SECOND-ORDER n-POINT EIGENVALUE PROBLEMS ON TIME SCALES

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We discuss conditions for the existence of at least one positive solution to a nonlinear second-order Sturm-Liouville-type multipoint eigenvalue problem on time scales. The results extend previous work on both the continuous case and more general time scales, and are based on the Guo-Krasnosel'skiĭ fixed point theorem.

1. Introduction

We are interested in the second-order multipoint time-scale eigenvalue problem

$$(py^{\nabla})^{\Delta}(t) - q(t)y(t) + \lambda h(t)f(y) = 0, \quad t_1 < t < t_n, \tag{1.1}$$

$$\alpha y(t_1) - \beta p(t_1)y^{\nabla}(t_1) = \sum_{i=2}^{n-1} a_i y(t_i), \qquad \gamma y(t_n) + \delta p(t_n)y^{\nabla}(t_n) = \sum_{i=2}^{n-1} b_i y(t_i), \tag{1.2}$$

where

$$p,q : [t_1,t_n] \longrightarrow (0,\infty), \quad p \in C^{\Delta}[t_1,t_n),\ q \in C[t_1,t_n]; \tag{1.3}$$

the points $t_i \in \mathbb{T}^{\kappa}_{\kappa}$ for $i \in \{1,2,\dots,n\}$ with $t_1 < t_2 < \cdots < t_n$;

$$\alpha,\beta,\gamma,\delta \in [0,\infty), \quad \alpha\gamma + \alpha\delta + \beta\gamma > 0, \qquad a_i,b_i \in [0,\infty), \quad i \in \{2,\dots,n-1\}. \tag{1.4}$$

The continuous function $f : [0,\infty) \to [0,\infty)$ is such that the following exist:

$$f_0 := \lim_{y\to 0^+} \frac{f(y)}{y}, \qquad f_\infty := \lim_{y\to\infty} \frac{f(y)}{y}; \tag{1.5}$$

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and the right-dense continuous function $h : [t_1, t_n] \to [0, \infty)$ satisfies some suitable conditions to be developed. Problem (1.1), (1.2) is a generalization to time scales of the problem when $\mathbb{T}$ is restricted to $\mathbb{R}$ on the unit interval in Ma and Thompson [19], and extends the type of time-scale boundary value problem found in Anderson [2], Atici and Guseinov [6], Kaufmann [15], Kaufmann and Raffoul [16], and Sun and Li [21, 22]. Other related three-point problems on time scales include Anderson and Avery [4], Anderson et al. [5], Peterson et al. [20], and a singular problem in DaCunha et al. [12]. Some of the work on multipoint time-scale problems includes Anderson [1, 3] and Kong and Kong [17], and a recent singular multipoint problem in Bohner and Luo [8]. For more general information concerning dynamic equations on time scales, introduced by Aulbach and Hilger [7] and Hilger [14], see the excellent text by Bohner and Peterson [9] and their edited text [10].

2. Time-scale primer

Any arbitrary nonempty closed subset of the reals $\mathbb{R}$ can serve as a time-scale $\mathbb{T}$; see [9, 10]. For $t \in \mathbb{T}$ define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. The graininess operators $\mu_\sigma, \mu_\rho : \mathbb{T} \to [0, \infty)$ are defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (rd-continuous) provided it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limit exists (is finite) at left-dense points of $\mathbb{T}$. The set of all right-dense continuous functions on $\mathbb{T}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Define the set $\mathbb{T}_\kappa$ by $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$ if $\mathbb{T}$ has a right scattered minimum m and $\mathbb{T}_\kappa = \mathbb{T}$ otherwise. In a similar vein, $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$ if $\mathbb{T}$ has a left scattered maximum M and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise. We take $\mathbb{T}_\kappa^\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

Definition 2.1 (delta derivative). Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left| [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \right| \le \epsilon \left| \sigma(t) - s \right| \quad \forall s \in U. \tag{2.1}$$

The function $f^\Delta(t)$ is the delta derivative of f at t.

Definition 2.2 (nabla derivative). For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s] \right| \le \epsilon \left| \rho(t) - s \right| \quad \forall s \in U. \tag{2.2}$$

The function $f^\nabla(t)$ is the nabla derivative of f at t.

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t) = f^\nabla(t)$. When $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = f(t+1) - f(t)$ and $f^\nabla(t) = f(t) - f(t-1)$.

Definition 2.3 (delta integral). Let $f:\mathbb{T}\to\mathbb{R}$ be a function, and let $a,b\in\mathbb{T}$. If there exists a function $F:\mathbb{T}\to\mathbb{R}$ such that $F^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}^{\kappa}$, then F is a delta antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(t)\Delta t = F(b)-F(a) \quad \text{for } a,b\in\mathbb{T}. \tag{2.3}$$

All right-dense continuous functions are delta integrable; see [9, Theorem 1.74].

3. Linear preliminaries

We first construct Green's function for the second-order boundary value problem

$$(py^{\nabla})^{\Delta}(t)-q(t)y(t)+u(t)=0, \quad t_1<t<t_n, \tag{3.1}$$

$$\alpha y(t_1)-\beta p(t_1)y^{\nabla}(t_1)=0, \qquad \gamma y(t_n)+\delta p(t_n)y^{\nabla}(t_n)=0, \tag{3.2}$$

where α, β, γ, δ are real numbers such that $|\alpha|+|\beta|\neq 0$, $|\gamma|+|\delta|\neq 0$. The techniques here are similar to those found in [6, 19].

