Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2008, Article ID 678402, 13 pages
doi:10.1155/2008/678402

Research Article

Dynamical Properties for a Class of Fourth-Order Nonlinear Difference Equations

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Received 6 May 2007; Revised 14 August 2007; Accepted 18 September 2007

Recommended by Jianshe Yu

We consider the dynamical properties for a kind of fourth-order rational difference equations. The key is for us to find that the successive lengths of positive and negative semicycles for nontrivial solutions of this equation periodically occur with same prime period 5. Although the period is same, the order for the successive lengths of positive and negative semicycles is completely different. The rule is $\ldots, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, \ldots$, or $\ldots, 2^+, 1^-, 1^+, 1^-, 2^+, 1^-, 1^+, 1^-, \ldots$, or $\ldots, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, \ldots$. By the use of the rule, the positive equilibrium point of this equation is proved to be globally asymptotically stable.

1. Introduction and preliminaries

Rational difference equation, as a kind of typical nonlinear difference equations, is always a subject studied in recent years. Especially, some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results of rational difference equations. For the systematical investigations of this aspect, refer to the monographs [1–3], the papers [4–9], and the references cited therein.

Motivated by the work [5–7], we consider in this paper the following fourth-order rational difference equation:

$$x_{n+1} = \frac{F(x_n, x_{n-1}, x_{n-2}, x_{n-3})}{G(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where

$$F(x,y,z,w) = x^u y^v + x^u z^k + x^u w^j + y^v z^k + y^v w^j + z^k w^j + x^u y^v z^k w^j + 1 + a,$$
$$G(x,y,z,w) = x^u + y^v + z^k + w^j + x^u y^v z^k + x^u y^v w^j + x^u z^k w^j + y^v z^k w^j + a, \tag{1.2}$$

the parameter $a \in [0,+\infty)$, $u \in (0,1]$, $v,k,j \in (0,+\infty)$, and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,+\infty)$.

Mainly, by analyzing the rule for the length of semicycle to occur successively, we clearly describe out the rule for the trajectory structure of its solutions. With the help of several key lemmas, we further derive the global asymptotic stability of positive equilibrium of (1.1). To the best of our knowledge, (1.1) has not been investigated so far; therefore, it is theoretically meaningful to study its qualitative properties.

It is easy to see that the positive equilibrium $\overline{x}$ of (1.1) satisfies

$$\overline{x} = \frac{1 + a + \overline{x}^{u+v} + \overline{x}^{u+k} + \overline{x}^{u+j} + \overline{x}^{v+k} + \overline{x}^{v+j} + \overline{x}^{k+j} + \overline{x}^{u+v+k+j}}{a + \overline{x}^{u} + \overline{x}^{v} + \overline{x}^{k} + \overline{x}^{j} + \overline{x}^{u+v+k} + \overline{x}^{u+v+j} + \overline{x}^{u+k+j} + \overline{x}^{v+k+j}}. \tag{1.3}$$

From this, we see that (1.1) possesses a unique positive equilibrium $\overline{x} = 1$.

It is essential in this note for us to obtain the general rule for the trajectory structure of solutions of (1.1) as follows.

Theorem 1.1. *The rule for the trajectory structure of any solution of* (1.1) *is as follows:*

(i) *the solution is either eventually trivial,*

(ii) *or the solution is eventually nontrivial,*

  (1) *and further, either the solution is eventually positive nonoscillatory,*

  (2) *or the solution is strictly oscillatory, and moreover, the successive lengths for positive and negative semicycles occur periodically with prime period 5, and the rule is* $\ldots, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, \ldots$, *or* $\ldots, 2^+, 1^-, 1^+, 1^-, 2^+, 1^-, 1^+, 1^-, \ldots$, *or* $\ldots, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, \ldots$.

The positive equilibrium point of (1.1) *is a global attractor of all its solutions.*

It follows from the results stated in the sequel that Theorem 1.1 is true.

For the corresponding concepts in this paper, see [3] or the papers [5–7].

2. Nontrivial solution

Theorem 2.1. *A positive solution* $\{x_n\}_{n=-3}^{\infty}$ *of* (1.1) *is eventually trivial if and only if*

$$(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) = 0. \tag{2.1}$$

Proof. Sufficiency. Assume that (2.1) holds. Then, according to (1.1), we know that the following conclusions are true: if $x_{-3} = 1$, $x_{-2} = 1$, $x_{-1} = 1$, or $x_0 = 1$, then $x_n = 1$ for $n \geq 1$.
Necessity. Conversely, assume that

$$(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) \neq 0. \tag{2.2}$$

Then, we can show that $x_n \neq 1$ for any $n \geq 1$. For the sake of contradiction, assume that for some $N \geq 1$,

$$x_N = 1, \quad x_n \neq 1 \quad \text{for any } -3 \leq n \leq N-1. \tag{2.3}$$

Clearly,

$$1 = x_N = \frac{F(x_{N-1}, x_{N-2}, x_{N-3}, x_{N-4})}{G(x_{N-1}, x_{N-2}, x_{N-3}, x_{N-4})}. \tag{2.4}$$

From this, we can know that

$$0 = x_N - 1 = \frac{(x_{N-1}^u - 1)(x_{N-2}^v - 1)(x_{N-3}^k - 1)(x_{N-4}^j - 1)}{G(x_{N-1}, x_{N-2}, x_{N-3}, x_{N-4})}, \tag{2.5}$$

which implies that $x_{N-1} = 1$, $x_{N-2} = 1$, $x_{N-3} = 1$, or $x_{N-4} = 1$. This contradicts (2.3). □

Remark 2.2. Theorem 2.1 actually demonstrates that a positive solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is eventually nontrivial if $(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1) \neq 0$. So, if a solution is a nontrivial one, then $x_n \neq 1$ for any $n \geq -3$.

