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Research Article

Multiple Twisted q-Euler Numbers and Polynomials Associated with p-Adic q-Integrals

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By using p-adic q-integrals on $\mathbb{Z}_p$, we define multiple twisted q-Euler numbers and polynomials. We also find Witt's type formula for multiple twisted q-Euler numbers and discuss some characterizations of multiple twisted q-Euler Zeta functions. In particular, we construct multiple twisted Barnes' type q-Euler polynomials and multiple twisted Barnes' type q-Euler Zeta functions. Finally, we define multiple twisted Dirichlet's type q-Euler numbers and polynomials, and give Witt's type formula for them.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ are, respectively, the ring of p-adic rational integers, the field of p-adic rational numbers, and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks about q-extension, q is variously considered as an indeterminate, a complex number, $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|1-q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. We use the notations

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.1}$$

(cf. [1–14]), for all $x \in \mathbb{Z}_p$. For a fixed odd positive integer d with $(p,d)=1$, set

$$X = X_d = \varprojlim_{n} \mathbb{Z}/dp^n\mathbb{Z}, \qquad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a(\operatorname{mod} dp^n)\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^n$. For any $n \in \mathbb{N}$,

$$\mu_q(a + dp^n\mathbb{Z}_p) = \frac{q^a}{[dp^n]_q} \tag{1.3}$$

is known to be a distribution on X (cf. [1–28]).

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \tag{1.4}$$

have a limit $l = f'(a)$ as $(x, y) \to (a, a)$ (cf. [25]).

The p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{x=0}^{p^n-1} f(x)q^x, \tag{1.5}$$

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{x=0}^{p^n-1} f(x)(-q)^x, \tag{1.6}$$

(cf. [4, 24, 25, 28]), from (1.6), we derive

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{1.7}$$

where $f_1(x) = f(x+1)$. If we take $f(x) = e^{tx}$, then we have $f_1(x) = e^{t(x+1)} = e^{tx}e^t$. From (1.7), we obtain that

$$I_{-q}(e^{tx}) = \frac{[2]_q}{qe^t + 1}. \tag{1.8}$$

In Section 2, we define the multiple twisted q-Euler numbers and polynomials on $\mathbb{Z}_p$ and find Witt's type formula for multiple twisted q-Euler numbers. We also have sums of consecutive multiple twisted q-Euler numbers. In Section 3, we consider multiple twisted q-Euler Zeta functions which interpolate new multiple twisted q-Euler polynomials at negative integers and investigate some characterizations of them. In Section 4, we construct the multiple twisted Barnes' type q-Euler polynomials and multiple twisted Barnes' type q-Euler Zeta functions which interpolate new multiple twisted Barnes' type q-Euler polynomials at negative integers. In Section 5, we define multiple twisted Dirichlet's type q-Euler numbers and polynomials and give Witt's type formula for them.

2. Multiple twisted q-Euler numbers and polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. For $n \in \mathbb{N}$, by the definition of p-adic q-integral on $\mathbb{Z}_p$, we have

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{x=0}^{n-1} (-1)^{n-1-x} q^x f(x), \tag{2.1}$$

where $f_n(x) = f(x+n)$. If n is odd positive integer, we have

$$q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{x=0}^{n-1} (-1)^{n-1-x} q^x f(x). \tag{2.2}$$

Let $T_p = \cup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n} = C_{p^\infty}$ be the locally constant space, where $C_{p^n} = \{w \mid w^{p^n} = 1\}$ is the cyclic group of order p^n. For $w \in T_p$, we denote the locally constant function by

$$\phi_w : \mathbb{Z}_p \longrightarrow \mathbb{C}_p, \; x \longrightarrow w^x, \tag{2.3}$$

(cf. [5, 7–14, 16, 18]). If we take $f(x) = \phi_w(x) e^{tx}$, then we have

$$\int_{\mathbb{Z}_p} e^{tx} \phi_w(x) d\mu_{-q}(x) = \frac{[2]_q}{qwe^t + 1}. \tag{2.4}$$

Now we define the twisted q-Euler numbers $E_{n,w}^q$ as follows:

$$F_w(t) = \frac{[2]_q}{qwe^t + 1} = \sum_{n=o}^{\infty} E_{n,w}^q \frac{t^n}{n!}. \tag{2.5}$$