Denote by ϕ and ψ the solutions of the corresponding homogeneous equation

$$(py^{\nabla})^{\Delta}(t)-q(t)y(t)=0, \quad t\in[t_1,t_n), \tag{3.3}$$

under the initial conditions

$$\psi(t_1)=\beta, \qquad p(t_1)\psi^{\nabla}(t_1)=\alpha, \tag{3.4}$$

$$\phi(t_n)=\delta, \qquad p(t_n)\phi^{\nabla}(t_n)=-\gamma, \tag{3.5}$$

so that ψ and ϕ satisfy the first and second boundary conditions in (3.2), respectively. Set

$$d=-W_t(\psi,\phi)=p(t)\psi^{\nabla}(t)\phi(t)-\psi(t)p(t)\phi^{\nabla}(t). \tag{3.6}$$

Since the Wronskian of any two solutions is independent of t, evaluating at $t=t_1$, $t=t_n$, and using the boundary conditions (3.4), (3.5) yields

$$d=\alpha\phi(t_1)-\beta p(t_1)\phi^{\nabla}(t_1)=\gamma\psi(t_n)+\delta p(t_n)\psi^{\nabla}(t_n). \tag{3.7}$$

In addition $d\neq 0$ if and only if the homogeneous equation (3.3) has only the trivial solution satisfying the boundary conditions (3.2). For the proof of the following theorem, see [6, Theorem 4.2].

Lemma 3.1. *Assume (1.3) and (1.4). If $d\neq 0$, then the nonhomogeneous boundary value problem (3.1)-(3.2) has a unique solution y for which the formula*

$$y(t)=\int_{t_1}^{t_n} G(t,s)u(s)\Delta s, \quad t\in[\rho(t_1),t_n] \tag{3.8}$$

holds, where the function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{d}\begin{cases} \psi(t)\phi(s), & \rho(t_1) \le t \le s \le t_n, \\ \psi(s)\phi(t), & \rho(t_1) \le s \le t \le t_n, \end{cases} \tag{3.9}$$

and $G(t,s)$ is Green's function of the boundary value problem (3.1)-(3.2). Furthermore Green's function is symmetric, that is, $G(t,s) = G(s,t)$ for $t,s \in [\rho(t_1),t_n]$.

Lemma 3.2. *Assume (1.3) and (1.4). Then the functions ψ and ϕ satisfy*

$$\begin{gathered} \psi(t) \ge 0, \quad t \in [\rho(t_1),t_n], \qquad \psi(t) > 0, \quad t \in (\rho(t_1),t_n], \\ p(t)\psi^{\nabla}(t) \ge 0, \quad t \in [\rho(t_1),t_n], \qquad \phi(t) \ge 0, \quad t \in [\rho(t_1),t_n], \\ \phi(t) > 0, \quad t \in [\rho(t_1),t_n), \qquad p(t)\phi^{\nabla}(t) \le 0, \quad t \in [\rho(t_1),t_n]. \end{gathered} \tag{3.10}$$

Proof. The proof is very similar to the proof of [6, Lemma 5.1] and is omitted. □

Set

$$D := \begin{vmatrix} -\sum_{i=2}^{n-1} a_i\psi(t_i) & d - \sum_{i=2}^{n-1} a_i\phi(t_i) \\ d - \sum_{i=2}^{n-1} b_i\psi(t_i) & -\sum_{i=2}^{n-1} b_i\phi(t_i) \end{vmatrix}. \tag{3.11}$$

Lemma 3.3. *Assume (1.3) and (1.4). If $D \ne 0$ and $u \in C_{rd}[t_1,t_n]$, then the nonhomogeneous dynamic equation (3.1) with boundary conditions (1.2) has a unique solution y for which the formula*

$$y(t) = \int_{t_1}^{t_n} G(t,s)u(s)\Delta s + A(u)\psi(t) + B(u)\phi(t), \quad t \in [\rho(t_1),t_n], \tag{3.12}$$

holds, where the function $G(t,s)$ is Green's function (3.9) of the boundary value problem (3.1)-(3.2) and the functionals A and B are defined by

$$A(u) := \frac{1}{D}\begin{vmatrix} \sum_{i=2}^{n-1} a_i \int_{t_1}^{t_n} G(t_i,s)u(s)\Delta s & d - \sum_{i=2}^{n-1} a_i\phi(t_i) \\ \sum_{i=2}^{n-1} b_i \int_{t_1}^{t_n} G(t_i,s)u(s)\Delta s & -\sum_{i=2}^{n-1} b_i\phi(t_i) \end{vmatrix}, \tag{3.13}$$

$$B(u) := \frac{1}{D}\begin{vmatrix} -\sum_{i=2}^{n-1} a_i\psi(t_i) & \sum_{i=2}^{n-1} a_i \int_{t_1}^{t_n} G(t_i,s)u(s)\Delta s \\ d - \sum_{i=2}^{n-1} b_i\psi(t_i) & \sum_{i=2}^{n-1} b_i \int_{t_1}^{t_n} G(t_i,s)u(s)\Delta s \end{vmatrix}. \tag{3.14}$$