3. Several key lemmas

We state several key lemmas in this section, which will be important in the proofs of the sequel. Denote $N_k = \{k, k+1, \dots\}$ for any integer k.

Lemma 3.1. *If the integer $i \in N_{-3}$, then*

$$x_{n+1} - x_i = \frac{K(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_i)}{G(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \quad n = 0, 1, \dots, \tag{3.1}$$

where

$$\begin{aligned} &K(x, y, z, w, p) \\ &\quad = (1 - x^u p)(1 + y^v z^k + y^v w^j + z^k w^j) + a(1 - x_i) + (x^u - p)(y^v + z^k + w^j + y^v z^k w^j). \end{aligned} \tag{3.2}$$

Lemma 3.2. *If the integer $i \in N_{-3}$, $t = 1, 2, \dots$, then*

$$1 - x_{n+1} x_i^{1/u^t} = \frac{M(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_i)}{G(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \quad n = 0, 1, \dots, \tag{3.3}$$

where

$$\begin{aligned} M(x, y, z, w, p) = {} & (x^u - p^{1/u^t})(1 + y^v z^k + y^v w^j + z^k w^j) + a(1 - p^{1/u^t}) \\ & + (1 - x^u p^{1/u^t})(y^v + z^k + w^j + y^v z^k w^j). \end{aligned} \tag{3.4}$$

Lemma 3.3. *If the integer $i \in N_{-3}$, $t = 1, 2, \ldots$, then*

$$x_{n+1} - x_i^{1/u^t} = \frac{N(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_i)}{G(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \quad n = 0, 1, \ldots, \tag{3.5}$$

where

$$\begin{aligned} N(x, y, z, w, p) = {} & \left(1 - x^u p^{1/u^t}\right)\left(1 + y^v z^k + y^v w^j + z^k w^j\right) + a\left(1 - p^{1/u^t}\right) \\ & + \left(x^u - p^{1/u^t}\right)\left(y^v + z^k + w^j + y^v z^k w^j\right). \end{aligned} \tag{3.6}$$

The results of Lemmas 3.1, 3.2, and 3.3 can be easily obtained from (1.1), and so we omit their proofs here.

Lemma 3.4. *Let $\{x_n\}_{n=-3}^{\infty}$ be a positive solution of* (1.1) *which is not eventually equal to 1, then the following conclusions are valid:*

(a) $(x_{n+1} - 1)(x_n - 1)(x_{n-1} - 1)(x_{n-2} - 1)(x_{n-3} - 1) > 0$, *for* $n \geq 0$;

(b) $(x_{n+1} - x_n)(x_n - 1) < 0$, *for* $n \geq 0$;

(c) $(x_{n+1} - x_{n-1})(x_{n-1} - 1) < 0$, *for* $n \geq 0$;

(d) $(x_{n+1} - x_{n-2})(x_{n-2} - 1) < 0$, *for* $n \geq 0$;

(e) $(x_{n+1} - x_{n-3})(x_{n-3} - 1) < 0$, *for* $n \geq 0$.

Proof. First, let us investigate (a). According to (1.1), it follows that

$$x_{n+1} - 1 = \frac{(x_n^u - 1)(x_{n-1}^v - 1)(x_{n-2}^k - 1)(x_{n-3}^j - 1)}{G(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \quad n = 0, 1, \ldots. \tag{3.7}$$

So,

$$(x_{n+1} - 1)(x_n^u - 1)(x_{n-1}^v - 1)(x_{n-2}^k - 1)(x_{n-3}^j - 1) > 0. \tag{3.8}$$

Noting that $u \in (0, 1]$, $v, k, j \in (0, +\infty)$, one has $(x_n^u - 1)(x_n - 1) > 0$, $(x_{n-1}^v - 1)(x_{n-1} - 1) > 0$, $(x_{n-2}^k - 1)(x_{n-2} - 1) > 0$, and $(x_{n-3}^j - 1)(x_{n-3} - 1) > 0$. From those, one can easily obtain the result of (a).

Second, (b) comes. From (3.1), we obtain

$$x_{n+1} - x_n = \frac{K(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_n)}{G(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \tag{3.9}$$

where

$$\begin{aligned} K(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_n) = {} & \left(1 - x_n^{u+1}\right)\left(1 + x_{n-1}^v x_{n-2}^k + x_{n-1}^v x_{n-3}^j + x_{n-2}^k x_{n-3}^j\right) + a(1 - x_n) \\ & + \left(x_n^u - x_n\right)\left(x_{n-1}^v + x_{n-2}^k + x_{n-3}^j + x_{n-1}^v x_{n-2}^k x_{n-3}^j\right) \\ = {} & \left(1 - x_n^{u+1}\right)\left(1 + x_{n-1}^v x_{n-2}^k + x_{n-1}^v x_{n-3}^j + x_{n-2}^k x_{n-3}^j\right) + a(1 - x_n) \\ & + \left(1 - x_n^{1-u}\right) x_n^u \left(x_{n-1}^v + x_{n-2}^k + x_{n-3}^j + x_{n-1}^v x_{n-2}^k x_{n-3}^j\right). \end{aligned} \tag{3.10}$$

From $u \in (0,1]$ and $\{x_n\}_{n=-3}^{\infty}$, being eventually not equal to 1, one can see that

$$\left(1-x_n^{u+1}\right)\left(1-x_n\right)>0, \qquad \left(1-x_n^{1-u}\right)\left(1-x_n\right)\geq 0, \qquad G\left(x_n,x_{n-1},x_{n-2},x_{n-3}\right)>0. \tag{3.11}$$

This tells us that $(x_{n+1}-x_n)(1-x_n)>0$, $n=0,1,\ldots$. That is, $(x_{n+1}-x_n)(x_n-1)<0$, $n=0,1,\ldots$. So, the proof of (b) is complete.

