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Research Article

Periodic and Almost Periodic Solutions of Functional Difference Equations with Finite Delay

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For periodic and almost periodic functional difference equations with finite delay, the existence of periodic and almost periodic solutions is obtained by using stability properties of a bounded solution.

1. Introduction

In this paper, we study periodic and almost periodic solutions of the following functional difference equations with finite delay:

$$x(n+1) = F(n,x_n), \quad n \geq 0, \tag{1.1}$$

under certain conditions for $F(n,\cdot)$ (see below), where n, j, and τ are integers, and x_n will denote the function $x(n+j)$, $j = -\tau, -\tau+1, \dots, 0$.

Equation (1.1) can be regarded as the discrete analogue of the following functional differential equation with bounded delay:

$$\frac{dx}{dt} = \mathcal{F}(t,x_t), \quad t \geq 0, \qquad x_t(0) = x(t+0) = \phi(t), \quad -\sigma \leq t \leq 0. \tag{1.2}$$

Almost periodic solutions of (1.2) have been discussed in [1]. The aim of this paper is to extend results in [1] to (1.1).

Delay difference equations or functional difference equations (no matter with finite or infinite delay), inspired by the development of the study of delay differential equations, have been studied extensively in the past few decades (see, [2–11], to mention a few, and references therein). Recently, several papers [12–17] are devoted to study almost periodic solutions of difference equations. To the best of our knowledge, little work has been done on almost periodic solutions of nonlinear functional difference equations with finite delay via uniform stability properties of a bounded solution. This motivates us to investigate almost periodic solutions of (1.1).

This paper is organized as follows. In Section 2, we review definitions of almost periodic and asymptotically almost periodic sequences and present some related properties for our purposes and some stability definitions of a bounded solution of (1.1). In Section 3, we discuss the existence of periodic solutions of (1.1). In Section 4, we discuss the existence of almost periodic solutions of (1.1).

2. Preliminaries

We formalize our notation. Denote by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$, respectively, the set of integers, the set of nonnegative integers, and the set of nonpositive integers. For any $a \in \mathbb{Z}$, let $\mathbb{Z}_a^+ = \{n : n \geq a,\ n \in \mathbb{Z}\}$. For any integers $a < b$, let $\operatorname{dis}[a,b] = \{j : a \leq j \leq b,\ j \in \mathbb{Z}\}$ and $\operatorname{dis}(a,b] = \{j : a < j \leq b,\ j \in \mathbb{Z}\}$ be discrete intervals of integers. Let $\mathbb{E}^d$ denote either $\mathbb{R}^d$, the d-dimensional real Euclidean space, or $\mathbb{C}^d$, the d-dimensional complex Euclidean space. In the following, we use $|\cdot|$ to denote a norm of a vector in $\mathbb{E}^d$.

2.1. Almost periodic sequences.

We review definitions of (uniformly) almost periodic and asymptotically almost periodic sequences, which have been discussed by several authors (see, e.g., [2, 18]), and present some related properties for our purposes. For almost periodic and asymptotically almost periodic functions, we recommend [19, 18].

Let X and Y be two Banach spaces with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let Ω be a subset of X.

Definition 2.1. Let $f : \mathbb{Z} \times \Omega \to Y$ and $f(n,\cdot)$ be continuous for each $n \in \mathbb{Z}$. Then f is said to be almost periodic in $n \in \mathbb{Z}$ uniformly for $w \in \Omega$ if for every $\varepsilon > 0$ and every compact $\Sigma \subset \Omega$ corresponds an integer $N_\varepsilon(\Sigma) > 0$ such that among $N_\varepsilon(\Sigma)$ consecutive integers there is one, call it p, such that

$$\left\|f(n+p,w) - f(n,w)\right\|_Y < \varepsilon \quad \forall n \in \mathbb{Z},\ w \in \Sigma. \tag{2.1}$$

Denote by $\mathcal{AP}(\mathbb{Z} \times \Omega : Y)$ the set of all such functions. We may call $f \in \mathcal{AP}(\mathbb{Z} \times \Omega : Y)$ a (uniformly) almost periodic sequence in Y. If Ω is an empty set and $Y = X$, then $f \in \mathcal{AP}(\mathbb{Z} : X)$ is called an almost periodic sequence in X.

Almost periodic sequences can be also defined for any sequence $\{f(n)\}_{n \geq a}$, or $f : \mathbb{Z}_a^+ \to X$ by requiring that any $N_\varepsilon(\Sigma)$ consecutive integers is in $\mathbb{Z}_a^+$.

For uniformly almost periodic sequences, we have the following results.

Theorem 2.2. *Let $f \in \mathcal{AP}(\mathbb{Z} \times \Omega : Y)$ and let Σ be any compact set in Ω. Then $f(n,\cdot)$ is continuous on Σ uniformly for $n \in \mathbb{Z}$ and the range $f(\mathbb{Z} \times \Sigma)$ is relatively compact, which implies that $f(\mathbb{Z} \times \Sigma)$ is a bounded subset in Y.*

THEOREM 2.3. *Let $f \in \mathscr{AP}(\mathbb{Z}\times\Omega : Y)$. Then for any integer sequence $\{\alpha'_k\}$, $\alpha'_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{\alpha_k\}$ of $\{\alpha'_k\}$, $\alpha_k \to \infty$ as $k \to \infty$, and a function $\xi : \mathbb{Z}\times\Omega \to Y$ such that*

$$f(n+\alpha_k, w) \longrightarrow \xi(n,w) \tag{2.2}$$

uniformly on $\mathbb{Z}\times\Sigma$ as $k \to \infty$, where Σ is any compact set in Ω. Moreover, $\xi \in \mathscr{AP}(\mathbb{Z}\times\Omega : Y)$, that is, $\xi(n,w)$ is also almost periodic in n uniformly for $w \in \Omega$.

If Ω is the empty set and $Y = X$ in Theorem 2.3, then $\{\xi(n)\}$ is an almost periodic sequence.

THEOREM 2.4. *If $f \in \mathscr{AP}(\mathbb{Z}\times\Omega : Y)$, then there exists a sequence $\{\alpha_k\}$, $\alpha_k \to \infty$ as $k \to \infty$, such that*

$$f(n+\alpha_k, w) \longrightarrow f(n,w) \tag{2.3}$$

uniformly on $\mathbb{Z}\times\Sigma$ as $k \to \infty$, where Σ is any compact set in Ω.

Obviously, $\{\alpha_k\}$ in Theorem 2.4 can be chosen to be a positive integer sequence.

Definition 2.5. A sequence $\{x(n)\}_{n\in\mathbb{Z}^+}$, $x(n) \in X$, or a function $x : \mathbb{Z}^+ \to X$, is called asymptotically almost periodic if $x = x_1|_{\mathbb{Z}^+} + x_2$, where $x_1 \in \mathscr{AP}(\mathbb{Z},X)$ and $x_2 : \mathbb{Z}^+ \to X$ satisfying $\|x_2(n)\|_X \to 0$ as $n \to \infty$. Denote by $\mathscr{AAP}(\mathbb{Z}^+,X)$ all such sequences.

THEOREM 2.6. *Let $x : \mathbb{Z}^+ \to X$. Then the following statements are equivalent.*

(1) $x \in \mathscr{AAP}(\mathbb{Z}^+,X)$.