Proof. It can be verified that for a solution y of the nonhomogeneous equation (3.1) under the nonhomogeneous boundary conditions (1.2), the formula (3.12) holds, where $G(t,s)$ is given by (3.9). We thus show that the function y given in (3.12) is a solution of (3.1) with conditions (1.2) only if A and B are given by (3.13) and (3.14), respectively. If y as in (3.12) is a solution of (3.1), (1.2), then

$$y(t) = \frac{1}{d}\int_{t_1}^{t}\phi(t)\psi(s)u(s)\Delta s + \frac{1}{d}\int_{t}^{t_n}\psi(t)\phi(s)u(s)\Delta s + A\psi(t) + B\phi(t) \tag{3.15}$$

for some constants A and B. Taking the nabla derivative and multiplying by p yields

$$py^{\nabla} = \frac{p\phi^{\nabla}}{d}\int_{t_1}^{t}\psi(s)u(s)\Delta s + \frac{p\psi^{\nabla}}{d}\int_{t}^{t_n}\phi(s)u(s)\Delta s + Ap\psi^{\nabla} + Bp\phi^{\nabla}; \tag{3.16}$$

the delta derivative of this expression is

$$\begin{aligned}(py^{\nabla})^{\Delta} = {}& \left(\frac{p\phi^{\nabla}}{d}\right)^{\Delta}\int_{t_1}^{\sigma(t)}\psi(s)u(s)\Delta s + \frac{p\phi^{\nabla}}{d}\psi(t)u(t) + A(p\psi^{\nabla})^{\Delta} + B(p\phi^{\nabla})^{\Delta} \\ & + \left(\frac{p\psi^{\nabla}}{d}\right)^{\Delta}\int_{\sigma(t)}^{t_n}\phi(s)u(s)\Delta s - \frac{p\psi^{\nabla}}{d}\phi(t)u(t).\end{aligned} \tag{3.17}$$

Using [9, Theorem 1.75], and the fact that ψ and ϕ are solutions to (3.3), we obtain

$$\begin{aligned}(py^{\nabla})^{\Delta}(t) = {}& \frac{q(t)}{d}\int_{t_1}^{t}\phi(t)\psi(s)u(s)\Delta s + \frac{q(t)}{d}\phi(t)\mu_{\sigma}(t)\psi(t)u(t) + \frac{u(t)}{d}p(t)\phi^{\nabla}(t)\psi(t) \\ & + \frac{q(t)}{d}\int_{t}^{t_n}\psi(t)\phi(s)u(s)\Delta s - \frac{q(t)}{d}\psi(t)\mu_{\sigma}(t)\phi(t)u(t) \\ & - \frac{u(t)}{d}p(t)\psi^{\nabla}(t)\phi(t) + q(t)\big(A\psi(t) + b\phi(t)\big).\end{aligned} \tag{3.18}$$

Recall that d is in terms of the Wronskian of ψ and ϕ in (3.6); it follows that

$$(py^{\nabla})^{\Delta}(t) = q(t)y(t) - u(t). \tag{3.19}$$

Now

$$\begin{aligned} y(t_1) &= \frac{\psi(t_1)}{d}\int_{t_1}^{t_n}\phi(s)u(s)\Delta s + A\psi(t_1) + B\phi(t_1), \\ p(t_1)y^{\nabla}(t_1) &= \frac{p(t_1)\psi^{\nabla}(t_1)}{d}\int_{t_1}^{t_n}\phi(s)u(s)\Delta s + Ap(t_1)\psi^{\nabla}(t_1) + Bp(t_1)\phi^{\nabla}(t_1);\end{aligned} \tag{3.20}$$

multiply the first line by α and the second by $-\beta$, and use (1.2) and (3.4) to see that

$$B[\alpha\phi(t_1)-\beta p(t_1)\phi^{\nabla}(t_1)]=\sum_{i=2}^{n-1}a_i\left(\int_{t_1}^{t_n}G(t_i,s)u(s)\Delta s+A\psi(t_i)+B\phi(t_i)\right). \tag{3.21}$$

At the other end,

$$\begin{aligned} y(t_n)&=\frac{\phi(t_n)}{d}\int_{t_1}^{t_n}\psi(s)u(s)\Delta s+A\psi(t_n)+B\phi(t_n),\\ p(t_n)y^{\nabla}(t_n)&=\frac{p(t_n)\phi^{\nabla}(t_n)}{d}\int_{t_1}^{t_n}\psi(s)u(s)\Delta s+Ap(t_n)\psi^{\nabla}(t_n)+Bp(t_n)\phi^{\nabla}(t_n); \end{aligned} \tag{3.22}$$

consequently

$$A[\gamma\psi(t_n)+\delta p(t_n)\psi^{\nabla}(t_n)]=\sum_{i=2}^{n-1}b_i\left(\int_{t_1}^{t_n}G\Big(t_i,s\Big)u(s)\Delta s+A\psi(t_i)+B\phi(t_i)\right). \tag{3.23}$$

Combining (3.21) and (3.23) and using (3.6), we arrive at the system of equations

$$\begin{aligned} -A\sum_{i=2}^{n-1}a_i\psi(t_i)+B\left[\alpha\phi(t_1)-\beta p(t_1)\phi^{\nabla}(t_1)-\sum_{i=2}^{n-1}a_i\phi(t_i)\right]&=\sum_{i=2}^{n-1}a_i\int_{t_1}^{t_n}G(t_i,s)u(s)\Delta s,\\ A\left[\gamma\psi(t_n)+\delta p(t_n)\psi^{\nabla}(t_n)-\sum_{i=2}^{n-1}b_i\psi(t_i)\right]-B\sum_{i=2}^{n-1}b_i\phi(t_i)&=\sum_{i=2}^{n-1}b_i\int_{t_1}^{t_n}G(t_i,s)u(s)\Delta s. \end{aligned} \tag{3.24}$$

Again using (3.6) at both t_1 and t_n, we verify (3.13) and (3.14). □

Lemma 3.4. *Let (1.3) and (1.4) hold, and assume*

$$D<0,\qquad d-\sum_{i=2}^{n-1}a_i\phi(t_i)>0,\qquad d-\sum_{i=2}^{n-1}b_i\psi(t_i)>0 \tag{3.25}$$

for D and d given in (3.11) and (3.6), respectively. If $u\in C_{rd}[t_1,t_n]$ with $u\geq 0$, the unique solution y as in (3.12) of the problem (3.1), (1.2) satisfies $y(t)\geq 0$ for $t\in[t_1,t_n]$.