Third, let us prove (c). From (3.1) one has

$$x_{n+1}-x_{n-1}=\frac{K\left(x_n,x_{n-1},x_{n-2},x_{n-3},x_{n-1}\right)}{G\left(x_n,x_{n-1},x_{n-2},x_{n-3}\right)}, \tag{3.12}$$

where

$$\begin{aligned} K\left(x_n,x_{n-1},x_{n-2},x_{n-3},x_{n-1}\right) &= \left(1-x_n^u x_{n-1}\right)\left(1+x_{n-1}^v x_{n-2}^k+x_{n-1}^v x_{n-3}^j+x_{n-2}^k x_{n-3}^j\right)+a\left(1-x_{n-1}\right) \\ &\quad + \left(x_n^u-x_{n-1}\right)\left(x_{n-1}^v+x_{n-2}^k+x_{n-3}^j+x_{n-1}^v x_{n-2}^k x_{n-3}^j\right). \end{aligned} \tag{3.13}$$

From (3.3), one gets

$$1-x_n x_{n-1}^{1/u}=\frac{M\left(x_{n-1},x_{n-2},x_{n-3},x_{n-4},x_{n-1}\right)}{G\left(x_{n-1},x_{n-2},x_{n-3},x_{n-4}\right)}, \tag{3.14}$$

where

$$\begin{aligned} &M\left(x_{n-1},x_{n-2},x_{n-3},x_{n-4},x_{n-1}\right) \\ &\quad = a\left(1-x_{n-1}^{1/u}\right)+\left(x_{n-1}^u-x_{n-1}^{1/u}\right)\left(1+x_{n-2}^v x_{n-3}^k+x_{n-2}^v x_{n-4}^j+x_{n-3}^k x_{n-4}^j\right) \\ &\qquad + \left(1-x_{n-1}^{u+1/u}\right)\left(x_{n-2}^v+x_{n-3}^k+x_{n-4}^j+x_{n-2}^v x_{n-3}^k x_{n-4}^j\right) \\ &\quad = \left(1-x_{n-1}^{(1-u^2)/u}\right)x_{n-1}^u\left(1+x_{n-2}^v x_{n-3}^k+x_{n-2}^v x_{n-4}^j+x_{n-3}^k x_{n-4}^j\right)+a\left(1-x_{n-1}^{1/u}\right) \\ &\qquad + \left(1-x_{n-1}^{u+1/u}\right)\left(x_{n-2}^v+x_{n-3}^k+x_{n-4}^j+x_{n-2}^v x_{n-3}^k x_{n-4}^j\right). \end{aligned} \tag{3.15}$$

According to $u \in (0,1]$ and $\{x_n\}_{n=-3}^{\infty}$, being eventually not equal to 1, one arrives at

$$\begin{gathered} \left(1-x_{n-1}^{1/u}\right)\left(1-x_{n-1}\right)>0, \qquad \left(1-x_{n-1}^{u+1/u}\right)\left(1-x_{n-1}\right)>0, \\ \left(1-x_{n-1}^{(1-u^2)/u}\right)\left(1-x_{n-1}\right)\geq 0. \end{gathered} \tag{3.16}$$

From (3.14), (3.15), and (3.16), we know that $(1-x_n x_{n-1}^{1/u})(1-x_{n-1})>0$. So, we can get immediately

$$\left(1-x_n^u x_{n-1}\right)\left(1-x_{n-1}\right)>0. \tag{3.17}$$

From (3.5), one can have

$$x_n-x_{n-1}^{1/u}=\frac{N\left(x_{n-1},x_{n-2},x_{n-3},x_{n-4},x_{n-1}\right)}{G\left(x_{n-1},x_{n-2},x_{n-3},x_{n-4}\right)}, \tag{3.18}$$

where

$$\begin{aligned} &N(x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-1}) \\ &\quad = \left(1 - x_{n-1}^{u+1/u}\right)\left(1 + x_{n-2}^{v}x_{n-3}^{k} + x_{n-2}^{v}x_{n-4}^{j} + x_{n-3}^{k}x_{n-4}^{j}\right) + a\left(1 - x_{n-1}^{1/u}\right) \\ &\qquad + \left(x_{n-1}^{u} - x_{n-1}^{1/u}\right)\left(x_{n-2}^{v} + x_{n-3}^{k} + x_{n-4}^{j} + x_{n-2}^{v}x_{n-3}^{k}x_{n-4}^{j}\right) \\ &\quad = \left(1 - x_{n-1}^{u+1/u}\right)\left(1 + x_{n-2}^{v}x_{n-3}^{k} + x_{n-2}^{v}x_{n-4}^{j} + x_{n-3}^{k}x_{n-4}^{j}\right) + a\left(1 - x_{n-1}^{1/u}\right) \\ &\qquad + \left(1 - x_{n-1}^{(1-u^2)/u}\right)x_{n-1}^{u}\left(x_{n-2}^{v} + x_{n-3}^{k} + x_{n-4}^{j} + x_{n-2}^{v}x_{n-3}^{k}x_{n-4}^{j}\right). \end{aligned} \tag{3.19}$$

