dt + \int_0^T \|\dot{u}(t)\|^p dt\right)^{1/p}. \tag{3.1}$$

We denote by H_T^1 the Hilbert space $W_T^{1,2}$. We recall that

$$\|u\|_{L^p} = \left(\int_0^T \|u(t)\|^p dt\right)^{1/p}, \qquad \|u\|_\infty = \max_{t\in[0,T]}\|u(t)\|. \tag{3.2}$$

For our aims, it is necessary to recall some very well-known results (for proof and details see [2]):

Proposition 3.1. *If $u \in W_T^{1,p}$ then*

$$\|u\|_\infty \le c\|u\|_{W_T^{1,p}}. \tag{3.3}$$

If $u \in W_T^{1,p}$ and $\int_0^T u(t)dt = 0$ then

$$\|u\|_\infty \le c\|\dot{u}\|_{L^p}. \tag{3.4}$$

If $u \in H_T^1$ and $\int_0^T u(t)dt = 0$ then

$$\begin{aligned} \|u\|_{L^2} &\le \frac{T}{2\pi}\|\dot{u}\|_{L^2} \quad \textit{(Wirtinger's inequality)},\\ \|u\|_\infty^2 &\le \frac{T}{12}\|\dot{u}\|_{L^2}^2 \quad \textit{(Sobolev inequality)}. \end{aligned} \tag{3.5}$$

Proposition 3.2. *If the sequence $(u_k)_k$ converges weakly to u in $W_T^{1,p}$, then $(u_k)_k$ converges uniformly to u on $[0,T]$.*

Let X be a Banach space. Now, following [1], for each $x, v \in X$, we define the *generalized directional derivative* at x in the direction v of a given $f \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R})$ as

$$f^0(x;v) = \limsup_{y\to x,\lambda\searrow 0} \frac{f(y+\lambda v)-f(y)}{\lambda} \tag{3.6}$$

and denote x by

$$\partial f(x) = \{x^* \in X^* : f^0(x;v) \ge \langle x^*, v\rangle,\ \forall v \in X\} \tag{3.7}$$

the *generalized gradient* of f at x (the Clarke subdifferential).

We recall the *Lebourg's mean value theorem* (see [1, Theorem 2.3.7]). *Let x and y be points in X, and suppose that f is Lipschitz on an open set containing the line segment $[x,y]$. Then there exists a point u in (x,y) such that*

$$f(y)-f(x) \in \langle \partial f(u), y-x\rangle. \tag{3.8}$$

Clarke considered in [1] the following abstract framework:

- let $(T,\mathcal{T},\mu)$ be a positive complete measure space with $\mu(T) < \infty$, and let Y be a separable Banach space;
- let Z be a closed subspace of $L^p(T;Y)$ (for some p in $[1,\infty)$), where $L^p(T;Y)$ is the space of p-integrable functions from T to Y;
- we define a functional f on Z via

$$f(x) = \int_T f_t(x(t))\mu(dt), \tag{3.9}$$

where $f_t : Y \to R$, $(t \in T)$ is a given family of functions;

- we suppose that for each y in Y the function $t \to f_t(y)$ is measurable, and that x is a point at which $f(x)$ is defined (finitely).

Hypothesis 3.3. There is a function k in $L^q(T,R)$, $(1/p+1/q=1)$ such that, for all $t \in T$,

$$\left|f_t(y_1) - f_t(y_2)\right| \le k(t)\left\|y_1 - y_2\right\|_Y \quad \forall y_1, y_2 \in Y. \tag{3.10}$$

Hypothesis 3.4. Each function f_t is Lipschitz (of some rank) near each point of Y, and for some constant c, for all $t \in T$, $y \in Y$, one has

$$\zeta \in \partial f_t(y) \Longrightarrow \|\zeta\|_{Y^*} \le c\left\{1 + \|y\|_Y^{p-1}\right\}. \tag{3.11}$$

Under the conditions described above Clarke proved (see [1, Theorem 2.7.5]):

THEOREM 3.5. *Under either of Hypotheses 3.3 or 3.4, f is uniformly Lipschitz on bounded subsets of Z, and there is*

$$\partial f(x) \subset \int_T \partial f_t\big(x(t)\big)\mu(dt). \tag{3.12}$$

Further, if each f_t is regular at $x(t)$ then f is regular at x and equality holds.

Remark 3.6. The function f is globally Lipschitz on Z when Hypothesis 3.3 holds.

Now we can prove the following result.