(2) For any sequence $\{\alpha_k\} \subset \mathbb{Z}^+$, $\alpha_k > 0$, and $\alpha_k \to \infty$ as $k \to \infty$, there is a subsequence $\{\beta_k\} \subset \{\alpha_k\}$ such that $\beta_k \to \infty$ as $k \to \infty$ and $\{x(n+\beta_k)\}$ converges uniformly on $\mathbb{Z}^+$ as $k \to \infty$.

Similarly, asymptotically almost periodic sequence can be defined for any sequence $\{x(n)\}_{n\geq a}$, or $x : \mathbb{Z}_a^+ \to X$.

The proof of the above results is omitted here because it is not difficult for readers giving proofs by the similar arguments in [19, 18] for continuous (uniformly) almost periodic function $\phi : \mathbb{R}\times\Omega \to X$ (see also [2] for the case that $X = Y = \mathbb{E}^d$).

2.2. Some assumptions and stability definitions. We now present some definitions and notations that will be used throughout this paper. For a given positive integer $\tau > 0$, we define C to be a Banach space with a norm $\|\cdot\|$ by

$$C = \{\phi \mid \phi : \text{dis}[-\tau,0] \longrightarrow \mathbb{E}^d, \|\phi\| = \max\{|\phi(j)|\} \text{ for } j \in \text{dis}[-\tau,0]\}. \tag{2.4}$$

It is clear that C is isometric to the space $\mathbb{E}^{d\times(\tau+1)}$.

Let $n_0 \in \mathbb{Z}^+$ and let $\{x(n)\}$, $n \geq n_0 - \tau$, be a sequence with $x(n) \in \mathbb{E}^d$. For each $n \geq n_0$, we define $x_n : \text{dis}[-\tau,0] \to \mathbb{E}^d$ by the relation

$$x_n(j) = x(n+j), \quad j \in \text{dis}[-\tau,0]. \tag{2.5}$$

Let us return to system (1.1), that is,

$$x(n+1) = F(n, x_n), \tag{2.6}$$

where $F : \mathbb{Z} \times C \to \mathbb{E}^d$ and $x_n : \text{dis}[-\tau, 0] \to C$.

Definition 2.7. Let $n_0 \in \mathbb{Z}^+$ and let ϕ be a given vector in C. A sequence $x = \{x(n)\}_{n \ge n_0}$ in $\mathbb{E}^d$ is said to be a solution of (2.6), passing through (n_0, ϕ), if $x_{n_0} = \phi$, that is, $x(n_0 + j) = \phi(j)$ for $j \in \text{dis}[-\tau, 0]$, $x(n+1)$, and x_n satisfy (2.6) for $n \ge n_0$, where x_n is defined by (2.5). Denote by $\{x(n,\phi)\}_{n \ge n_0}$ a solution of (2.6) such that $x_{n_0} = \phi$. No loss of clarity arises if we refer to the solution $\{x(n,\phi)\}_{n \ge n_0}$ as $x = \{x(n)\}_{n \ge n_0}$.

We make the following assumptions on (2.6) throughout this paper.

(H1) $F : \mathbb{Z} \times C \to \mathbb{E}^d$ and $F(n, \cdot)$ is continuous on C for each $n \in \mathbb{Z}$.

(H2) System (2.6) has a bounded solution $u = \{u(n)\}_{n \ge 0}$, passing through $(0, \phi^0)$, $\phi^0 \in C$.

For this bounded solution $\{u(n)\}_{n \ge 0}$, there is an $\alpha > 0$ such that $|u(n)| \le \alpha$ for all $n \ge -\tau$, which implies that $\|u_n\| \le \alpha$ and $u_n \in S_\alpha = \{\phi : \|\phi\| \le \alpha \text{ and } \phi \in C\}$ for all $n \ge 0$.

Definition 2.8. A bounded solution $\mathfrak{x} = \{\mathfrak{x}(n)\}_{n \ge 0}$ of (2.6) is said to be

(i) uniformly stable, abbreviated to read "$\mathfrak{x}$ is $\mathcal{US}$," if for any $\varepsilon > 0$ and any integer $n_0 \ge 0$, there exists $\delta(\varepsilon) > 0$ such that $\|\mathfrak{x}_{n_0} - x_{n_0}\| < \delta(\varepsilon)$ implies that $\|\mathfrak{x}_n - x_n\| < \varepsilon$ for all $n \ge n_0$, where $\{x(n)\}_{n \ge n_0}$ is any solution of (2.6);

(ii) uniformly asymptotically stable, abbreviated to read "$\mathfrak{x}$ is $\mathcal{UAS}$," if it is uniformly stable and there exists $\delta_0 > 0$ such that for any $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon) > 0$ such that if $n_0 \ge 0$ and $\|\mathfrak{x}_{n_0} - x_{n_0}\| < \delta_0$, then $\|\mathfrak{x}_n - x_n\| < \varepsilon$ for all $n \ge n_0 + N$, where $\{x(n)\}_{n \ge n_0}$ is any solution of (2.6);

(iii) globally uniformly asymptotically stable, abbreviated to read "$\mathfrak{x}$ is $\mathcal{GUAS}$," if it is uniformly stable and $\|\mathfrak{x}_n - x_n\| \to 0$ as $n \to \infty$, whenever $\{x(n)\}_{n \ge n_0}$ is any solution of (2.6).

Remark 2.9. It is easy to see that an equivalent definition for $\mathfrak{x} = \{\mathfrak{x}(n)\}_{n \ge 0}$ being $\mathcal{UAS}$ is the following:

(ii*) $\mathfrak{x} = \{\mathfrak{x}(n)\}_{n \ge 0}$ is $\mathcal{UAS}$, if it is uniformly stable and there exists $\delta_0 > 0$ such that if $n_0 \ge 0$ and $\|\mathfrak{x}_{n_0} - x_{n_0}\| < \delta_0$, then $\|\mathfrak{x}_n - x_n\| \to 0$ as $n \to \infty$, where $\{x(n)\}_{n \ge n_0}$ is any solution of (2.6).

3. Periodic systems

In this section, we discuss the existence of periodic solutions of (2.6), namely,

$$x(n+1) = F(n, x_n), \quad n \ge 0, \tag{3.1}$$

under a periodic condition (H3) as follows.

(H3) The $F(n, \cdot)$ in (3.1) is periodic in $n \in \mathbb{Z}$, that is, there exists a positive integer ω such that $F(n + \omega, v) = F(n, v)$ for all $n \in \mathbb{Z}$ and $v \in C$.

We are now in a position to give our main results in this section. We first show that if the bounded solution $\{u(n)\}_{n\geq 0}$ of (3.1) is uniformly stable, then $\{u(n)\}_{n\geq 0}$ is an asymptotically almost periodic sequence.