Proof. From the previous lemmas and assumptions we know that Green's function (3.9) satisfies $G(t,s)\geq 0$ on $[\rho(t_1),t_n]\times[\rho(t_1),t_n]$. Hypotheses (1.3), (1.4), and (3.25) applied to (3.13) and (3.14) imply that $A(u),B(u)\geq 0$. □

Suppose (3.25) does not hold. For example, let $n=3$, $p(t)\equiv 1=\alpha=\gamma$, $q(t)\equiv 0=\beta=\delta=a_2$, and $t_1=0$. Then (3.1), (1.2) becomes

$$y^{\nabla\Delta}(t)+u(t)=0,\quad t_1<t<t_3,\quad y(t_1)=0,\quad y(t_3)=b_2y(t_2). \tag{3.26}$$

Note that $\psi(t) = t$, $d = t_3$, and $D = t_3(b_2t_2 - t_3)$. If $D > 0$, then $b_2t_2 > t_3$, and there is no positive solution; see [15, Lemma 4].

Lemma 3.5. *Let (1.3), (1.4), and (3.25) hold, and fix*

$$\xi_1, \xi_2 \in \mathbb{T}_\kappa^\kappa, \quad \rho(t_1) < \xi_1 < \xi_2 < t_n. \tag{3.27}$$

If $u \in C_{rd}[t_1, t_n]$ with $u \geq 0$, the unique solution y as in (3.12) of the time-scale boundary value problem (3.1), (1.2) satisfies

$$\min_{t\in[\xi_1,\xi_2]} y(t) \geq \Gamma\|y\|, \quad \|y\| := \max_{t\in[\rho(t_1),t_n]} y(t), \tag{3.28}$$

where

$$\Gamma := \min\left\{\frac{\phi(\xi_2)}{\phi(\rho(t_1))}, \frac{\psi(\xi_1)}{\psi(t_n)}\right\} \in (0,1). \tag{3.29}$$

Proof. From (1.3), (3.9), and Lemma 3.2,

$$0 \leq G(t,s) \leq G(s,s), \quad t \in [\rho(t_1), t_n], \tag{3.30}$$

so that

$$y(t) \leq \int_{t_1}^{t_n} G(s,s)u(s)\Delta s + A(u)\psi(t_n) + B(u)\phi(\rho(t_1)) \quad \forall t \in [\rho(t_1), t_n]. \tag{3.31}$$

For $t \in [\xi_1, \xi_2]$, Green's function (3.9) satisfies

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{\phi(t)}{\phi(s)}: & \rho(t_1) \leq s \leq t \leq t_n \\ \dfrac{\psi(t)}{\psi(s)}: & \rho(t_1) \leq t \leq s \leq t_n \end{cases} \geq \begin{cases} \dfrac{\phi(\xi_2)}{\phi(\rho(t_1))}: & \rho(t_1) \leq s \leq t \leq t_n \\ \dfrac{\psi(\xi_1)}{\psi(t_n)}: & \rho(t_1) \leq t \leq s \leq t_n \end{cases} \geq \Gamma \tag{3.32}$$

for Γ as in (3.29), and

$$\begin{aligned} y(t) &= \int_{t_1}^{t_n} \frac{G(t,s)}{G(s,s)} G(s,s)u(s)\Delta s + A(u)\psi(t) + B(u)\phi(t) \\ &\geq \int_{t_1}^{t_n} \Gamma G(s,s)u(s)\Delta s + A(u)\psi(\xi_1) + B(u)\phi(\xi_2) \\ &\geq \Gamma\left(\int_{t_1}^{t_n} G(s,s)u(s)\Delta s + A(u)\psi(t_n) + B(u)\phi(\rho(t_1))\right) \geq \Gamma\|y\|. \end{aligned} \tag{3.33}$$

□

4. Eigenvalue intervals

To establish eigenvalue intervals we will employ the following fixed point theorem due to Krasnosel'skiĭ [18]; for more on the establishment of eigenvalue intervals for time-scale boundary value problems, see, for example, Chyan and Henderson [11] and Davis et al. [13].

THEOREM 4.1. *Let E be a Banach space, $P \subseteq E$ a cone, and suppose that Ω_1, Ω_2 are bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $L : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ly\| \le \|y\|$, $y \in P \cap \partial\Omega_1$ *and* $\|Ly\| \ge \|y\|$, $y \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ly\| \ge \|y\|$, $y \in P \cap \partial\Omega_1$ *and* $\|Ly\| \le \|y\|$, $y \in P \cap \partial\Omega_2$

holds. Then L has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Assume that the right-dense continuous function h satisfies

$$h : [t_1, t_n] \longrightarrow [0, \infty), \quad \exists t_* \in (\sigma(t_1), \rho(t_n)) \ni h(t_*) > 0. \tag{4.1}$$

Then there exist ξ_1, ξ_2 as in Lemma 3.5 such that

$$\xi_1 < t_* < \xi_2, \quad \int_{\xi_1}^{\xi_2} G(t,s)h(s)\Delta s > 0, \quad t \in (\rho(t_1), t_n). \tag{4.2}$$