We note that by substituting $w = 1$, $\lim_{q\to 1} E_{n,1}^q = E_n$ are the familiar Euler numbers. Over five decades ago, Carlitz defined q-extension of Euler numbers (cf. [15]). From (2.4) and (2.5), we note that Witt's type formula for a twisted q-Euler number is given by

$$\int_{\mathbb{Z}_p} x^n w^x d\mu_{-q}(x) = E_{n,w}^q. \tag{2.6}$$

for each $w \in T_p$ and $n \in \mathbb{N}$.

Twisted q-Euler polynomials $E_{n,w}^q(x)$ are defined by means of the generating function

$$F_w^q(t,x) = \frac{[2]_q}{qwe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,w}^q(x) \frac{t^n}{n!}, \tag{2.7}$$

where $E_{n,w}^q(0) = E_{n,w}^q$. By using the hth iterative fermionic p-adic q-integral on $\mathbb{Z}_p$, we define multiple twisted q-Euler number as follows:

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} w^{x_1+\cdots+x_h} e^{(x_1+x_2+\cdots+x_h)t} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) = \left(\frac{[2]_q}{qwe^t + 1}\right)^h = \sum_{n=0}^{\infty} E_{n,w}^{(h,q)} \frac{t^n}{n!}. \tag{2.8}$$

Thus we give Witt's type formula for multiple twisted q-Euler numbers as follows.

Theorem 2.1. *For each $w \in T_p$ and $h, n \in \mathbb{N}$,*

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} w^{x_1+\cdots+x_h} \left(x_1 + \cdots + x_h\right)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) = E_{n,w}^{(h,q)}, \tag{2.9}$$

where

$$\left(x_1 + \cdots + x_h\right)^n = \sum_{\substack{l_1+\cdots+l_h=n \\ l_1,\ldots,l_h \geq 0}} \frac{n!}{l_1! \cdots l_h!} x_1^{l_1} \cdots x_h^{l_h}. \tag{2.10}$$

From (2.8) and (2.9), we obtain the following theorem.

Theorem 2.2. *For $w \in T_p$ and $h, k \in \mathbb{N}$,*

$$E_{k,w}^{(h,q)} = \sum_{\substack{l_1+\cdots+l_h=k \\ l_1,\ldots,l_h \geq 0}} \frac{k!}{l_1! \cdots l_h!} E_{l_1,w}^{q} \cdots E_{l_h,w}^{q}. \tag{2.11}$$

From these formulas, we consider multivariate fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} w^{x_1+\cdots+x_h} e^{(x_1+\cdots+x_h+x)t} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) &= \left(\frac{[2]_q}{qwe^t + 1}\right) \cdots \left(\frac{[2]_q}{qwe^t + 1}\right) e^{xt} \\ &= \left(\frac{[2]_q}{qwe^t + 1}\right)^h e^{xt}.\end{aligned} \tag{2.12}$$

Then we can define the multiple twisted q-Euler polynomials $E_{n,w}^{(h,q)}(x)$ as follows:

$$F_w^{(h,q)}(t, x) = \left(\frac{[2]_q}{qwe^t + 1}\right)^h e^{xt} = \sum_{n=0}^{\infty} E_{n,w}^{(h,q)}(x) \frac{t^n}{n!}. \tag{2.13}$$

From (2.12) and (2.13), we note that

$$\sum_{n=0}^{\infty} \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} w^{x_1+\cdots+x_h} \left(x_1 + \cdots + x_h + x\right)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,w}^{(h,q)}(x) \frac{t^n}{n!}. \tag{2.14}$$

Then by the kth differentiation on both sides of (2.14), we obtain the following.