Theorem 3.1. *Suppose conditions (H1)–(H3) hold. If the bounded solution $\{u(n)\}_{n\geq 0}$ of (3.1) is $\mathcal{US}$, then $\{u(n)\}_{n\geq 0}$ is an asymptotically almost periodic sequence in $\mathbb{E}^d$, equivalently, (3.1) has an asymptotically almost periodic solution.*

Proof. Since $\|u_n\| \leq \alpha$ for $n \in \mathbb{Z}^+$, there is bounded (or compact) set $S_\alpha \subset C$ such that $u_n \in S_\alpha$ for all $n \geq 0$. Let $\{n_k\}_{k\geq 1}$ be any integer sequence such that $n_k > 0$ and $n_k \to \infty$ as $k \to \infty$. For each n_k, there exists a nonnegative integer l_k such that $l_k\omega \leq n_k \leq (l_k+1)\omega$. Set $n_k = l_k\omega + \tau_k$. Then $0 \leq \tau_k < \omega$ for all $k \geq 1$. Since $\{\tau_k\}_{k\geq 1}$ is bounded set, we can assume that, taking a subsequence if necessary, $\tau_k = j_*$ for all $k \geq 1$, where $0 \leq j_* < \omega$. Now, set $u^k(n) = u(n+n_k)$. Notice that $u_{n+n_k}(j) = u(n+n_k+j) = u^k(n+j) = u^k_n(j)$ and hence, $u_{n+n_k} = u^k_n$. Thus,

$$u^k(n+1) = u(n+n_k+1) = F(n+n_k, u_{n+n_k}) = F(n+n_k, u^k_n) = F(n+j_*, u^k_n), \tag{3.2}$$

which implies that $\{u^k(n)\}$ is a solution of the system

$$x(n+1) = F(n+j_*, x_n) \tag{3.3}$$

through $(0, u_{n_k})$. It is readily shown that if $\{u(n)\}_{n\geq 0}$ is $\mathcal{US}$, then $\{u^k(n)\}_{n\geq 0}$ is also $\mathcal{US}$ with the same pair $(\varepsilon, \delta(\varepsilon))$ as the one for $\{u(n)\}_{n\geq 0}$.

Since $\{u(n+n_k)\}$ is bounded for all $n \geq -\tau$ and n_k, we can use the diagonal method to get a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ such that $u(n+n_{k_j})$ converges for each $n \geq -\tau$ as $j \to \infty$. Thus, we can assume that the sequence $u(n+n_k)$ converges for each $n \geq -\tau$ as $k \to \infty$. Notice that $u^k_0(j) = u^k(0+j) = u(j+n_k)$. Then for any $\varepsilon > 0$ there exists a positive integer $N_1(\varepsilon)$ such that if $k, m \geq N_1(\varepsilon)$, then

$$||u^k_0 - u^m_0|| < \delta(\varepsilon), \tag{3.4}$$

where $\delta(\varepsilon)$ is the number for the uniform stability of $\{u(n)\}_{n\geq 0}$. Notice that $\{u^m(n) = u(n+n_m)\}_{n\geq 0}$ is also a solution of (3.3) and that $\{u^k(n)\}_{n\geq 0}$ is uniformly stable. It follows from Definition 2.8 and (3.4) that

$$||u^k_n - u^m_n|| < \varepsilon \quad \forall n \geq 0, \tag{3.5}$$

and hence,

$$|u^k(n) - u^m(n)| < \varepsilon \quad \forall n \geq 0,\ k, m \geq N_1(\varepsilon). \tag{3.6}$$

This implies that for any positive integer sequence n_k, $n_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ for which $\{u(n+n_{k_j})\}$ converges uniformly on $\mathbb{Z}^+$ as $j \to \infty$. Thus, $\{u(n)\}_{n\geq 0}$ is an asymptotically almost periodic sequence by Theorem 2.6 and the proof is completed. □

Lemma 3.2. *Suppose that (H1)–(H3) hold and $\{u(n)\}_{n\geq 0}$, the bounded solution of (3.1), is $\mathcal{US}$. Let $\{n_k\}_{k\geq 1}$ be an integer sequence such that $n_k > 0$, $n_k \to \infty$ as $k \to \infty$, $u(n+n_k) \to \eta(n)$ for each $n \in \mathbb{Z}^+$ and $F(n+n_k,\nu) \to G(n,\nu)$ uniformly for $n \in \mathbb{Z}^+$ and Σ as $k \to \infty$, where Σ is any compact set in C. Then $\{\eta(n)\}_{n\geq 0}$ is a solution of the system*

$$x(n+1) = G(n,x_n), \quad n \geq 0, \tag{3.7}$$

and is $\mathcal{US}$. Moreover, if $\{u(n)\}_{n\geq 0}$ is $\mathcal{UAS}$, then $\{\eta(n)\}_{n\geq 0}$ is also $\mathcal{UAS}$.

Proof. Since $u^k(n) = u(n+n_k)$ is uniformly bounded for $n \geq -\tau$ and $k \geq 1$, we can assume that, taking a subsequence if necessary, $u(n+n_k)$ also converges for each $n \in \text{dis}[-\tau,-1]$. Define $\eta(j) = \lim_{k\to\infty} u(j+n_k)$ for $j \in \text{dis}[-\tau,-1]$. Then $u(n+n_k) \to \eta(n)$ for each $n \in \text{dis}[-\tau,\infty)$, and hence, $u_n^k \to \eta_n$ as $k \to \infty$ for each $n \geq 0$. Notice that $u_n^k \in S_\alpha$ for all $n \geq 0$, $k \geq 1$, and $\eta_n \in S_\alpha$ for $n \geq 0$. It follows from Theorem 2.4 that there exists a subsequence $\{n_{k_j}\}$ of $\{n_k\}$, $n_{k_j} \to \infty$ as $j \to \infty$, such that $F(n+n_{k_j},\nu) \to G(n,\nu)$ uniformly on $\mathbb{Z} \times S_\alpha$ as $j \to \infty$ and $G(n,\cdot)$ is continuous on S_α uniformly for all $n \in \mathbb{Z}$. Since $u(n+n_{k_j}+1) = F(n+n_{k_j},u_n^{k_j})$ and

$$\begin{aligned} &F(n+n_{k_j},u_n^{k_j}) - G(n,\eta_n) \\ &\quad = F(n+n_{k_j},u_n^{k_j}) - G(n,u_n^{k_j}) + G(n,u_n^{k_j}) - G(n,\eta_n) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \end{aligned} \tag{3.8}$$

we have $\eta(n+1) = G(n,\eta_n)$ $(n \geq 0)$. This shows that $\{\eta(n)\}_{n\geq 0}$ is a solution of (3.7).

To prove that $\{\eta(n)\}_{n\geq 0}$ is $\mathcal{US}$, we set $n_k = l_k\omega + j_*$ as before, where $0 \leq j_* < \omega$. Then $u^{k_j}(n) = u(n+n_{k_j}) \to \eta(n)$ for each $n \in \mathbb{Z}^+$ as $j \to \infty$. Since $F(n+n_{k_j},\nu) = F(n+j_*,\nu) \to G(n,\nu)$ as $j \to \infty$, we have $G(n,\nu) = F(n+j_*,\nu)$. For any $\varepsilon > 0$, let $\delta(\varepsilon) > 0$ be the one for uniform stability of $\{u(n)\}_{n\geq 0}$. For any $n_0 \in \mathbb{Z}^+$, let $\{x(n)\}_{n\geq 0}$ be a solution of (3.7) such that $\|\eta_{n_0} - x_{n_0}\| = \mu < \delta(\varepsilon)$. Since $u_n^{k_j} \to \eta_n$ as $j \to \infty$ for each $n \geq 0$, there is a positive integer $J_1 > 0$ such that if $j \geq J_1$, then

$$||u_{n_0}^{k_j} - \eta_{n_0}|| < \delta(\varepsilon) - \mu. \tag{3.9}$$

Thus, for $j \geq J_1$, we have

$$||u_{n_0+j_*+l_{k_j}\omega} - x_{n_0}|| \leq ||u_{n_0+j_*+l_{k_j}\omega} - \eta_{n_0}|| + ||\eta_{n_0} - x_{n_0}|| < \delta(\varepsilon). \tag{3.10}$$