From (3.16), (3.18), and (3.19), we can obtain $(x_n - x_{n-1}^{1/u})(1 - x_{n-1}) > 0$, that is,

$$\left(x_n^u - x_{n-1}\right)\left(1 - x_{n-1}\right) > 0. \tag{3.20}$$

By virtue of (3.12), (3.13), (3.17), and (3.20), we see that (c) is true.

The proofs of (d) and (e) are similar to those of (c). The proof for this lemma is complete. □

Lemma 3.5. *Let $\{x_n\}_{n=-3}^{\infty}$ be a positive solution of* (1.1) *which is not eventually equal to 1, then* $(x_{n+1} - x_{n-4})(x_{n-4} - 1) < 0$, *for* $n \geq 1$.

Proof. From (3.1), one has

$$x_{n+1} - x_{n-4} = \frac{K(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4})}{G(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \quad n = 0, 1, \ldots, \tag{3.21}$$

where

$$\begin{aligned} K(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}) &= \left(1 - x_n^u x_{n-4}\right)\left(1 + x_{n-1}^{v}x_{n-2}^{k} + x_{n-1}^{v}x_{n-3}^{j} + x_{n-2}^{k}x_{n-3}^{j}\right) + a\left(1 - x_{n-4}\right) \\ &\quad + \left(x_n^u - x_{n-4}\right)\left(x_{n-1}^{v} + x_{n-2}^{k} + x_{n-3}^{j} + x_{n-1}^{v}x_{n-2}^{k}x_{n-3}^{j}\right). \end{aligned} \tag{3.22}$$

From (3.3), one has

$$1 - x_{n-3}x_{n-4}^{1/u^4} = \frac{M(x_{n-4}, x_{n-5}, x_{n-6}, x_{n-7}, x_{n-4})}{G(x_{n-4}, x_{n-5}, x_{n-6}, x_{n-7})}, \tag{3.23}$$

where

$$\begin{aligned} M(x_{n-4}, x_{n-5}, x_{n-6}, x_{n-7}, x_{n-4}) &= \left(x_{n-4}^{u} - x_{n-4}^{1/u^4}\right)\left(1 + x_{n-5}^{v}x_{n-6}^{k} + x_{n-5}^{v}x_{n-7}^{j} + x_{n-6}^{k}x_{n-7}^{j}\right) + a\left(1 - x_{n-4}^{1/u^4}\right) \\ &\quad + \left(1 - x_{n-4}^{u+1/u^4}\right)\left(x_{n-5}^{v} + x_{n-6}^{k} + x_{n-7}^{j} + x_{n-5}^{v}x_{n-6}^{k}x_{n-7}^{j}\right). \end{aligned} \tag{3.24}$$

Noticing that $u \in (0, 1]$, we have

$$\begin{gathered} \left(x_{n-4}^{u} - x_{n-4}^{1/u^4}\right)\left(1 - x_{n-4}\right) = x_{n-4}^{u}\left(1 - x_{n-4}^{(1-u^5)/u^4}\right)\left(1 - x_{n-4}\right) \geq 0, \\ \left(1 - x_{n-4}^{1/u^4}\right)\left(1 - x_{n-4}\right) > 0, \quad \left(1 - x_{n-4}^{u+1/u^4}\right)\left(1 - x_{n-4}\right) > 0. \end{gathered} \tag{3.25}$$

From (3.23), (3.24), and (3.25), we know that $(1 - x_{n-3}x_{n-4}^{1/u^4})(1 - x_{n-4}) > 0$. So,

$$(1 - x_{n-3}^{u}x_{n-4}^{1/u^3})(1 - x_{n-4}) > 0. \tag{3.26}$$

From (3.5), one can recognize that

$$x_{n-3} - x_{n-4}^{1/u^4} = \frac{N(x_{n-4}, x_{n-5}, x_{n-6}, x_{n-7}, x_{n-4})}{G(x_{n-4}, x_{n-5}, x_{n-6}, x_{n-7})}, \tag{3.27}$$

where

$$\begin{aligned} N(x_{n-4}, x_{n-5}, x_{n-6}, x_{n-7}, x_{n-4}) = {} & (1 - x_{n-4}^{u+1/u^4})(1 + x_{n-5}^{v}x_{n-6}^{k} + x_{n-5}^{v}x_{n-7}^{j} + x_{n-6}^{k}x_{n-7}^{j}) + a(1 - x_{n-4}^{1/u^4}) \\ & + (x_{n-4}^{u} - x_{n-4}^{1/u^4})(x_{n-5}^{v} + x_{n-6}^{k} + x_{n-7}^{j} + x_{n-5}^{v}x_{n-6}^{k}x_{n-7}^{j}). \end{aligned} \tag{3.28}$$