THEOREM 3.7. *Let $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that $F = F_1 + F_2$ where F_1, F_2 are measurable in t for each $x \in \mathbb{R}^n$, and there exist $k_1 \in L^2(0,T;\mathbb{R})$, $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$, c_1, $c_2 > 0$, and $\alpha \in [0.1)$ such that*

$$\left|F_1(t,x_1) - F_1(t,x_2)\right| \le k_1(t)\left\|x_1 - x_2\right\|, \tag{3.13}$$

$$\left|F_2(t,x)\right| \le a\big(\|x\|\big)b(t), \tag{3.14}$$

$$\zeta \in \partial F_2(t,x) \Longrightarrow \|\zeta\| \le c_1\|x\|^\alpha + c_2, \tag{3.15}$$

for all $t \in [0,T]$ and all $x, x_1, x_2 \in \mathbb{R}^n$. We suppose that $L : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by $L(t,x,y) = (1/2)\|y\|^2 + F(t,x)$.

Then, the functional $f : Z \in \mathbb{R}$, where

$$Z = \left\{(u,v) \in L^2(0,T;Y) : u(t) = \int_0^t v(s)ds + c,\ c \in \mathbb{R}^n\right\} \tag{3.16}$$

given by $f(u,v) = \int_0^T L(t,u(t),v(t))dt$, is uniformly Lipschitz on bounded subsets of Z and

$$\partial f(u,v) \subset \int_0^T \left\{\partial F_1\big(t,u(t)\big) + \partial F_2\big(t,u(t)\big)\right\} \times \left\{v(t)\right\}dt. \tag{3.17}$$

Proof. Let $L_1(t,x,y) = F_1(t,x)$, $L_2(t,x,y) = (1/2)\|y\|^2 + F_2(t,x)$, and $f_1, f_2 : Z \to \mathbb{R}$ given by $f_1(u,v) = \int_0^T L_1(t,u(t),v(t))dt$, $f_2(u,v) = \int_0^T L_2(t,u(t), v(t))dt$. For f_1 we can apply Theorem 3.5 under Hypothesis 3.3, with the following cast of characters:

- $(T,\mathcal{T},\mu) = [0,T]$ with Lebesgue measure, $Y = \mathbb{R}^n \times \mathbb{R}^n$ is the Hilbert product space (hence is separable);
- $p = 2$ and

$$Z = \left\{(u,v) \in L^2(0,T;Y) : u(t) = \int_0^t v(s)ds + c,\ c \in \mathbb{R}^n\right\} \tag{3.18}$$

  is a closed subspace of $L^2(0,T;Y)$;
- $f_t(x,y) = L_1(t,x,y) = F_1(t,x)$; in our assumptions it results that the integrand $L_1(t,x,y)$ is measurable in t for a given element (x,y) of Y and there exists $k \in L^2(0,T;\mathbb{R})$ such that

$$\begin{aligned} \left|L_1(t,x_1,y_1) - L_1(t,x_2,y_2)\right| &= \left|F_1(t,x_1) - F_1(t,x_2)\right| \\ &\le k_1(t)\left\|x_1 - x_2\right\| \\ &\le k_1(t)\left(\left\|x_1 - x_2\right\| + \left\|y_1 - y_2\right\|\right) \\ &= k_1(t)\left\|(x_1,y_1) - (x_2,y_2)\right\|_Y, \end{aligned} \tag{3.19}$$

  for all $t \in [0,T]$ and all $(x_1,y_1), (x_2,y_2) \in Y$. Hence f_1 is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f_1(u,v) \subset \int_0^T \partial L_1\big(t,u(t),v(t)\big)dt. \tag{3.20}$$

For f_2 we can apply Theorem 3.5 under Hypothesis 3.4 with the same cast of characters, but now $f_t(x,y) = L_2(t,x,y) = (1/2)\|y\|^2 + F_2(t,x)$. In our assumptions, it results that the integrand $L_2(t,x,y)$ is measurable in t for a given element (x,y) of Y and locally Lipschitz in (x,y) for each $t \in [0,T]$.

Proposition 2.3.15 in [1] implies

$$\partial L_2(t,x,y) \subset \partial_x L_2(t,x,y) \times \partial_y L_2(t,x,y) = \partial F_2(t,x) \times y. \tag{3.21}$$

Using (3.15) and (3.21), if $\zeta = (\zeta_1,\zeta_2) \in \partial L_2(t,x,y)$ then $\zeta_1 \in \partial F_2(t,x)$ and $\zeta_2 = y$, and hence

$$\|\zeta\| = \left\|\zeta_1\right\| + \left\|\zeta_2\right\| \le c_1\|x\|^\alpha + c_2 + \|y\| \le \tilde{c}\left\{1 + \left\|(x,y)\right\|\right\}, \tag{3.22}$$

for each $t \in [0,T]$. Hence f_2 is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f_2(u,v) \subset \int_0^T \partial L_2\big(t,u(t),v(t)\big)dt. \tag{3.23}$$

It follows that $f = f_1 + f_2$ is uniformly Lipschitz on the bounded subsets of Z.