Notice that $\{u(n+j_*+l_{k_j}\omega)\}$ $(n \geq 0)$ is a uniformly stable solution of (3.7) with $G(n,x_n) = F(n+j_*,x_n)$. Then,

$$||u_{n+j_*+l_{k_j}\omega} - x_n|| < \varepsilon \quad \forall n \geq n_0. \tag{3.11}$$

Since $\{\eta(n)\}$ is also a solution of (3.7) and $u_n^{k_j} \to \eta_n$ for each $n \geq 0$ as $j \to \infty$, for an arbitrary $\nu > 0$, there is $J_2 > 0$ such that if $j \geq J_2$, then

$$||\eta_{n_0} - u_{n_0+j_*+l_{k_j}\omega}|| < \delta(\nu), \tag{3.12}$$

and hence, $\|\eta_n - u_{n+j_*+l_{k_j}\omega}\| < \nu$ for all $n \geq n_0$, where $(\nu,\delta(\nu))$ is a pair for the uniform stability of $u(n+j_*+l_{k_j}\omega)$. This shows that if $j \geq \max(J_1,J_2)$, then

$$||\eta_n - x_n|| \leq ||\eta_n - u_{n+j_*+l_{k_j}\omega}|| + ||u_{n+j_*+l_{k_j}\omega} - x_n|| < \varepsilon + \nu \tag{3.13}$$

for all $n \geq n_0$, which implies that $\|\eta_n - x_n\| \leq \varepsilon$ for all $n \geq n_0$ if $\|\eta_{n_0} - x_{n_0}\| < \delta(\varepsilon)$ because ν is arbitrary. This proves that $\{\eta(n)\}_{n\geq 0}$ is uniformly stable.

To prove that $\{\eta(n)\}_{n\geq 0}$ is $\mathcal{UAS}$, we use definition (ii*) in Remark 2.9. Let $\{x(n)\}$ be a solution of (3.7) such that $\|\eta_{n_0} - x_{n_0}\| < \delta_0$, where δ_0 is the number for the uniformly asymptotic stability of $\{u(n)\}$. Notice that $u(n+j_*+l_{k_j}\omega) = u^{k_j}(n)$ is a uniformly asymptotically stable solution of (3.7) with $G(n,\phi) = F(n+j_*,\phi)$ and with the same δ_0 as the one for $\{u(n)\}$. Set $\|\eta_{n_0} - x_{n_0}\| = \mu < \delta_0$. Again, for sufficient large j, we have the similar relations (3.10) and (3.12) with $\|u_{n_0+j_*+l_{k_j}\omega} - x_{n_0}\| < \delta_0$ and $\|u_{n_0+j_*+l_{k_j}\omega} - \eta_{n_0}\| < \delta_0$. Thus,

$$||\eta_n - x_n|| \leq ||\eta_n - u_{n+j_*+l_{k_j}\omega}|| + ||u_{n+j_*+l_{k_j}\omega} - x_n|| \longrightarrow 0 \tag{3.14}$$

as $n \to \infty$ if $\|u_{n_0} - x_{n_0}\| < \delta_0$, because $\{u^{k_j}(n)\}$, $\{x(n)\}$, and $\{\eta(n)\}$ satisfy (3.7) with $G(n,\phi) = F(n+j_*,\phi)$. This completes the proof. □

Using Theorem 3.1 and Lemma 3.2, we can show that (3.1) has an almost periodic solution.

Theorem 3.3. *If the bounded solution $\{u(n)\}_{n\geq 0}$ of (3.1) is $\mathcal{US}$, then system (3.1) has an almost periodic solution, which is also $\mathcal{US}$.*

Proof. It follows from Theorem 3.1 that $\{u(n)\}_{n\geq 0}$ is asymptotically almost periodic. Set $u(n) = p(n) + q(n)$ $(n \geq 0)$, where $\{p(n)\}_{n\geq 0}$ is almost periodic sequence and $q(n) \to 0$ as $n \to \infty$. For positive integer sequence $\{n_k\omega\}$, there is a subsequence $\{n_{k_j}\omega\}$ of $\{n_k\omega\}$ such that $p(n+n_{k_j}\omega) \to p^*(n)$ uniformly on $\mathbb{Z}$ as $j \to \infty$ and $\{p^*(n)\}$ is almost periodic. Then $u(n+n_{k_j}\omega) \to p^*(n)$ uniformly for $n \geq -\tau$, and hence, $u_{n+n_{k_j}\omega} \to p^*_n$ for all $n \geq 0$ as $j \to \infty$. Since

$$u(n+n_{k_j}\omega+1) = F(n+n_{k_j}\omega, u_{n+n_{k_j}\omega}) = F(n, u_{n+n_{k_j}\omega}) \longrightarrow F(n, p^*_n) \tag{3.15}$$

as $j \to \infty$, we have $p^*(n+1) = F(n,p^*_n)$ for $n \geq 0$, that is, system (3.1) has an almost periodic solution, which is also $\mathcal{US}$ by Lemma 3.2. □

Now, we show that if the bounded solution $\{u(n)\}$ is uniformly asymptotically stable, then (3.1) has a periodic solution of period $m\omega$ for some positive integer m.

Theorem 3.4. *If the bounded solution $\{u(n)\}_{n\geq 0}$ of (3.1) is $\mathcal{UAS}$, then system (3.1) has a periodic solution of period $m\omega$ for some positive integer m, which is also $\mathcal{UAS}$.*

Proof. Set $u^k(n) = u(n+k\omega)$, $k = 1,2,\ldots$. By the proof of Theorem 3.1, there is a subsequence $\{u^{k_j}(n)\}$ converges to a solution $\{\eta(n)\}$ of (3.3) for each $n \geq -\tau$ and hence, $u_0^{k_j} \to \eta_0$ as $j \to \infty$. Thus, there is a positive integer p such that $\|u_0^{k_p} - u_0^{k_{p+1}}\| < \delta_0$ $(0 \leq k_p < k_{p+1})$, where δ_0 is the one for uniformly asymptotic stability of $\{u(n)\}_{n\geq 0}$. Let $m = k_{p+1} - k_p$

and notice that $u^m(n) = u(n+m\omega)$ is a solution of (3.1). Since $u^m_{k_p\omega}(j) = u^m(k_p\omega + j) = u(k_{p+1}\omega + j) = u_{k_{p+1}\omega}(j)$ for $j \in \text{dis}[-\tau, 0]$, we have

$$||u^m_{k_p\omega} - u_{k_p\omega}|| = ||u_{k_{p+1}\omega} - u_{k_p\omega}|| = ||u_0^{k_{p+1}} - u_0^{k_p}|| < \delta_0, \tag{3.16}$$

and hence,

$$||u^m_n - u_n|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.17}$$

because $\{u(n)\}_{n\geq 0}$ is $\mathcal{UAS}$ (see also Remark 2.9). On the other hand, $\{u(n)\}_{n\geq 0}$ is asymptotically almost periodic by Theorem 3.1, then

$$u(n) = p(n) + q(n), \quad n \geq 0, \tag{3.18}$$

where $\{p(n)\}_{n\in\mathbb{Z}}$ is almost periodic and $q(n) \to 0$ as $n \to \infty$. It follows from (3.17) and (3.18) that

$$|p(n) - p(n+m\omega)| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.19}$$

which implies that $p(n) = p(n+m\omega)$ for all $n \in \mathbb{Z}$ because $\{p(n)\}$ is almost periodic.