From (3.25), (3.27), and (3.28), we derive $(x_{n-3} - x_{n-4}^{1/u^4})(1 - x_{n-4}) > 0$. So,

$$(x_{n-3}^{u} - x_{n-4}^{1/u^3})(1 - x_{n-4}) > 0. \tag{3.29}$$

Equation (3.5) shows that

$$x_{n-2} - x_{n-4}^{1/u^3} = \frac{N(x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}, x_{n-4})}{G(x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6})}, \tag{3.30}$$

where

$$\begin{aligned} N(x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}, x_{n-4}) = {} & (1 - x_{n-3}^{u}x_{n-4}^{1/u^3})(1 + x_{n-4}^{v}x_{n-5}^{k} + x_{n-4}^{v}x_{n-6}^{j} + x_{n-5}^{k}x_{n-6}^{j}) + a(1 - x_{n-4}^{1/u^3}) \\ & + (x_{n-3}^{u} - x_{n-4}^{1/u^3})(x_{n-4}^{v} + x_{n-5}^{k} + x_{n-6}^{j} + x_{n-4}^{v}x_{n-5}^{k}x_{n-6}^{j}). \end{aligned} \tag{3.31}$$

By using (3.26), (3.29), (3.30), and (3.31), and noting that $(1 - x_{n-4}^{1/u^3})(1 - x_{n-4}) > 0$ when $u \in (0, 1]$, we get $(x_{n-2} - x_{n-4}^{1/u^3})(1 - x_{n-4}) > 0$. Hence,

$$(x_{n-2}^{u} - x_{n-4}^{1/u^2})(1 - x_{n-4}) > 0. \tag{3.32}$$

It follows from (3.3) that

$$1 - x_{n-2}x_{n-4}^{1/u^3} = \frac{M(x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}, x_{n-4})}{G(x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6})}, \tag{3.33}$$

where

$$\begin{aligned} M(x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}, x_{n-4}) = {} & (x_{n-3}^{u} - x_{n-4}^{1/u^3})(1 + x_{n-4}^{v}x_{n-5}^{k} + x_{n-4}^{v}x_{n-6}^{j} + x_{n-5}^{k}x_{n-6}^{j}) + a(1 - x_{n-4}^{1/u^3}) \\ & + (1 - x_{n-3}^{u}x_{n-4}^{1/u^3})(x_{n-4}^{v} + x_{n-5}^{k} + x_{n-6}^{j} + x_{n-4}^{v}x_{n-5}^{k}x_{n-6}^{j}). \end{aligned} \tag{3.34}$$

By virtue of (3.26), (3.29), (3.33), and (3.34), as well as $(1-x_{n-4}^{1/u^3})(1-x_{n-4})>0$ for $u\in(0,1]$, one has $(1-x_{n-2}x_{n-4}^{1/u^3})(1-x_{n-4})>0$. Accordingly,

$$\left(1-x_{n-2}^{u}x_{n-4}^{1/u^2}\right)\left(1-x_{n-5}\right)>0. \tag{3.35}$$

Equation (3.3) instructs us that

$$1-x_{n-1}x_{n-4}^{1/u^2}=\frac{M\left(x_{n-2},x_{n-3},x_{n-4},x_{n-5},x_{n-4}\right)}{G\left(x_{n-2},x_{n-3},x_{n-4},x_{n-5}\right)}, \tag{3.36}$$

where

$$\begin{aligned}M\left(x_{n-2},x_{n-3},x_{n-4},x_{n-5},x_{n-4}\right)=&\left(x_{n-2}^{u}-x_{n-4}^{1/u^2}\right)\left(1+x_{n-3}^{v}x_{n-4}^{k}+x_{n-3}^{v}x_{n-5}^{j}+x_{n-4}^{k}x_{n-5}^{j}\right)+a\left(1-x_{n-4}^{1/u^2}\right)\\&+\left(1-x_{n-2}^{u}x_{n-4}^{1/u^2}\right)\left(x_{n-3}^{v}+x_{n-4}^{k}+x_{n-5}^{j}+x_{n-3}^{v}x_{n-4}^{k}x_{n-5}^{j}\right).\end{aligned} \tag{3.37}$$

By virtue of (3.32), (3.35), (3.36), and (3.37), together with $(1-x_{n-4}^{1/u^2})(1-x_{n-4})>0$ when $u\in(0,1]$, one sees that $(1-x_{n-1}x_{n-4}^{1/u^2})(1-x_{n-4})>0$. So,

$$\left(1-x_{n-1}^{u}x_{n-4}^{1/u}\right)\left(1-x_{n-4}\right)>0. \tag{3.38}$$

From (3.5), one obtains

$$x_{n-1}-x_{n-4}^{1/u^2}=\frac{N\left(x_{n-2},x_{n-3},x_{n-4},x_{n-5},x_{n-4}\right)}{G\left(x_{n-2},x_{n-3},x_{n-4},x_{n-5}\right)}, \tag{3.39}$$

where

$$\begin{aligned}N\left(x_{n-2},x_{n-3},x_{n-4},x_{n-5},x_{n-4}\right)=&\left(1-x_{n-2}^{u}x_{n-4}^{1/u^2}\right)\left(1+x_{n-3}^{v}x_{n-4}^{k}+x_{n-3}^{v}x_{n-5}^{j}+x_{n-4}^{k}x_{n-5}^{j}\right)+a\left(1-x_{n-4}^{1/u^2}\right)\\&+\left(x_{n-2}^{u}-x_{n-4}^{1/u^2}\right)\left(x_{n-3}^{v}+x_{n-4}^{k}+x_{n-5}^{j}+x_{n-3}^{v}x_{n-4}^{k}x_{n-5}^{j}\right).\end{aligned} \tag{3.40}$$