Theorem 2.3. *For each $w \in T_p$ and $k, h \in \mathbb{N}$,*

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} w^{x_1+\cdots+x_h} \left(x_1 + \cdots + x_h + x\right)^k d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) = E_{k,w}^{(h,q)}(x). \tag{2.15}$$

Note that

$$(x_1+\cdots+x_h+x)^n=\sum_{\substack{l_1+\cdots+l_h=n\\ l_1,\dots,l_h\geq 0}}\frac{n!}{l_1!\cdots l_h!}x_1^{l_1}\cdot x_2^{l_2}\cdots(x_h+x)^{l_h}. \tag{2.16}$$

Then we see that

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{h\text{-times}} w^{x_1+\cdots+x_h}(x_1+\cdots+x_h+x)^k d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_h)\\
&\quad=\sum_{\substack{l_1+\cdots+l_h=k\\ l_1,\dots,l_h\geq 0}}\frac{k!}{l_1!\cdots l_h!}\int_{\mathbb{Z}_p}w^{x_1}x_1^{l_1}d\mu_{-q}(x_1)\cdots\int_{\mathbb{Z}_p}w^{x_{h-1}}x_{h-1}^{l_{h-1}}d\mu_{-q}(x_{h-1})\int_{\mathbb{Z}_p}(x+x_h)^{l_h}d\mu_{-q}(x_h)\\
&\quad=\sum_{\substack{l_1+\cdots+l_h=k\\ l_1,\dots,l_h\geq 0}}\frac{k!}{l_1!\cdots l_h!}E_{l_1,w}^q\cdots E_{l_{h-1},w}^q E_{l_h,w}^q(x).
\end{aligned} \tag{2.17}$$

From (2.15) and (2.17), we obtain the sums of powers of consecutive q-Euler numbers as follows.

Theorem 2.4. *For each $w\in T_p$ and $k,h\in\mathbb{N}$,*

$$E_{k,w}^{(h,q)}(x)=\sum_{\substack{l_1+\cdots+l_h=k\\ l_1,\dots,l_h\geq 0}}\frac{k!}{l_1!\cdots l_h!}E_{l_1,w}^q\cdots E_{l_{h-1},w}^q\cdot E_{l_h,w}^q(x). \tag{2.18}$$

3. Multiple twisted q-Euler Zeta functions

For $q\in\mathbb{C}$ with $|q|<1$ and $w\in T_p$, the multiple twisted q-Euler numbers can be considered as follows:

$$F_w^h(t)=\left(\frac{[2]_q}{qwe^t+1}\right)^h=\sum_{n=0}^{\infty}E_{n,w}^{(h,q)}\frac{t^n}{n!},\quad \left|t+\log(qw)\right|<\pi. \tag{3.1}$$

From (3.1), we notethat

$$\begin{aligned}
\sum_{n=0}^{\infty}E_{n,w}^{(h,q)}\frac{t^n}{n!}=F_w^h(t)&=\left(\frac{[2]_q}{qwe^t+1}\right)^h=[2]_q^h\left(\frac{[2]_q}{qwe^t+1}\right)\cdots\left(\frac{[2]_q}{qwe^t+1}\right)\\
&=[2]_q^h\sum_{n_1=0}^{\infty}(-1)^{n_1}q^{n_1}w^{n_1}e^{n_1t}\cdots\sum_{n_h=0}^{\infty}(-1)^{n_h}q^{n_h}w^{n_h}e^{n_ht}\\
&=[2]_q^h\sum_{n_1,\dots,n_h=0}(-1)^{n_1+\cdots+n_h}q^{n_1+\cdots+n_h}w^{n_1+\cdots+n_h}e^{(n_1+\cdots+n_h)t}.
\end{aligned} \tag{3.2}$$

By the kth differentiation on both sides of (3.2) at $t=0$, we obtain that

$$E_{k,w}^{(h,q)}=[2]_q^h\sum_{\substack{n_1+\cdots+n_h\neq 0\\ n_1,\ldots,n_h\geq 0}}(-1)^{n_1+\cdots+n_h}q^{n_1+\cdots+n_h}w^{n_1+\cdots+n_h}\left(n_1+\cdots+n_h\right)^k. \tag{3.3}$$

From (3.3), we derive multiple twisted q-Euler Zeta function as follows:

$$\zeta_w^{(h,q)}(s)=[2]_q^h\sum_{\substack{n_1+\cdots+n_h\neq 0\\ n_1,\ldots,n_h\geq 0}}\frac{(-1)^{n_1+\cdots+n_h}q^{n_1+\cdots+n_h}w^{n_1+\cdots+n_h}}{\left(n_1+\cdots+n_h\right)^s} \tag{3.4}$$

for all $s\in\mathbb{C}$. We also obtain the following theorem in which multiple twisted q-Euler Zeta functions interpolate multiple twisted q-Euler polynomials.