For integer sequence $\{km\omega\}$, $k = 1,2,\ldots$, we have $u(n+km\omega) = p(n) + q(n+km\omega)$. Then $u(n+km\omega) \to p(n)$ uniformly for all $n \geq -\tau$ as $k \to \infty$, and hence, $u_{n+km\omega} \to p_n$ for $n \geq 0$ as $k \to \infty$. Since $u(n+km\omega+1) = F(n, u_{n+km\omega})$, we have $p(n+1) = F(n, p_n)$ for $n \geq 0$, which implies that (3.1) has a periodic solution $\{p(n)\}_{n\geq 0}$ of period $m\omega$. The uniformly asymptotic stability of $\{p(n)\}_{n\geq 0}$ follows from Lemma 3.2. □

Finally, we show that if the bounded solution $\{u(n)\}$ is $\mathcal{GUAS}$, then (3.1) has a periodic solution of period ω.

Theorem 3.5. *If the bounded solution $\{u(n)\}_{n\geq 0}$ of (3.1) is $\mathcal{GUAS}$, then system (3.1) has a periodic solution of period ω.*

Proof. By Theorem 3.1, $\{u(n)\}_{n\geq 0}$ is asymptotically almost periodic. Then $u(n) = p(n) + q(n)$ $(n \geq 0)$, where $\{p(n)\}$ $(n \in \mathbb{Z})$ is an almost periodic sequence and $q(n) \to 0$ as $n \to \infty$. Notice that $u(n+\omega)$ is also a solution of (3.1) satisfying $u_\omega \in S_\alpha$. Since $\{u(n)\}$ is $\mathcal{GUAS}$, we have $\|u_n - u_{n+\omega}\| \to 0$ as $n \to \infty$, which implies that $p(n) = p(n+\omega)$ for all $n \in \mathbb{Z}$. By the same technique in the proof of Theorem 3.4, we can show that $\{p(n)\}$ is an ω-periodic solution of (3.1). □

4. Almost periodic systems

In this section, we discuss the existence of asymptotically almost periodic solutions of (2.6), that is,

$$x(n+1) = F(n, x_n), \quad n \geq 0, \tag{4.1}$$

under the condition (H4) as follows.

(H4) $F \in \mathcal{AP}(\mathbb{Z} \times C : \mathbb{E}^d)$, that is, $F(n,v)$ is almost periodic in $n \in \mathbb{Z}$ uniformly for $v \in C$.

By $H(F)$ we denote the uniform closure of $F(n,v)$, that is, $G \in H(F)$ if there is an integer sequence $\{\alpha_k\}$ such that $\alpha_k \to \infty$ and $F(n+\alpha_k,v) \to G(n,v)$ uniformly on $\mathbb{Z}\times\Sigma$ as $k\to\infty$, where Σ is any compact set in C. Note that $H(F) \subset \mathcal{AP}(\mathbb{Z}\times C:\mathbb{E}^d)$ by Theorem 2.3 and $F \in H(F)$ by Theorem 2.4.

We first show that if (4.1) has a bounded asymptotically almost periodic solution, then (4.1) has an almost periodic solution. In fact, we have a more general result in the following.

Theorem 4.1. *Suppose (H1), (H2), and (H4) hold. If the bounded solution $\{u(n)\}_{n\geq 0}$ of (4.1) is asymptotically almost periodic, then for any $G \in H(F)$, the system*

$$x(n+1) = G(n,x_n) \tag{4.2}$$

has an almost periodic solution for $n \geq 0$. Consequently, (4.1) has an almost periodic solution.

Proof. Since the solution $\{u(n)\}_{n\geq 0}$ is asymptotically almost periodic, it follows from Theorem 2.6 that it has the decomposition $u(n) = p(n)+q(n)$ $(n \geq 0)$, where $\{p(n)\}_{n\in\mathbb{Z}}$ is almost periodic and $q(n) \to 0$ as $n\to\infty$. Notice that $\{u(n)\}$ is bounded. There is compact set $S_\alpha \in C$ such that $u_n \in S_\alpha$ and $p_n \in S_\alpha$ for all $n \geq 0$. For any $G \in H(F)$, there is an integer sequence $\{n_k\}$, $n_k > 0$, such that $n_k \to \infty$ as $k\to\infty$ and $F(n+n_k,v) \to G(n,v)$ uniformly on $\mathbb{Z}\times S_\alpha$ as $k\to\infty$. Taking a subsequence if necessary, we can also assume that $p(n+n_k) \to p^*(n)$ uniformly on $\mathbb{Z}$ and $\{p^*(n)\}$ is also an almost periodic sequence. For any $j \in \operatorname{dis}[-\tau,0]$, there is positive integer k_0 such that if $k > k_0$, then $j+n_k \geq 0$ for any $j \in \operatorname{dis}[-\tau,0]$. In this case, we see that $u(n+n_k) \to p^*(n)$ uniformly for all $n \geq -\tau$ as $k\to\infty$, and hence, $u_{n+n_k} \to p_n^*$ in C uniformly for $n \in \mathbb{Z}^+$ as $k\to\infty$. Since

$$\begin{aligned} u(n+n_k+1) &= F(n+n_k,u_{n+n_k}) \\ &= [F(n+n_k,u_{n+n_k}) - F(n+n_k,p_n^*)] \\ &\quad + [F(n+n_k,p_n^*) - G(n,p_n^*)] + G(n,p_n^*), \end{aligned} \tag{4.3}$$

the first term of right-hand side of (4.3) tends to zero as $k\to\infty$ by Theorem 2.2 and $F(n+n_k,p_n^*) - G(n,p_n^*) \to 0$ as $k\to\infty$, we have $p^*(n+1) = G(n,p_n^*)$ for all $n \in \mathbb{Z}^+$, which implies that (4.2) has an almost periodic solution $\{p^*(n)\}_{n\geq 0}$, passing through $(0,p_0^*)$, where $p_0^*(j) = p^*(j)$ for $j \in \operatorname{dis}[-\tau,0]$. □

To deal with almost periodic solutions of (4.1) in terms of uniform stability, we assume that for each $G \in H(F)$, system (4.2) has a unique solution for a given initial condition.