By virtue of (3.32), (3.35), (3.39), and (3.40), in addition to $(1-x_{n-4}^{1/u^2})(1-x_{n-4})>0$ when $u\in(0,1]$, one can see that $(x_{n-1}-x_{n-4}^{1/u^2})(1-x_{n-4})>0$. Thus,

$$\left(x_{n-1}^{u}-x_{n-4}^{1/u}\right)\left(1-x_{n-4}\right)>0. \tag{3.41}$$

From (3.3), we can see that

$$1-x_{n}x_{n-4}^{1/u}=\frac{M\left(x_{n-1},x_{n-2},x_{n-3},x_{n-4},x_{n-4}\right)}{G\left(x_{n-1},x_{n-2},x_{n-3},x_{n-4}\right)}, \tag{3.42}$$

where

$$\begin{aligned}M\left(x_{n-1},x_{n-2},x_{n-3},x_{n-4},x_{n-4}\right)=&\left(x_{n-1}^{u}-x_{n-4}^{1/u}\right)\left(1+x_{n-2}^{v}x_{n-3}^{k}+x_{n-2}^{v}x_{n-4}^{j}+x_{n-3}^{k}x_{n-4}^{j}\right)+a\left(1-x_{n-4}^{1/u}\right)\\&+\left(1-x_{n-1}^{u}x_{n-4}^{1/u}\right)\left(x_{n-2}^{v}+x_{n-3}^{k}+x_{n-4}^{j}+x_{n-2}^{v}x_{n-3}^{k}x_{n-4}^{j}\right).\end{aligned} \tag{3.43}$$

Utilizing (3.38), (3.41), (3.42), and (3.43), and adding $(1 - x_{n-4}^{1/u})(1 - x_{n-4}) > 0$ when $u \in (0,1]$, we know that the following is true: $(1 - x_n x_{n-4}^{1/u})(1 - x_{n-4}) > 0$. So,

$$\left(1 - x_n^u x_{n-4}\right)\left(1 - x_{n-4}\right) > 0. \tag{3.44}$$

Similar to (3.44), by virtue of (3.5), (3.38), (3.41), and $(1 - x_{n-4}^{1/u})(1 - x_{n-4}) > 0$ when $u \in (0,1]$, we know that $(x_n - x_{n-4}^{1/u})(1 - x_{n-4}) > 0$ is true. So,

$$\left(x_n^u - x_{n-4}\right)\left(1 - x_{n-4}\right) > 0. \tag{3.45}$$

From (3.21), (3.22), (3.44), and (3.45), one knows that the following is true:

$$\left(x_n - x_{n-4}\right)\left(1 - x_{n-4}\right) > 0. \tag{3.46}$$

This shows that Lemma 3.5 is true. □

4. Oscillation and nonoscillation

Theorem 4.1. *There exist nonoscillatory solutions of* (1.1) *with* x_{-3}, x_{-2}, x_{-1}, $x_0 \in (1,+\infty)$, *which must be eventually positive. There are not eventually negative nonoscillatory solutions of* (1.1).

Proof. Consider a solution of (1.1) with $x_{-3}, x_{-2}, x_{-1}, x_0 \in (1,+\infty)$. We then know from Lemma 3.4(a) that $x_n > 1$ for $n \in N_{-3}$. So, this solution is just a nonoscillatory solution and it is, furthermore, eventually positive.

Suppose that there exist eventually negative nonoscillatory solutions of (1.1). Then, there exists a positive integer N such that $x_n < 1$ for $n \geq N$. Thereout, for $n \geq N+3$, $(x_{n+1} - 1)(x_n - 1)(x_{n-2} - 1)(x_{n-3} - 1) \leqslant 0$. This contradicts Lemma 3.4(a). So, there are not eventually negative nonoscillatory solutions of (1.1), as desired. □

5. Rule of cycle length

Theorem 5.1. *Let* $\{x_n\}_{-3}^{\infty}$ *be a strictly oscillatory solution of* (1.1), *then the rule for the lengths of positive and negative semicycles of this solution to occur successively is* $\ldots, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, \ldots$, *or* $\ldots, 2^+, 1^-, 1^+, 1^-, 2^+, 1^-, 1^+, 1^-, \ldots$, *or* $\ldots, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, \ldots$.

Proof. By Lemma 3.4(a), one can see that the length of a negative semicycle is at most 4, and that of a positive semicycle is at most 3. On the basis of the strictly oscillatory character of the solution, we see, for some integer $p \geq 0$, that one of the following sixteen cases must occur:

(1) $x_p > 1$, $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} > 1$;

(2) $x_p > 1$, $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} < 1$;

(3) $x_p > 1$, $x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} > 1$;

(4) $x_p > 1$, $x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} < 1$;

(5) $x_p > 1$, $x_{p+1} < 1$, $x_{p+2} > 1$, $x_{p+3} > 1$;

(6) $x_p > 1$, $x_{p+1} < 1$, $x_{p+2} > 1$, $x_{p+3} < 1$;

(7) $x_p > 1$, $x_{p+1} < 1$, $x_{p+2} < 1$, $x_{p+3} > 1$;

(8) $x_p > 1$, $x_{p+1} < 1$, $x_{p+2} < 1$, $x_{p+3} < 1$;

(9) $x_p < 1$, $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} > 1$;

(10) $x_p < 1$, $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} < 1$;

(11) $x_p < 1$, $x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} > 1$;

(12) $x_p < 1$, $x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} < 1$;

(13) $x_p < 1$, $x_{p+1} < 1$, $x_{p+2} > 1$, $x_{p+3} > 1$;

(14) $x_p < 1$, $x_{p+1} < 1$, $x_{p+2} > 1$, $x_{p+3} < 1$;

(15) $x_p < 1$, $x_{p+1} < 1$, $x_{p+2} < 1$, $x_{p+3} > 1$;

(16) $x_p < 1$, $x_{p+1} < 1$, $x_{p+2} < 1$, $x_{p+3} < 1$.