Theorem 3.1. *For $w\in T_p$ and $k,h\in\mathbb{N}$,*

$$\zeta_w^{(h,q)}(-k)=E_{k,w}^{(h,q)}. \tag{3.5}$$

4. Multiple twisted Barnes' type q-Euler polynomials

In this section, we consider the generating function of multiple twisted q-Euler polynomials:

$$F_w^h(t,x)=\left(\frac{[2]_q}{qwe^t+1}\right)^h e^{xt}=\sum_{n=0}^{\infty}E_{n,w}^{(h,q)}(x)\frac{t^n}{n!},$$
$$\left|t+\log(qw)\right|<\pi,\qquad \operatorname{Re}(x)>0. \tag{4.1}$$

We note that

$$\sum_{n=0}^{\infty}E_{n,w}^{(h,q)}(x)\frac{t^n}{n!}=F_w^h(t,x)=[2]_q^h\sum_{n_1,\ldots,n_h=0}(-1)^{n_1+\cdots+n_h}q^{n_1+\cdots+n_h}w^{n_1+\cdots+n_h}e^{(n_1+\cdots+n_h+x)t}. \tag{4.2}$$

By the kth differentiation on both sides of (4.2) at $t=0$, we obtain that

$$E_{k,w}^{(h,q)}(x)=[2]_q^h\sum_{n_1,\ldots,n_h=0}(-1)^{n_1+\cdots+n_h}q^{n_1+\cdots+n_h}w^{n_1+\cdots+n_h}(n_1+\cdots+n_h+x)^k. \tag{4.3}$$

Thus we can consider multiple twisted Hurwitz's type q-Euler Zeta function as follows:

$$\zeta_w^{(h,q)}(s,x)=[2]_q^h\sum_{\substack{n_1+\cdots+n_h\neq 0\\ n_1,\ldots,n_h\geq 0}}\frac{(-1)^{n_1+\cdots+n_h}q^{n_1+\cdots+n_h}w^{n_1+\cdots+n_h}}{\left(n_1+\cdots+n_h+x\right)^s} \tag{4.4}$$

for all $s\in\mathbb{C}$ and $\operatorname{Re}(x)>0$. We note that $\zeta_w^{(h,q)}(s,x)$ is analytic function in the whole complex s-plane and $\zeta_w^{(h,q)}(s,0)=\zeta_w^{(h,q)}(s)$. We also remark that if $w=1$ and $h=1$, then $\zeta_1^{(1,q)}(s,x)=\zeta^q(s,x)$ is Hurwitz's type q-Euler Zeta function (see [7, 27]). The following theorem means that multiple twisted q-Euler Zeta functions interpolate multiple twisted q-Euler polynomials at negative integers.

Theorem 4.1. *For $w \in T_p$, $k, h \in \mathbb{N}$, $s \in \mathbb{C}$, and* $\mathrm{Re}(x) > 0$,

$$\zeta_w^{(h,q)}(-k, x) = E_{k,w}^{(h,q)}(x). \tag{4.5}$$

Let us consider

$$\begin{aligned} F_w^h(a_1, \dots, a_h \mid t, x) &= \left(\frac{[2]_q}{qwe^{a_1 t} + 1}\right) \cdots \left(\frac{[2]_q}{qwe^{a_h t} + 1}\right) e^{xt} \\ &= [2]_q^h \sum_{n_1, \dots, n_h = 0}^{\infty} (-1)^{n_1 + \cdots + n_h} q^{n_1 + \cdots + n_h} w^{n_1 + \cdots + n_h} e^{(a_1 n_1 + \cdots + a_h n_h + x)t} \\ &= \sum_{n=0}^{\infty} E_{n,w}^{(h,q)}(a_1, \dots, a_h \mid x) \frac{t^n}{n!}, \end{aligned} \tag{4.6}$$

where $a_1, \dots, a_h \in \mathbb{C}$ and $\max_{1 \le i \le k}\{|\log(q + a_i t)|\} < \pi$. Then $E_{n,w}^{(h,q)}(a_1, \dots, a_h \mid x)$ will be called multiple twisted Barnes' type q-Euler polynomials. We note that

$$E_{n,w}^{(h,q)}(1, 1, \dots, 1 \mid x) = E_{n,w}^{(h,q)}(x). \tag{4.7}$$

By the kth differentiation of both sides of (4.6), we obtain the following theorem.