Lemma 4.2. *Suppose (H1), (H2), and (H4) hold. Let $\{u(n)\}_{n\geq 0}$ be the bounded solution of (4.1). Let $\{n_k\}_{k\geq 1}$ be a positive integer sequence such that $n_k\to\infty$, $u_{n_k}\to\psi$, and $F(n+n_k,v) \to G(n,v)$ uniformly on $\mathbb{Z}\times\Sigma$ as $k\to\infty$, where Σ is any compact subset in C and $G\in H(F)$. If the bounded solution $\{u(n)\}_{n\geq 0}$ is $\mathcal{US}$, then the solution $\{\eta(n)\}_{n\geq 0}$ of (4.2), through $(0,\psi)$, is $\mathcal{US}$. In addition, if $\{u(n)\}_{n\geq 0}$ is $\mathcal{UAS}$, then $\{\eta(n)\}_{n\geq 0}$ is also $\mathcal{UAS}$.*

Proof. Set $u^k(n) = u(n+n_k)$. It is easy to see that $u^k(n)$ is a solution of

$$x(n+1) = F(n+n_k,x_n), \quad n \geq 0, \tag{4.4}$$

passing though $(0, u_{n_k})$ and $u_n^k \in S_\alpha$ for all k, where S_α is compact subset of C such that $\|u_n\| < \alpha$ for all $n \geq 0$. Since $\{u(n)\}_{n\geq 0}$ is $\mathcal{US}$, $\{u^k(n)\}$ is also $\mathcal{US}$ with the same pair $(\varepsilon, \delta(\varepsilon))$ as the one for $\{u(n)\}_{n\geq 0}$. Taking a subsequence if necessary, we can assume that $\{u^k(n)\}_{k\geq 1}$ converges to a vector $\eta(n)$ for all $n \geq 0$ as $k \to \infty$. From (4.3) with $p_n^* = \eta_n$, we can see that $\{\eta(n)\}_{n\geq 0}$ is the unique solution of (4.2), through $(0, \psi)$.

To show that the solution $\{\eta(n)\}_{n\geq 0}$ of (4.2) is $\mathcal{US}$, we need to prove that if for any $\varepsilon > 0$ and any integer $n_0 \geq 0$, there exists $\delta^*(\varepsilon) > 0$ such that $\|\eta_{n_0} - y_{n_0}\| < \delta^*(\varepsilon)$ implies that $\|\eta_n - y_n\| < \varepsilon$ for all $n \geq n_0$, where $\{y(n)\}_{n\geq n_0}$ is a solution of (4.2) passing through (n_0, ϕ) with $y_{n_0} = \phi \in C$.

For any given $n_0 \in \mathbb{Z}^+$, if k is sufficiently large, say $k \geq k_0 > 0$, we have

$$||u_{n_0}^k - \eta_{n_0}|| < \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right), \tag{4.5}$$

where $\delta(\varepsilon)$ is the one for uniform stability of $\{u(n)\}_{n\geq 0}$. Let $\phi \in C$ such that

$$||\phi - \eta_{n_0}|| < \frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right) \tag{4.6}$$

and let $\{x(n)\}_{n\geq n_0}$ be the solution of (4.1) such that $x_{n_0+n_k} = \phi$. Then $\{x^k(n) = x(n+n_k)\}$ is a solution of (4.4) with $x_{n_0}^k = \phi$. Since $\{u^k(n)\}$ is $\mathcal{US}$ and $\|x_{n_0}^k - u_{n_0}^k\| < \delta(\varepsilon/2)$ for $k \geq k_0$, we have

$$||u_n^k - x_n^k|| < \frac{\varepsilon}{2} \quad \forall n \geq n_0,\ k \geq k_0. \tag{4.7}$$

It follows from (4.7) that

$$||x_n^k|| \leq ||u_n^k|| + \frac{\varepsilon}{2} < \alpha + \frac{\varepsilon}{2} \quad \forall n \geq n_0,\ k \geq k_0. \tag{4.8}$$

Then there exists a number $\alpha^* > 0$ such that $x_n^k \in S_{\alpha^*}$ for all $n \geq 0$ and $k \geq k_0$, which implies that there is subsequence of $\{x^k(n)\}_{k\geq k_0}$ for each $n \geq n_0 - \tau$, denoted by $\{x^k(n)\}$ again, such that $x^k(n) \to y(n)$ for each $n \geq n_0 - \tau$, and hence, $x_n^k \to y_n$ for all $n \geq n_0$ as $k \to \infty$. Clearly, $y_{n_0} = \phi$ and the set S_{α^*} is compact set in C. Since $F(n, v)$ is almost periodic in n uniformly for $v \in C$, we can assume that, taking a subsequence if necessary, $F(n + n_k, v) \to G(n, v)$ uniformly on $\mathbb{Z} \times S_{\alpha^*}$ as $k \to \infty$. Taking $k \to \infty$ in $x^k(n+1) = F(n + n_{n_k}, x_n^k)$, we have $y(n+1) = G(n, y_n)$, namely, $\{y(n)\}$ is the unique solution of (4.2), passing through (n_0, ϕ) with $y_{n_0} = \phi \in C$. On the other hand, for any integer $N > 0$, there exists $k_N \geq k_0$ such that if $k \geq k_N$, then

$$||x_n^k - y_n|| < \frac{\varepsilon}{4}, \quad ||u_n^k - \eta_n|| < \frac{\varepsilon}{4} \quad \text{for } n_0 \leq n \leq n_0 + N. \tag{4.9}$$

From (4.7) and (4.9), we obtain

$$||\eta_n - y_n|| < \varepsilon \quad \text{for } n_0 \leq n \leq n_0 + N. \tag{4.10}$$

Since N is arbitrary, we have $\|\eta_n - y_n\| < \varepsilon$ for all $n \geq n_0$ if $\|\phi - \eta_{n_0}\| < [\delta(\varepsilon/2)]/2$ and $\phi \in C$, which implies that the solution $\{\eta(n)\}_{n\geq 0}$ of (4.2) is $\mathcal{US}$.

Now, we assume that $\{u(n)\}_{n\geq 0}$ is $\mathcal{UAS}$. Then the solution $\{u^k(n)\}$ of (4.4) is also $\mathcal{UAS}$ with the same pair $(\delta_0,\varepsilon,N(\varepsilon))$ as the one for $\{u(n)\}_{n\geq 0}$. Let $(\delta^*(\varepsilon),\varepsilon)$ be the pair for uniform stability of $\{\eta(n)\}$.

For any given $n_0\in\mathbb{Z}^+$, if k is sufficiently large, say $k\geq k_0>0$, we have

$$||u^k_{n_0}-\eta_{n_0}||<\frac{1}{2}\delta_0, \tag{4.11}$$

where δ_0 is the one for uniformly asymptotic stability of $\{u(n)\}_{n\geq 0}$. Let $\phi\in C$ such that $\|\phi-\eta_{n_0}\|<(\delta_0/2)$ and let $\{x(n)\}_{n\geq n_0}$, for each fixed $k\geq k_0$, be the solution of (4.1) such that $x_{n_0+n_k}=\phi$. Then $\{x^k(n)=x(n+n_k)\}$ is a solution of (4.4) with $x^k_{n_0}=\phi$. Since $\{u^k(n)\}$ is $\mathcal{UAS}$ and $\|x^k_{n_0}-u^k_{n_0}\|<(\delta_0/2)$ for each fixed $k\geq k_0$, we have

$$||u^k_n-x^k_n||<\frac{\varepsilon}{2}\quad \forall n\geq n_0+N\left(\frac{\varepsilon}{2}\right),\ k\geq k_0. \tag{4.12}$$