If case (1) occurs, of course, it will be a nonoscillatory solution of (1.1).

If case (2) occurs, it follows from Lemma 3.4(a) that $x_{p+4} < 1$, $x_{p+5} > 1$, $x_{p+6} > 1$, $x_{p+7} > 1$, $x_{p+8} < 1$, $x_{p+9} < 1$, $x_{p+10} > 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} > 1$, $x_{p+17} > 1$, $x_{p+18} < 1$, $x_{p+19} < 1, \ldots$.

This means that the rule for the lengths of positive and negative semicycles of the solution of (1.1) to occur successively is

$$\ldots, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^- \ldots. \tag{5.1}$$

If case (3) occurs, it follows from Lemma 3.4(a) that $x_{p+4} < 1$, $x_{p+5} > 1$, $x_{p+6} > 1$, $x_{p+7} < 1$, $x_{p+8} > 1$, $x_{p+9} < 14$, $x_{p+10} > 1$, $x_{p+11} > 1$, $x_{p+12} < 1$, $x_{p+13} > 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} > 1$, $x_{p+17} < 1$, $x_{p+18} > 1$, $x_{p+19} < 1, \ldots$, which means that the rule for the lengths of positive and negative semicycles of the solution of (1.1) to occur successively is

$$\ldots, 2^+, 1^-, 1^+, 1^-, 2^+, 1^-, 1^+, 1^-, 2^+, 1^-, 1^+, 1^-, 2^+, 1^-, 1^+, 1^-, \ldots. \tag{5.2}$$

If case 8 is reached, Lemma 3.4(a) tells us that $x_{p+4} < 1$, $x_{p+5} > 1$, $x_{p+6} < 1$, $x_{p+7} < 1$, $x_{p+8} < 1$, $x_{p+9} < 1$, $x_{p+10} > 1$, $x_{p+11} < 1$, $x_{p+12} < 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} < 1$, $x_{p+17} < 1$, $x_{p+18} < 1$, $x_{p+19} < 1, \ldots$.

This implies that the rule for the lengths of positive and negative semicycles of the solution of (1.1) to occur successively is

$$\ldots, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, \ldots. \tag{5.3}$$

Moreover, the rule for the cases (2.3), (3.5), (3.13), (3.15), and (3.17) is the same as that of case (2.1). And cases (3.1), (3.3), and (3.15) are completely similar to case (3) except possibly for the first semicycle. And cases (3.16), (3.18), (3.19), and (3.20) are like case (8) with a possible exception for the first semicycle.

Up to now, the proof of Theorem 5.1 is complete. □

6. Global asymptotic stability

First, we consider the local asymptotic stability for unique positive equilibrium point $\overline{x}$ of (1.1). We have the following result.

Theorem 6.1. *The positive equilibrium point of* (1.1) *is locally asymptotically stable.*

Proof. The linearized equation of (1.1) about thepositive equilibrium point $\overline{x}$ is

$$y_{n+1} = 0\cdot y_n + 0\cdot y_{n-1} + 0\cdot y_{n-2} + 0\cdot y_{n-3}, \quad n = 0, 1, \ldots, \tag{6.1}$$

and so it is clear from [3, Remark 1.3.7] that the positive equilibrium point $\overline{x}$ of (1.1) is locally asymptotically stable. The proof is complete. □

We are now in a position to study the global asymptotic stability of positive equilibrium point $\overline{x}$.

Theorem 6.2. *The positive equilibrium point of* (1.1) *is globally asymptotically stable.*

Proof. We must prove that the positive equilibrium point $\overline{x}$ of (1.1) is both locally asymptotically stable and globally attractive. Theorem 6.1 has shown the local asymptotic stability of $\overline{x}$. Hence, it remains to verify that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) converges to $\overline{x}$ as $n\to\infty$. Namely, we want to prove that

$$\lim_{n\to\infty} x_n = \overline{x} = 1. \tag{6.2}$$

We can divide the solutions into two types:

(i) trivial solutions;

(ii) nontrivial solutions.

If a solution is a trivial one, then it is obvious for (6.2) to hold because $x_n = 1$ holds eventually.

If the solution is a nontrivial one, then we can further divide the solution into two cases:

(a) nonoscillatory solution;

(b) oscillatory solution.

Consider now $\{x_n\}$ to be nonoscillatory about the positive equilibrium point $\overline{x}$ of (1.1). By virtue of Lemma 3.4(b), it follows that the solution is monotonic and bounded. So, $\lim_{n\to\infty} x_n$ exists and is finite. Taking limits on both sides of (1.1), one can easily see that (6.2) holds.