In the following, let Γ be the constant defined in (3.29) with respect to such constants ξ_1, ξ_2. Let $\tau \in [\rho(t_1), t_n]$ be determined by

$$\int_{\xi_1}^{\xi_2} G(\tau,s)h(s)\Delta s = \max_{\rho(t_1) \le t \le t_n} \int_{\xi_1}^{\xi_2} G(t,s)h(s)\Delta s > 0. \tag{4.3}$$

For $G(t,s)$ in (3.9) and A, B as in (3.13), (3.14), respectively, define the constant

$$K := \int_{t_1}^{t_n} G(s,s)h(s)\Delta s + A(h)\psi(t_n) + B(h)\phi(\rho(t_1)). \tag{4.4}$$

Let $\mathcal{B}$ denote the Banach space $C[\rho(t_1), t_n]$ with the norm $\|y\| = \sup_{t \in [\rho(t_1), t_n]} |y(t)|$. Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{ y \in \mathcal{B} : y(t) \ge 0 \text{ on } [\rho(t_1), t_n],\ y(t) \ge \Gamma \|y\| \text{ on } [\xi_1, \xi_2] \}, \tag{4.5}$$

where Γ is given in (3.29). Since y is a solution of (1.1), (1.2) if and only if

$$y(t) = \lambda \left(\int_{t_1}^{t_n} G(t,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(t) + B(hf(y))\phi(t) \right), \quad t \in [\rho(t_1), t_n], \tag{4.6}$$

define for $y \in \mathcal{P}$ the operator $T : \mathcal{P} \to \mathcal{B}$ by

$$(Ty)(t) := \lambda\left(\int_{t_1}^{t_n} G(t,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(t) + B(hf(y))\phi(t)\right). \tag{4.7}$$

We seek a fixed point of T in $\mathcal{P}$ by establishing the hypotheses of Theorem 4.1.

Theorem 4.2. *Suppose (1.3), (1.4), (3.25), (4.1), and (4.3) hold. Then for each λ satisfying*

$$\frac{1}{f_\infty \Gamma \int_{\xi_1}^{\xi_2} G(\tau,s)h(s)\Delta s} < \lambda < \frac{1}{f_0 K}, \tag{4.8}$$

there exists at least one positive solution of (1.1), (1.2) in $\mathcal{P}$.

Proof. Let ξ_1, ξ_2 be as in Lemma 3.5, let τ be as in (4.3), let K be as in (4.4), let λ be as in (4.8), and let $\epsilon > 0$ be such that

$$\frac{1}{(f_\infty - \epsilon)\Gamma \int_{\xi_1}^{\xi_2} G(\tau,s)h(s)\Delta s} \le \lambda \le \frac{1}{(f_0 + \epsilon)K}. \tag{4.9}$$

Consider the integral operator T in (4.7). If $y \in \mathcal{P}$, then by (3.30) we have

$$\begin{aligned}(Ty)(t) &= \lambda\left(\int_{t_1}^{t_n} G(t,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(t) + B(hf(y))\phi(t)\right)\\ &\le \lambda\left(\int_{t_1}^{t_n} G(s,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(t_n) + B(hf(y))\phi(\rho(t_1))\right),\end{aligned} \tag{4.10}$$

so that for $t \in [\xi_1, \xi_2]$,

$$\begin{aligned}(Ty)(t) &= \lambda\left(\int_{t_1}^{t_n} G(t,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(t) + B(hf(y))\phi(t)\right)\\ &\ge \lambda\left(\int_{t_1}^{t_n} \frac{G(t,s)}{G(s,s)} G(s,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(\xi_1) + B(hf(y))\phi(\xi_2)\right)\\ &\ge \lambda\Gamma\left(\int_{t_1}^{t_n} G(s,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(t_n) + B(hf(y))\phi(\rho(t_1))\right) \ge \Gamma\|Ty\|.\end{aligned} \tag{4.11}$$

Therefore $T : \mathcal{P} \to \mathcal{P}$. Moreover, T is completely continuous by a typical application of the Ascoli-Arzela theorem.

Now consider f_0. There exists an $R_1 > 0$ such that $f(y) \leq (f_0 + \epsilon)y$ for $0 < y \leq R_1$ by the definition of f_0. Pick $y \in \mathcal{P}$ with $\|y\| = R_1$. From (3.13) and (3.14),

$$|A(hf(y))| \leq A(h)\|f(y)\|, \qquad |B(hf(y))| \leq B(h)\|f(y)\|. \tag{4.12}$$

Using (3.30), we have

$$\begin{aligned}(Ty)(t) &= \lambda\left(\int_{t_1}^{t_n} G(t,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(t) + B(hf(y))\phi(t)\right)\\ &\leq \lambda\|f(y)\|\left(\int_{t_1}^{t_n} G(s,s)h(s)\Delta s + A(h)\psi(t_n) + B(h)\phi(\rho(t_1))\right)\\ &\leq \lambda(f_0+\epsilon)\|y\|K \leq \|y\|\end{aligned} \tag{4.13}$$

from the right-hand side of (4.9). As a result, $\|Ty\| \leq \|y\|$. Thus, take

$$\Omega_1 := \{y \in \mathcal{B} : \|y\| < R_1\} \tag{4.14}$$

so that $\|Ty\| \leq \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_1$.