By the same argument as the above, we can assume that, taking a subsequence if necessary, $\{x^k(n)\}$ converges to the solution $\{y(n)\}$ of (4.2) through (n_0,ϕ) and $F(n+n_k,v)\to G(n,v)$ uniformly on $\mathbb{Z}\times S_{\alpha^*}$ as $k\to\infty$, where S_{α^*} is compact set in C with $|x^k(n)|\leq\alpha^*$ for all $k\geq k_0$ and $n\geq n_0-\tau$. Then $\{y(n)\}$ is the unique solution of (4.2), passing through (n_0,ϕ) with $y_{n_0}=\phi\in C$. On the other hand, for any integer $N>0$ there exists $k_N\geq k_0$ such that if $k\geq k_N$, then

$$||x^k_n-y_n||<\frac{\varepsilon}{4},\quad ||u^k_n-\eta_n||<\frac{\varepsilon}{4}\quad \text{for } n_0+N\left(\frac{\varepsilon}{2}\right)\leq n\leq n_0+N\left(\frac{\varepsilon}{2}\right)+N \tag{4.13}$$

and hence, $\|y_n-\eta_n\|<\varepsilon$ for $n_0+N(\varepsilon/2)\leq n\leq n_0+N(\varepsilon/2)+N$. Since N is arbitrary, we have

$$||y_n-\eta_n||<\varepsilon\quad \forall n\geq n_0+N\left(\frac{\varepsilon}{2}\right) \tag{4.14}$$

if $\|\phi-\eta_{n_0}\|<(\delta_0/2)$ and $\phi\in C$. The proof is completed. □

Before dealing with the asymptotic almost periodicity of $\{u(n)\}$, we need the following lemma.

Lemma 4.3. *Suppose that assumptions (H1), (H2), and (H4) hold, the bounded solution* $\{u(n)=u(n,\psi^0)\}$ *of (4.1) is* $\mathcal{UAS}$ *and for each* $G\in H(F)$*, the solution of (4.2) is unique for any given initial data. Let* $S\supseteq S_\alpha$ *be a given compact set in* C*. Then for any* $\varepsilon>0$*, there exists* $\delta=\delta(\varepsilon)>0$ *such that if* $n_0\geq 0$, $\|u_{n_0}-x_{n_0}\|<\delta$*, and* $\{h(n)\}$ *is a sequence with* $|h(n)|\leq\delta$ *for* $n\geq n_0$*, one has* $\|u_n-x_n\|<\varepsilon$ *for all* $n\geq n_0$*, where* $\{x(n)\}$ *is any bounded solution of the system*

$$x(n+1)=F(n,x_n)+h(n),\quad n\geq n_0, \tag{4.15}$$

passing through (n_0,x_{n_0}) *such that* $x_n\in S$ *for all* $n\geq n_0$.

Proof. Suppose that the bounded solution $\{u(n)\}_{n\geq 0}$ of (4.1) is $\mathcal{UAS}$ with the triple $(\delta(\cdot),\delta_0,N(\cdot))$. In order to establish Lemma 4.3 by a contradiction, we assume that Lemma 4.3 is not true. Then for some compact set $S_* \supseteq S_\alpha$, there exist an ε, $0<\varepsilon<\delta_0$, sequences $\{n_k\}\subset\mathbb{Z}^+$, $\{r_k\}\subset\mathbb{Z}^+$, mapping sequences $h_k : \mathrm{dis}[n_k,\infty)\to\mathbb{E}^d$, $\varphi^k : \mathrm{dis}[n_k-\tau,n_k]\to\mathbb{E}^d$, and

$$\begin{gathered} \|u_{n_k}-x^k_{n_k}\|<\frac{1}{k},\quad |h_k(n)|\leq\frac{1}{k}\quad \text{for } n\geq n_k,\\ \|u_n-x^k_n\|\leq\varepsilon\quad \text{for } n_k\leq n\leq n_k+r_k-1,\quad \|u_{n_k+r_k}-x^k_{n_k+r_k}\|\geq\varepsilon \end{gathered}\tag{4.16}$$

for sufficiently large k, where $\{x^k(n)\}$ is a solution of

$$x(n+1)=F(n,x_n)+h_k(n),\quad n\geq n_k,\tag{4.17}$$

passing through (n_k,φ^k) such that $x^k_n\in S_*$ for all $n\geq n_k$ and $k\geq 1$. Since S_* is bounded subset of C, it follows that $\{x^k(n_k+r_k+n)\}_{k\geq1}$ and $\{x^k(n_k+n)\}_{k\geq1}$ are uniformly bounded for all n_k and $n\geq-\tau$. We first consider the case where $\{r_k\}_{k\geq1}$ contains an unbounded subsequence. Set $N=N(\varepsilon)>1$. Taking a subsequence if necessary, we may assume that there is $G\in H(F)$ such that $F(n+n_k+r_k-N,v)\to G(n,v)$ uniformly on $\mathbb{Z}^+\times S_*$, $x^k(n+n_k+r_k-N)\to z(n)$, and $u(n+n_k+r_k-N)\to w(n)$ for $n\in\mathbb{Z}^+$ as $k\to\infty$, where $z,w:\mathbb{Z}^+\to\mathbb{E}^d$ are some bounded functions. Since

$$x^k(n+n_k+r_k-N+1)=F(n+n_k+r_k-N,x^k_{n+n_k+r_k-N})+h_k(n+n_k+r_k-N),\tag{4.18}$$

passing to limit as $k\to\infty$, by the similar arguments in the proof of Theorem 4.1, we conclude that $\{z(n)\}_{n\geq0}$ is the solution of the following equation:

$$x(n+1)=G(n,x_n),\quad n\in\mathbb{Z}^+.\tag{4.19}$$

Similarly, $\{w(n)\}_{n\in\mathbb{Z}^+}$ is also a solution of (4.19). Since $x^k_{n_k+r_k-N}(j)=x^k(n_k+r_k-N+j)\to z(j)=z_0(j)$ and $u_{n_k+r_k-N}(j)=u(n_k+r_k-N+j)\to w(j)=w_0(j)$ as $k\to\infty$ for all $j\in\mathrm{dis}[-\tau,0]$, it follows from (4.16) that $\|w_0-z_0\|\leq\lim_{k\to\infty}\|w_{n_k+r_k-N}-z_{n_k+r_k-N}\|\leq\varepsilon<\delta_0$. Notice that $\{w(n)\}_{n\in\mathbb{Z}^+}$ is a solution of (4.19), passing through $(0,w_0)$, and is $\mathcal{UAS}$ by Lemma 4.2. We have $\|w_N-z_N\|<\varepsilon$. On the other hand, since

$$\begin{gathered} u_{n_k+r_k}(j)=u(N+j+n_k+r_k-N)\longrightarrow w(N+j)=w_N(j),\\ x^k_{n_k+r_k}(j)=x^k(N+j+n_k+r_k-j)\longrightarrow z(N+j)=z_N(j) \end{gathered}\tag{4.20}$$

as $k\to\infty$ for each $j\in\mathrm{dis}[-\tau,0]$, it follows from (4.16) that

$$\|w_N-z_N\|=\lim_{k\to\infty}\|u_{n_k+r_k}-x^k_{n_k+r_k}\|\geq\varepsilon.\tag{4.21}$$