Propositions 2.3.3 and 2.3.15 in [1] imply that

$$\begin{aligned}
\partial f(u,v) &\subset \partial f_1(u,v)+\partial f_2(u,v)\\
&\subset \int_0^T \big[\partial L_1\big(t,u(t),v(t)\big)+\partial L_2\big(t,u(t),v(t)\big)\big]dt\\
&\subset \int_0^T \big[\big(\partial_x L_1\big(t,u(t),v(t)\big)\times\partial_y L_1\big(t,u(t),v(t)\big)\big)\\
&\qquad\quad +\big(\partial_x L_2\big(t,u(t),v(t)\big)\times\partial_y L_2\big(t,u(t),v(t)\big)\big)\big]dt\\
&\subset \int_0^T \big[\big(\partial_x L_1\big(t,u(t),v(t)\big)+\partial_x L_2\big(t,u(t),v(t)\big)\big)\\
&\qquad\quad \times\big(\partial_y L_1\big(t,u(t),v(t)\big)+\partial_y L_2\big(t,u(t),v(t)\big)\big)\big]dt\\
&= \int_0^T \big(\partial F_1\big(t,u(t)\big)+\partial F_2\big(t,u(t)\big)\big)\times\big\{v(t)\big\}dt.
\end{aligned}\tag{3.24}$$

Moreover, Corollary 1 of Proposition 2.3.3 in [1] implies that, if at least one of the functions F_1, F_2 is strictly differentiable in x for all $t\in[0,T]$ then

$$\partial f(u,v)\subset\int_0^T \partial F\big(t,u(t)\big)\times\big\{v(t)\big\}dt. \tag{3.25}$$

□

Remark 3.8. The interpretation of expression (3.25) is that if (u_0,v_0) is an element of Z (so that $v_0=\dot u_0$) and if $\zeta\in\partial f(u_0,v_0)$, we deduce the existence of a measurable function $(q(t),p(t))$ such that

$$q(t)\in\partial F\big(t,u_0(t)\big),\qquad p(t)=v_0(t)\quad \text{a.e. on } [0,T] \tag{3.26}$$

and for any (u,v) in Z, one has

$$\big\langle\zeta,(u,v)\big\rangle=\int_0^T \big\{\big\langle q(t),u(t)\big\rangle+\big\langle p(t),v(t)\big\rangle\big\}dt. \tag{3.27}$$

In particular, if $\zeta=0$ (so that u_0 is a critical point for $\varphi(u)=\int_0^T[(1/2)\|\dot u(t)\|^2+F(t,u(t))]dt$), it then follows easily that $q(t)=\dot p(t)$ a.e., or taking into account (3.26)

$$\ddot u_0(t)\in\partial F\big(t,u_0(t)\big)\quad\text{a.e. on } [0,T], \tag{3.28}$$

so that u_0 satisfies the inclusions system (2.1).

Remark 3.9. Of course, if F is continuously differentiable in x, then system (2.1) becomes system (1.1).

4. Proofs of the theorems

Proof of Theorem 2.1. From assumption (A′) it follows immediately that there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$\left|F_1(t,x)\right| \le a\left(\|x\|\right)b(t), \tag{4.1}$$

for all $x \in \mathbb{R}^n$ and all $t \in [0,T]$. Like, in [4], we obtain

$$F_1(t,x) \le \left(2\mu\|x\|^{\beta}+1\right)a_0 b(t), \tag{4.2}$$

for all $x \in \mathbb{R}^n$ and all $t \in [0,T]$, where $\beta < 2$ and $a_0 = \max_{0\le s\le 1} a(s)$.