Next consider f_∞. Again by definition, there exists an $R_2' > R_1$ such that $f(y) \geq (f_\infty - \epsilon)y$ for $y \geq R_2'$; take $R_2 = \max\{2R_1, R_2'/\Gamma\}$. If $y \in \mathcal{P}$ with $\|y\| = R_2$, then for $s \in [\xi_1, \xi_2]$ we have

$$y(s) \geq \Gamma\|y\| = \Gamma R_2. \tag{4.15}$$

Define $\Omega_2 := \{y \in \mathcal{B} : \|y\| < R_2\}$; using (4.3) and (4.15) for $s \in [\xi_1, \xi_2]$, we get

$$\begin{aligned}(Ty)(\tau) &= \lambda\left(\int_{t_1}^{t_n} G(\tau,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(\tau) + B(hf(y))\phi(\tau)\right)\\ &\geq \lambda\int_{\xi_1}^{\xi_2} G(\tau,s)h(s)f(y(s))\Delta s \geq \lambda(f_\infty-\epsilon)\int_{\xi_1}^{\xi_2} G(\tau,s)h(s)y(s)\Delta s\\ &\geq \lambda(f_\infty-\epsilon)\Gamma R_2\int_{\xi_1}^{\xi_2} G(\tau,s)h(s)\Delta s \geq R_2 = \|y\|,\end{aligned} \tag{4.16}$$

where we have used the left-hand side of (4.9). Hence we have shown that

$$\|Ty\| \geq \|y\|, \quad y \in \mathcal{P} \cap \partial\Omega_2. \tag{4.17}$$

An application of Theorem 4.1 yields the conclusion of the theorem; in other words, T has a fixed point y in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $R_1 \leq \|y\| \leq R_2$. □

Theorem 4.3. *Suppose (1.3), (1.4), (3.25), (4.1), and (4.3) hold. Then for each* λ *satisfying*

$$\frac{1}{f_0\Gamma\int_{\xi_1}^{\xi_2} G(\tau,s)h(s)\Delta s} < \lambda < \frac{1}{f_\infty K}, \tag{4.18}$$

there exists at least one positive solution of (1.1), (1.2) in $\mathcal{P}$.

Proof. Let λ be as in (4.18) and let $\eta > 0$ be such that

$$\frac{1}{(f_0-\eta)\Gamma\int_{\xi_1}^{\xi_2} G(\tau,s)h(s)\Delta s} \le \lambda \le \frac{1}{(f_\infty+\eta)K}. \tag{4.19}$$

Again let T be the operator defined in (4.7). We once more seek a fixed point of T in $\mathcal{P}$ by establishing the hypotheses of Theorem 4.1.

First, consider f_0. There exists an $R_1 > 0$ such that $f(y) \ge (f_0-\eta)y$ for $0 < y \le R_1$ by the definition of f_0. Pick $y \in \mathcal{P}$ with $\|y\| = R_1$. For $s \in [\xi_1, \xi_2]$, where ξ_1, ξ_2 are as in Lemma 3.5, we have

$$y(s) \ge \Gamma\|y\| = \Gamma R_1. \tag{4.20}$$

Using the left-hand side of (4.19) and (4.20) we get, for $s \in [\xi_1, \xi_2]$,

$$\begin{aligned}(Ty)(\tau) &= \lambda\left(\int_{t_1}^{t_n} G(\tau,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(\tau) + B(hf(y))\phi(\tau)\right)\\ &\ge \lambda(f_0-\eta)\int_{\xi_1}^{\xi_2} G(\tau,s)h(s)y(s)\Delta s \ge \lambda(f_0-\eta)R_1\Gamma\int_{\xi_1}^{\xi_2} G(\tau,s)h(s)\Delta s\\ &\ge R_1 = \|y\|.\end{aligned} \tag{4.21}$$

Therefore $\|Ty\| \ge \|y\|$. This motivates us to define

$$\Omega_1 := \{y \in \mathcal{B} : \|y\| < R_1\}, \tag{4.22}$$

whereby our work above confirms

$$\|Ty\| \ge \|y\|, \quad y \in \mathcal{P} \cap \partial\Omega_1. \tag{4.23}$$

Next consider f_∞. Again by definition there exists an $R_2' > R_1$ such that $f(y) \le (f_\infty+\eta)y$ for $y \ge R_2'$. If f is bounded, there exists $M > 0$ with $f(y) \le M$ for all $y \in (0,\infty)$. Let

$$R_2 := \max\left\{2R_2', \lambda M\left(\int_{t_1}^{t_n} G(s,s)h(s)\Delta s + A(h)\psi(t_n) + B(h)\phi(\rho(t_1))\right)\right\}. \tag{4.24}$$

If $y \in \mathcal{P}$ with $\|y\| = R_2$, then we have

$$(Ty)(t) \le \lambda\left(\int_{t_1}^{t_n} G(s,s)h(s)f(y(s))\Delta s + A(hf(y))\psi(t_n) + B(hf(y))\phi(\rho(t_1))\right)$$
$$\le \lambda M\left(\int_{t_1}^{t_n} G(s,s)h(s)\Delta s + A(h)\psi(t_n) + B(h)\phi(\rho(t_1))\right) \le R_2 = \|y\|. \tag{4.25}$$

As a result, $\|Ty\| \le \|y\|$. Thus, take

$$\Omega_2 := \{y \in \mathcal{B} : \|y\| < R_2\} \tag{4.26}$$

so that $\|Ty\| \le \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_2$. If f is unbounded, take $R_2 := \max\{2R_1, R_2'\}$ such that $f(y) \le f(R_2)$ for $0 < y \le R_2$. If $y \in \mathcal{P}$ with $\|y\| = R_2$, then we have

$$\begin{aligned}(Ty)(t) &\le \lambda f(R_2)\left(\int_{t_1}^{t_n} G(s,s)h(s)\Delta s + A(h)\psi(t_n) + B(h)\phi(\rho(t_1))\right)\\ &\le \lambda(f_\infty + \eta)R_2K \le R_2 = \|y\|,\end{aligned} \tag{4.27}$$

where we have used the left-hand side of (4.19). Hence we have shown that

$$\|Ty\| \le \|y\|, \quad y \in \mathcal{P} \cap \partial\Omega_2 \tag{4.28}$$

if we take

$$\Omega_2 := \{y \in \mathcal{B} : \|y\| < R_2\}. \tag{4.29}$$

As before, an application of Theorem 4.1 yields the conclusion that T has a fixed point y in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $R_1 \le \|y\| \le R_2$. □