Now, let $\{x_n\}$ be strictly oscillatory about the positive equilibrium point of (1.1). By virtue of Theorem 5.1, one understands that the rule for the lengths of positive and negative semicycles occurring successively is

(i) $\ldots, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, 3^+, 2^-, \ldots,$

(ii) $\ldots, 2^+, 1^-, 1^+, 1^-, 2^+, 1^-, 1^+, 1^-, \ldots,$ or

(iii) $\ldots, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, 1^+, 4^-, \ldots.$

Now, we consider the case (i). For simplicity, for some nonnegative integer p, we denote by $\{x_p, x_{p+1}, x_{p+2}\}^+$ the terms of a positive semicycle of the length three, and by $\{x_{p+3}, x_{p+4}\}^-$ a negative semicycle with semicycle length of two, then a positive semicycle and a negative semicycle, and so on. Namely, the rule for the lengths of positive and negative semicycles to occur successively can be periodically expressed as follows:

$$\{x_{p+5n}, x_{p+5n+1}, x_{p+5n+2}\}^{+}, \quad \{x_{p+5n+3}, x_{p+5n+4}\}^{-}, \quad n = 0, 1, 2, \ldots. \tag{6.3}$$

Lemma 3.4(b), (c), (d), (e) and Lemma 3.5 teach us that the following results are true:

(a) $x_{p+5n} > x_{p+5n+1} > x_{p+5n+2} > x_{p+5n+5}$, $n = 0, 1, 2, \ldots$;

(b) $x_{p+5n+3} < x_{p+5n+4} < x_{p+5n+8}$, $n = 0, 1, 2, \ldots$.

So, by virtue of (a), one can see that $\{x_{p+5n}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So, its limit exists and is finite, denoted by S. Moreover, the limits of $\{x_{p+5n+1}\}_{n=0}^{\infty}$ and $\{x_{p+5n+2}\}_{n=0}^{\infty}$ are all equal to that of $\{x_{p+5n}\}_{n=0}^{\infty}$.

Similarly, using (b), one can see that $\{x_{p+5n+3}\}_{n=0}^{\infty}$ is increasing with upper bound 1. So, its limit exists and is finite too. Furthermore, the limits of $\{x_{p+5n+4}\}_{n=0}^{\infty}$ are equal to that of $\{x_{p+5n+3}\}_{n=0}^{\infty}$, and one can assume the limit of it to be T. It is easy to see that $S \geq 1 \geq T$. It suffices to show that $S = 1 = T$.

Noting that

$$x_{p+5n+5} = \frac{F(x_{p+5n+4}, x_{p+5n+3}, x_{p+5n+2}, x_{p+5n+1})}{G(x_{p+5n+4}, x_{p+5n+3}, x_{p+5n+2}, x_{p+5n+1})}, \tag{6.4}$$

$$x_{p+5n+4} = \frac{F(x_{p+5n+3}, x_{p+5n+2}, x_{p+5n+1}, x_{p+5n})}{G(x_{p+5n+3}, x_{p+5n+2}, x_{p+5n+1}, x_{p+5n})}, \tag{6.5}$$

and taking limits on both sides of (6.4) and (6.5), respectively, we get

$$\begin{aligned} S &= \frac{T^{u+v} + T^u S^k + T^u S^j + T^v S^k + T^v S^j + S^{k+j} + T^{u+v} S^{k+j} + 1 + a}{T^u + T^v + S^k + S^j + T^{u+v} S^k + T^{u+v} S^j + T^u S^{k+j} + T^v S^{k+j} + a}, \\ T &= \frac{T^u S^v + T^u S^k + T^u S^j + S^{v+k} + S^{v+j} + S^{k+j} + T^u S^{v+k+j} + 1 + a}{T^u + S^v + S^k + S^j + T^u S^{v+k} + T^u S^{v+j} + T^u S^{k+j} + S^{v+k+j} + a}. \end{aligned} \tag{6.6}$$

From (6.6), we can show that $S = T = 1$. Otherwise, assume that

$$S > 1. \tag{6.7}$$

From (3.1), with both n and i being replaced by $5n$, we get

$$x_{5n+1} - x_{5n} = \frac{K(x_{5n}, x_{5n-1}, x_{5n-2}, x_{5n-3}, x_{5n})}{G(x_{5n}, x_{5n-1}, x_{5n-2}, x_{5n-3})}. \tag{6.8}$$

Taking limits on both sides of the above equation, we can obtain

$$a(1 - S) + (1 - S^{u+1})(1 + T^{v+k} + T^v S^j + T^k S^j) + (S^u - S)(T^v + T^k + S^j) = 0. \tag{6.9}$$

By virtue of (6.7), one has $1 - S < 0, 1 - S^{u+1} < 0$, and $S^u - S \leq 0$ when $u \in (0, 1]$; and so, one will get

$$a(1 - S) + (1 - S^{u+1})(1 + T^{v+k} + T^v S^j + T^k S^j) + (S^u - S)(T^v + T^k + S^j) < 0. \tag{6.10}$$

This contradicts (6.9). Thus, $S = 1$. Similarly, we can get $T = 1$. Therefore, (6.2) holds when case (i) occurs.

When case (ii) or (iii) occurs, using the method similar to that proving case (i), we can prove that (6.2) is also true. Thus, the proof of Theorem is complete. □

Acknowledgments

This work was partly supported by NNSF of China (Grant no. 10771094), and the Foundations for "New Century Engineering of 121 Talents in Hunan Province" and "Chief Professor of Mathematical Discipline in Hunan Province."

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