For $u \in H_T^1$, let $\bar{u} = (1/T)\int_0^T u(t)dt$ and $\tilde{u} = u - \bar{u}$. From Lebourg's mean value theorem it follows that for each $t \in [0,T]$ there exist $z(t)$ in $(\bar{u}, u(t))$ and $\zeta \in \partial F_2(t, z(t))$ such that $F_2(t,u(t)) - F_2(t,\bar{u}) = \langle \zeta, \tilde{u}(t)\rangle$. It follows from (2.3) and Sobolev's inequality that

$$\begin{aligned}
&\left|\int_0^T \left[F_2\left(t,u(t)\right) - F_2(t,\bar{u})\right]dt\right| \\
&\quad \le \int_0^T \left|F_2\left(t,u(t)\right) - F_2(t,\bar{u})\right|dt \le \int_0^T \|\zeta\|\left\|\tilde{u}(t)\right\|dt \\
&\quad \le \int_0^T \left[2c_1\left(\|\bar{u}\|^{\alpha} + \left\|\tilde{u}(t)\right\|^{\alpha}\right) + c_2\right]\left\|\tilde{u}(t)\right\|dt \\
&\quad \le 2c_1 T\|\tilde{u}\|_\infty\|\bar{u}\|^{\alpha} + 2c_1T\|\tilde{u}\|_\infty^{\alpha+1} + c_2T\|\tilde{u}\|_\infty \\
&\quad \le \frac{3}{T}\|\tilde{u}\|_\infty^2 + \frac{T^3}{3}c_1^2\|\bar{u}\|^{2\alpha} + 2c_1T\|\tilde{u}\|_\infty^{\alpha+1} + c_2T\|\tilde{u}\|_\infty \\
&\quad \le \frac{1}{4}\|\dot{u}\|_{L^2}^2 + C_1\|\dot{u}\|_{L^2}^{\alpha+1} + C_2\|\dot{u}\|_{L^2} + C_3\|\bar{u}\|^{2\alpha},
\end{aligned} \tag{4.3}$$

for all $u \in H_T^1$ and some positive constants C_1, C_2, and C_3. Hence we have

$$\begin{aligned}
\varphi(u) \ge{}& \frac{1}{2}\int_0^T \left\|\dot{u}(t)\right\|^2dt + \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})dt - \int_0^T F_1\left(t,-\tilde{u}(t)\right)dt \\
&+ \int_0^T F_2(t,\bar{u})dt + \int_0^T \left[F_2\left(t,u(t)\right) - F_2(t,\bar{u})\right]dt \\
\ge{}& \frac{1}{4}\|\dot{u}\|_{L^2}^2 - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - C_2\|\dot{u}\|_{L^2} - C_3\|\bar{u}\|^{2\alpha} - \left(2\mu\|\tilde{u}\|_\infty^{\beta}+1\right)\int_0^T a_0 b(t)dt \\
&+ \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})dt + \int_0^T F_2(t,\bar{u})dt
\end{aligned}$$

$$\geq \frac{1}{4}\|\dot{u}\|_{L^2}^2 - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - C_2\|\dot{u}\|_{L^2} - C_4\|\dot{u}\|_{L^2}^{\beta} - C_5$$

$$+\|\bar{u}\|^{2\alpha}\left\{\frac{1}{\|\bar{u}\|^{2\alpha}}\left[\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})dt + \int_0^T F_2(t,\bar{u})dt\right] - C_3\right\} \tag{4.4}$$

for all $u \in H_T^1$, which implies that $\varphi(u) \to \infty$ as $\|u\| \to \infty$ by (2.4) because $\alpha < 1$, $\beta < 2$, and the norm $\|u\| = (\|\bar{u}\|^2 + \|\dot{u}\|_{L^2}^2)^{1/2}$ is an equivalent norm on H_T^1. Now we write $\varphi(u) = \varphi_1(u) + \varphi_2(u)$ where

$$\varphi_1(u) = \frac{1}{2}\int_0^T \|\dot{u}(t)\|^2 dt, \qquad \varphi_2(u) = \int_0^T F\big(t, u(t)\big)dt. \tag{4.5}$$

The function φ_1 is weakly lower semi-continuous (w.l.s.c.) on H_T^1. From (i), (ii), and Theorem 3.5, taking into account Remark 3.6 and Proposition 3.2, it follows that φ_2 is w.l.s.c. on H_T^1. By [2, Theorem 1.1], it follows that φ has a minimum u_0 on H_T^1. Evidently, $Z \simeq H_T^1$ and $\varphi(u) = f(u,v)$ for all $(u,v) \in Z$. From Theorem 3.7, it results that f is uniformly Lipschitz on bounded subsets of Z, and therefore φ possesses the same properties relative to H_T^1. Proposition 2.3.2 in [1] implies that $0 \in \partial\varphi(u_0)$ (so that u_0 is a critical point for φ). Now from Theorem 3.7 and Remark 3.8 it follows that problem (2.1) has at least one solution $u \in H_T^1$. □