Corollary 4.4. *Suppose (1.3), (1.4), (3.25), and (4.1) hold. If f is sublinear (i.e., $f_0 = \infty$ and $f_\infty = 0$), or if f is superlinear (i.e., $f_0 = 0$ and $f_\infty = \infty$), then for any $\lambda > 0$ the boundary value problem (1.1)-(1.2) has at least one positive solution in $\mathcal{P}$.*

Proof. For the superlinear claim, use (4.8) of Theorem 4.2; for the sublinear claim, use (4.18) of Theorem 4.3. □

5. Examples

Example 5.1. Let $\mathbb{T} = \mathbb{R}$, and consider the three-point boundary value problem

$$\begin{gathered} y'' - y + \lambda f(y) = 0, \quad -1 < t < 1, \\ y(-1) = ay(0) = y(1), \end{gathered} \tag{5.1}$$

where $a := \sinh(2)/4\sinh(1)$ and $f \in C([0,\infty),[0,\infty))$ such that f_0 and f_∞ exist.

It is easy to check that

$$\psi(t)=\frac{e^{t+1}-e^{-t-1}}{2}=\sinh(1+t),\qquad \phi(t)=\frac{e^{1-t}-e^{t-1}}{2}=\sinh(1-t),$$
$$d=\begin{vmatrix}\phi(1) & \psi(1)\\ \phi'(1) & \psi'(1)\end{vmatrix}=\sinh(2). \tag{5.2}$$

Since

$$D=\begin{vmatrix}-a\psi(0) & d-a\phi(0)\\ d-a\psi(0) & -a\phi(0)\end{vmatrix}=-\frac{1}{2}\sinh^2(2)<0,$$
$$d-a\phi(0)=d-a\psi(0)=\frac{3}{4}\sinh(2)>0, \tag{5.3}$$

(3.25) holds. We take $[\xi_1,\xi_2]=[-1/2,1/2]$, so that

$$\Gamma=\min\left\{\frac{\phi(1/2)}{\phi(-1)},\frac{\psi(-1/2)}{\psi(1)}\right\}=\frac{\sinh(1/2)}{\sinh(2)}, \tag{5.4}$$

$$A(1)=\frac{1}{D}\begin{vmatrix}a\int_{-1}^{1}G(0,s)ds & d-a\phi(0)\\ a\int_{-1}^{1}G(0,s)ds & -a\phi(0)\end{vmatrix}=\frac{(e-1)^2}{2e\sinh(2)},$$
$$B(1)=\frac{1}{D}\begin{vmatrix}-a\psi(0) & a\int_{-1}^{1}G(0,s)ds\\ d-a\psi(0) & a\int_{-1}^{1}G(0,s)ds\end{vmatrix}=\frac{(e-1)^2}{2e\sinh(2)}, \tag{5.5}$$

$$K=\frac{1}{d}\int_{-1}^{1}\psi(s)\phi(s)ds+A(1)\psi(1)+B(1)\phi(-1)=\frac{\cosh(2)}{\sinh(2)}+e+\frac{1}{e}-\frac{5}{2}. \tag{5.6}$$

Note that τ in (4.3) is determined by

$$\max\left\{t\in\left[-1,-\frac{1}{2}\right]:\frac{\psi(t)}{d}\int_{-1/2}^{1/2}\phi(s)ds,\ t\in\left[\frac{1}{2},1\right]:\frac{\phi(t)}{d}\int_{-1/2}^{1/2}\psi(s)ds,\right.$$
$$\left.t\in\left(-\frac{1}{2},\frac{1}{2}\right):\frac{\phi(t)}{d}\int_{-1/2}^{t}\psi(s)ds+\frac{\psi(t)}{d}\int_{t}^{1/2}\phi(s)ds\right\}, \tag{5.7}$$

which is

$$\frac{\phi(0)}{d}\int_{-1/2}^{0}\psi(s)ds+\frac{\psi(0)}{d}\int_{0}^{1/2}\phi(s)ds=2\frac{\sinh(1)}{\sinh(2)}\left(\cosh(1)-\cosh\left(\frac{1}{2}\right)\right). \tag{5.8}$$

Applying (5.4) and (5.6), we can find the interval in (4.8):

$$\frac{\sinh^2(2)}{2\sinh(1)\sinh(1/2)\left(\cosh(1)-\cosh(1/2)\right)f_\infty}<\lambda<\frac{1}{Kf_0}, \tag{5.9}$$

approximately

$$\frac{25.8511}{f_\infty}<\lambda<\frac{0.615962}{f_0}. \tag{5.10}$$

Example 5.2. Let $\mathbb{T}=h\mathbb{Z}$ for $h=2^{-10}$, and consider the four-point boundary value problem

$$\begin{gathered}\left(py^{\nabla}\right)^{\Delta}(t)+\lambda f(y)=0,\quad 0<t<1,\\ y(0)-p(0)y^{\nabla}(0)=\frac{2}{5}\left(y\left(\frac{1}{4}\right)+y\left(\frac{3}{4}\right)\right),\\ y(1)+p(1)y^{\nabla}(1)=\frac{2}{5}\left(y\left(\frac{1}{4}\right)+y\left(\frac{3}{4}\right)\right),\end{gathered} \tag{5.11}$$

where $p(t):=1/(t+h)(t+2h)$ and $f\in C([0,\infty),[0,\infty))$ such that f_0 and f_∞ exist.