This is a contradiction. Thus, the sequence $\{r_k\}$ must be bounded. Taking a subsequence if necessary, we can assume that $0<r_k\equiv r_0<\infty$. Moreover, we may assume that $x^k(n_k+n)\to\tilde{z}(n)$ and $u(n_k+n)\to\tilde{w}(n)$ for each $n\geq-\tau$, and $F(n+n_k,v)\to\tilde{G}(n,v)$ uniformly on $\mathbb{Z} \times S_*$, for some functions $\widetilde{z}(n)$, $\widetilde{w}(n)$ on $\mathbb{Z}^+$, and $\widetilde{G} \in H(F)$. Since $u_{n_k}(j) = u(n_k + j) \to \widetilde{w}(j) = \widetilde{w}_0(j)$ and $x^k_{n_k}(j) = x^k(n_k + j) \to \widetilde{z}(j) = \widetilde{z}_0(j)$ as $k \to \infty$ for all $j \in \text{dis}[-\tau,0]$, we have $\|\widetilde{w}_0 - \widetilde{z}_0\| = \lim_{k\to\infty} \|u_{n_k} - x^k_{n_k}\| = \lim_{k\to\infty} \|u_{n_k} - \varphi^k\| = 0$ by (4.16), and hence, $\widetilde{w}_0 \equiv \widetilde{z}_0$, that is, $\widetilde{w}(j) = \widetilde{z}(j)$ for all $j \in \text{dis}[-\tau,0]$. Moreover, $\widetilde{z}(n)$ and $\widetilde{w}(n)$ satisfy the same relation

$$x(n+1) = \widetilde{G}(n, x_n), \quad n \in \mathbb{Z}^+. \tag{4.22}$$

The uniqueness of the solutions for the initial value problems implies that $\widetilde{z}(n) \equiv \widetilde{w}(n)$ for $n \in \mathbb{Z}^+$, and hence, $\|\widetilde{w}_{r_0} - \widetilde{z}_{r_0}\| = 0$. On the other hand, since $u_{n_k+r_0}(j) = u(n_k + r_0 + j) \to \widetilde{w}(r_0 + j) = \widetilde{w}_{r_0}(j)$ and $x^k_{n_k+r_0}(j) = x^k(n_k + r_0 + j) \to \widetilde{z}(r_0 + j) = \widetilde{z}_{r_0}(j)$ as $k \to \infty$ for all $j \in \text{dis}[-\tau,0]$, from (4.16) we have

$$||\widetilde{w}_{r_0} - \widetilde{z}_{r_0}|| = \lim_{k\to\infty} ||u_{n_k+r_k} - x^k_{n_k+r_k}|| \geq \varepsilon \tag{4.23}$$

This is a contradiction. This contradiction shows that Lemma 4.3 is true. □

We are now in a position to prove the following result.

Theorem 4.4. *Suppose that for each $G \in H(F)$, the solution of (4.2) is unique for the initial condition. If the bounded solution $\{u(n)\}_{n\geq 0}$ of (4.1) is $\mathcal{UAS}$, then $\{u(n)\}_{n\geq 0}$ is asymptotically almost periodic. Consequently, (4.1) has an almost periodic solution which is $\mathcal{UAS}$.*

Proof. Let the bounded solution $\{u(n)\}$ of (4.1) be $\mathcal{UAS}$ with the triple $(\delta(\cdot), \delta_0, N(\cdot))$. Let $\{n_k\}_{k\geq 1}$ be any positive integer such that $n_k \to \infty$ as $k \to \infty$. Set $u^k(n) = u(n + n_k)$. Then $u^k(n)$ is a solution of

$$x(n+1) = F(n + n_k, x_n) \tag{4.24}$$

and $\{u^k(n)\}$ is $\mathcal{UAS}$ with the same triple $(\delta(\cdot), \delta_0, N(\cdot))$. By Lemma 4.3, for the set S_α and any $0 < \varepsilon < 1$ there exists $\delta_1(\varepsilon) > 0$ such that $|h(n)| < \delta_1(\varepsilon)$ and $\|x^k_{n_0} - x_{n_0}\| < \delta_1(\varepsilon)$ for some $n_0 \geq 0$ imply that $\|x^k_n - x_n\| < (\varepsilon/2)$ for all $n \geq n_0$, where $\{x(n)\}$ $(n \geq n_0)$ is a solution of

$$x(n+1) = F(n + n_k, x_n) + h(n), \tag{4.25}$$

through (n_0, x_{n_0}) and $x_n \in S_\alpha$ for $n \geq n_0$. Since $\{u^k(j) = u(n_k + j)\}$ is uniformly bounded for all $k \geq 1$ and $j \geq -\tau$, taking a subsequence if necessary, we can assume that $\{u^k(j)\}$ is convergent for each $j \in \text{dis}[-\tau, \infty)$, $F(n + n^k, v) \to G(n, v)$ uniformly on $\mathbb{Z}^+ \times S_\alpha$. In this case, there is a positive integer $k_1(\varepsilon)$ such that if $m, k \geq k_1(\varepsilon)$, then

$$||u^k_0 - u^m_0|| < \delta_1(\varepsilon). \tag{4.26}$$

On the other hand, $\{u^m(n) = u(n + n_m)\}$, $u^m_n \in S_\alpha$ for $n \in \mathbb{Z}^+$, is a solution of (4.25) with $h(n) = h_{k,m}(n)$, that is,

$$x(n+1) = F(n + n_k, x_n) + h_{k,m}(n), \tag{4.27}$$

where $h_{k,m}(n)$ is defined by the relation

$$h_{k,m}(n) = F(n+n_m, u_n^m) - F(n+n_k, u_n^m), \quad n \in \mathbb{Z}^+. \tag{4.28}$$

To apply Lemma 4.3 to (4.24) and its associated (4.27), we will discuss the properties of the sequence $\{h_{k,m}(n)\}_{n\geq 0}$. Since $F(n+n_k, v) \to G(n, v)$ uniformly on $\mathbb{Z}^+ \times S_\alpha$, for the above $\delta_1(\varepsilon) > 0$, there is a positive integer $k_2(\varepsilon) > k_1(\varepsilon)$ such that if $k, m \geq k_2(\varepsilon)$, then

$$|F(n+n_m, v) - F(n+n_k, v)| < \delta_1(\varepsilon) \quad \forall n \in \mathbb{Z}^+,\ v \in S_\alpha, \tag{4.29}$$

which implies that $|h_{k,m}(n)| = |F(n+n_m, u_n^m) - F(n+n_k, u_n^m)| < \delta_1(\varepsilon)$ for all $n \in \mathbb{Z}$. Applying Lemma 4.3 to (4.24) and its associated (4.27), with the above arguments and condition (4.26), we conclude that for any positive integer sequence $\{n_k\}_{k\geq 1}$, $n_k \to \infty$ as $k \to \infty$, and $\varepsilon > 0$, there is a positive integer $k_2(\varepsilon) > 0$ such that

$$||u_n^k - u_n^m|| < \varepsilon \quad \forall n \geq 0 \text{ if } k, m > k_2(\varepsilon), \tag{4.30}$$

and hence, $|u^k(n) - u^m(n)| = |u_n^k(0) - u_n^m(0)| < \varepsilon$ for all $n \geq 0$ if $k, m > k_2(\varepsilon)$. This implies that the bounded solution $\{u(n)\}_{n\geq 0}$ of (4.1) is asymptotically almost periodic sequence by Theorem 2.6. Furthermore, (4.1) has an almost periodic solution, which is $\mathcal{UAS}$ by Theorem 4.1. This ends the proof. □

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