Proof of Theorem 2.3. Let (u_k) be a minimizing sequence of φ. It follows from (iv), (v), Lebourg's mean value theorem, and Sobolev inequality, that

$$\begin{aligned}
\varphi(u_k) &\geq \frac{1}{2}\|\dot{u}_k\|_{L^2}^2 + \int_0^T \langle h(t), u_k(t)\rangle dt + \int_0^T \gamma(t)dt \\
&\quad + \int_0^T F_2(t,\bar{u}_k)dt - \int_0^T \|\zeta\|\,\|\tilde{u}_k(t)\|dt \\
&\geq \frac{1}{2}\|\dot{u}_k\|_{L^2}^2 - \|\tilde{u}_k\|_\infty \int_0^T \|h(t)\|dt \\
&\quad + \int_0^T \gamma(t)dt - c_1\|\tilde{u}_k\|_\infty + c_0 \\
&\geq \frac{1}{2}\|\dot{u}_k\|_{L^2}^2 - c_2\|\dot{u}_k\|_{L^2} - c_3,
\end{aligned} \tag{4.6}$$

for all k and some constants c_2, c_3, which implies that $(\tilde{u}_k)$ is bounded. On the other hand, in a way similar to the proof of Theorem 2.1, one has

$$\left|\int_0^T \big[F_2\big(t,u(t)\big) - F_2(t,\bar{u})\big]dt\right| \leq \frac{1}{4}\|\dot{u}\|_{L^2}^2 + C_1\|\dot{u}\|_{L^2}, \tag{4.7}$$

for all k and some positive constant C_1, which implies that

$$\begin{aligned}\varphi(u_k) \geq& \frac{1}{2}\|\dot{u}_k\|_{L^2}^2 + \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u}_k)dt - \int_0^T F_1(t,-\tilde{u}_k(t))dt \\ &+ \int_0^T F_2(t,\bar{u}_k)dt + \int_0^T \left[F_2(t,u(t)) - F_2(t,\bar{u}_k)\right]dt \\ \geq& \frac{1}{4}\|\dot{u}_k\|_{L^2}^2 - a(\|\tilde{u}_k\|_\infty)\int_0^T b(t)dt - C_1\|\dot{u}_k\|_{L^2} \\ &+ \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u}_k)dt + \int_0^T F_2(t,\bar{u}_k)dt,\end{aligned} \tag{4.8}$$

for all k and some positive constant C_1. It follows from (vi) and the boundedness of $(\tilde{u}_k)$ that $(\bar{u}_k)$ is bounded. Hence φ has a bounded minimizing sequence (u_k). This completes the proof. □

Proof of Theorem 2.4. From (vii), (3.26), and Sobolev's inequality it follows that

$$\begin{aligned}\varphi(u) \geq& \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T \langle h(t),u(t)\rangle dt + \int_0^T \gamma(t)dt \\ &+ \int_0^T F_2(t,\bar{u})dt + \int_0^T \left[F_2(t,u(t)) - F_2(t,\bar{u})\right]dt \\ \geq& \frac{1}{4}\|\dot{u}\|_{L^2}^2 - \|\tilde{u}\|_\infty \int_0^T \|h(t)\|dt + \int_0^T \gamma(t)dt \\ &- C_1\|\dot{u}\|_{L^2}^{\alpha+1} - C_2\|\dot{u}\|_{L^2} + \int_0^T F_2(t,\bar{u})dt - C_3\|\bar{u}\|^{2\alpha} \\ \geq& \frac{1}{4}\|\dot{u}\|_{L^2}^2 - C_1\|\dot{u}\|_{L^2}^{\alpha+1} - C_4(\|\dot{u}\|_{L^2}+1) \\ &+ \|\bar{u}\|^{2\alpha}\left[\frac{1}{\|\bar{u}\|^{2\alpha}}\int_0^T F_2(t,\bar{u})dt - C_3\right],\end{aligned} \tag{4.9}$$

for all $u \in H_T^1$ and some positive constants C_1, C_3, and C_4. Now it follows like in the proof of Theorem 2.1 that φ is coercive by (ix), which completes the proof. □

References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, 2nd ed., Classics in Applied Mathematics, vol. 5, SIAM, Pennsylvania, 1990. MR 91e:49001. Zbl 696.49002.

[2] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, vol. 74, Springer-Verlag, New York, 1989. MR 90e:58016. Zbl 676.58017.

[3] D. Pasca, *Periodic solutions for second order differential inclusions with sublinear nonlinearity*, Panamer. Math. J. **10** (2000), no. 4, 35–45. CMP 1 801 529.

[4] X.-P. Wu and C.-L. Tang, *Periodic solutions of a class of non-autonomous second-order systems*, J. Math. Anal. Appl. **236** (1999), no. 2, 227–235. MR 2000g:34069. Zbl 991.21227.

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