Then direct calculation verifies that

$$\begin{aligned}\psi(t)&=\frac{1}{3}(t+h)(t+2h)(t+3h)+1-2h^3,\\ \phi(t)&=\frac{1}{3}(1+h)(1+2h)(1+3h)+1-\frac{1}{3}(t+h)(t+2h)(t+3h),\\ d&=\psi(1)+p(1)\frac{(\psi(1)-\psi(1-h))}{h}=\frac{1}{3}\left(11h^2+6h+7\right),\\ D&=\begin{vmatrix}-\frac{2}{5}\left(\psi\left(\frac{1}{4}\right)+\psi\left(\frac{3}{4}\right)\right) & d-\frac{2}{5}\left(\phi\left(\frac{1}{4}\right)+\phi\left(\frac{3}{4}\right)\right)\\ d-\frac{2}{5}\left(\psi\left(\frac{1}{4}\right)+\psi\left(\frac{3}{4}\right)\right) & -\frac{2}{5}\left(\phi\left(\frac{1}{4}\right)+\phi\left(\frac{3}{4}\right)\right)\end{vmatrix}=\frac{-d^2}{5}.\end{aligned} \tag{5.12}$$

Moreover, since

$$\begin{aligned}d-\frac{2}{5}\left(\psi\left(\frac{1}{4}\right)+\psi\left(\frac{3}{4}\right)\right)&=\frac{1}{40}\left(59+60h+88h^2\right)>0,\\ d-\frac{2}{5}\left(\phi\left(\frac{1}{4}\right)+\phi\left(\frac{3}{4}\right)\right)&=\frac{1}{40}\left(53+36h+88h^2\right)>0,\end{aligned} \tag{5.13}$$

(3.25) holds. Let $[\xi_1,\xi_2]=[0,1/2]$, so that

$$\Gamma=\min\left\{\frac{\phi(1/2)}{\phi(-h)},\frac{\psi(0)}{\psi(1)}\right\}=\frac{\psi(0)}{\psi(1)}=\frac{3}{11h^2+6h+4}, \tag{5.14}$$

$$\begin{gathered}
A(1)=\frac{1}{D}\begin{vmatrix}\frac{2}{5}\sum_{s=0}^{1/h-1}G\left(\frac{1}{4},sh\right)h+\frac{2}{5}\sum_{s=0}^{1/h-1}G\left(\frac{3}{4},sh\right)h & d-\frac{2}{5}\left(\phi\left(\frac{1}{4}\right)+\phi\left(\frac{3}{4}\right)\right)\\ \frac{2}{5}\sum_{s=0}^{1/h-1}G\left(\frac{1}{4},sh\right)h+\frac{2}{5}\sum_{s=0}^{1/h-1}G\left(\frac{3}{4},sh\right)h & -\frac{2}{5}\left(\phi\left(\frac{1}{4}\right)+\phi\left(\frac{3}{4}\right)\right)\end{vmatrix},\\
B(1)=\frac{1}{D}\begin{vmatrix}-\frac{2}{5}\left(\psi\left(\frac{1}{4}\right)+\psi\left(\frac{3}{4}\right)\right) & \frac{2}{5}\sum_{s=0}^{1/h-1}G\left(\frac{1}{4},sh\right)h+\frac{2}{5}\sum_{s=0}^{1/h-1}G\left(\frac{3}{4},sh\right)h\\ d-\frac{2}{5}\left(\psi\left(\frac{1}{4}\right)+\psi\left(\frac{3}{4}\right)\right) & \frac{2}{5}\sum_{s=0}^{1/h-1}G\left(\frac{1}{4},sh\right)h+\frac{2}{5}\sum_{s=0}^{1/h-1}G\left(\frac{3}{4},sh\right)h\end{vmatrix},
\end{gathered} \tag{5.15}$$

$$K=\frac{1}{d}\sum_{s=0}^{1/h-1}\psi(sh)\phi(sh)h+A(1)\psi(1)+B(1)\phi(-h)\approx 3.02392. \tag{5.16}$$

As in the previous example, we determine τ in (4.3) from

$$\begin{gathered}
\max\left\{t\in[-h,0]:\frac{\psi(t)h}{d}\sum_{s=0}^{(1/2h)-1}\phi(sh),\ t\in\left[\frac{1}{2},1\right]:\frac{\phi(t)h}{d}\sum_{s=0}^{(1/2h)-1}\psi(sh),\right.\\
\left.t\in\left(0,\frac{1}{2}\right):\frac{\phi(t)h}{d}\sum_{s=0}^{t/h-1}\psi(sh)+\frac{\psi(t)h}{d}\sum_{s=t/h}^{(1/2h)-1}\phi(sh)\right\},
\end{gathered} \tag{5.17}$$

which is

$$\frac{\phi(290h)h}{d}\sum_{s=0}^{289}\psi(sh)+\frac{\psi(290h)h}{d}\sum_{s=290}^{(1/2h)-1}\phi(sh)\approx 0.284188. \tag{5.18}$$

Applying (5.14) and (5.15), we can find an approximate interval for (4.8):

$$\frac{4.69862}{f_\infty}<\lambda<\frac{0.330697}{f_0}. \tag{5.19}$$

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