Theorem 4.2. *For each $w \in T_p$, $a_1, \dots, a_h \in \mathbb{C}$, $k, h \in \mathbb{N}$, and* $\mathrm{Re}(x) > 0$,

$$E_{k,w}^{(h,q)}(a_1, \dots, a_h \mid x) = [2]_q^h \sum_{\substack{n_1 + \cdots + n_h \neq 0 \\ n_1, \dots, n_h \ge 0}} (-1)^{n_1 + \cdots + n_h} q^{n_1 + \cdots + n_h} w^{n_1 + \cdots + n_h} (a_1 n_1 + \cdots + a_h n_h + x)^k, \tag{4.8}$$

where

$$(a_1 n_1 + \cdots + a_h n_h + x)^k = \sum_{\substack{l_1 + \cdots + l_h = k \\ l_1, \dots, l_h \ge 0}} \frac{k!}{l_1! \cdots l_h!} a_1^{l_1} \cdots a_{h-1}^{l_{h-1}} n_1^{l_1} \cdots n_{h-1}^{l_{h-1}} (a_h n_h + x)^{l_h}. \tag{4.9}$$

From (4.8), we consider multiple twisted Barnes' type q-Euler Zeta function defined as follows: for each $w \in T_p$, $a_1, \dots, a_h \in \mathbb{C}$, $k, h \in \mathbb{N}$, and $\mathrm{Re}(x) > 0$,

$$\zeta_{k,w}^{(h,q)}(a_1, \dots, a_h \mid s, x) = [2]_q^h \sum_{\substack{n_1 + \cdots + n_h \neq 0 \\ n_1, \dots, n_h \ge 0}} \frac{(-1)^{n_1 + \cdots + n_h} q^{n_1 + \cdots + n_h} w^{n_1 + \cdots + n_h}}{(a_1 n_1 + \cdots + a_h n_h + x)^s}. \tag{4.10}$$

We note that $\zeta_{k,w}^{(h,q)}(a_1, \dots, a_h \mid s, x)$ is analytic function in the whole complex s-plane. We also see that multiple twisted Barnes' type q-Euler Zeta functions interpolate multiple twisted Barnes' type q-Euler polynomials at negative integers as follows.

Theorem 4.3. *For each $w \in T_p$, $a_1, \dots, a_h \in \mathbb{C}$, $k, h \in \mathbb{N}$, and* $\mathrm{Re}\,(x) > 0$,

$$\zeta_{k,w}^{(h,q)}(a_1, \dots, a_h \mid -k, x) = E_{k,w}^{(h,q)}(a_1, \dots, a_h \mid x). \tag{4.11}$$

5. Multiple twisted Dirichlet's type q-Euler numbers and polynomials

Let χ be a Dirichlet's character with conductor $d(=\text{odd}) \in \mathbb{N}$ and $w \in T_p$. If we take $f(x) = \chi(x)\phi_w(x)e^{tx}$, then we have $f_d(x) = f(x+d) = \chi(x)w^d e^{td} w^x e^{tx}$. From (2.2), we derive

$$\int_X \chi(x) w^x e^{tx} d\mu_{-q}(x) = \frac{[2]_q \sum_{i=0}^{d-1} (-1)^{d-1-i} q^i \chi(i) w^i e^{ti}}{q^d w^d e^{td} + 1}. \tag{5.1}$$

In view of (5.1), we can define twisted Dirichlet's type q-Euler numbers as follows:

$$F_{w,\chi}^q(t) = \frac{[2]_q \sum_{i=0}^{d-1} (-1)^{d-1-i} q^i \chi(i) w^i e^{ti}}{q^d w^d e^{td} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,w}^q \frac{t^n}{n!}, \left|t + \log(qw)\right| < \frac{\pi}{d}, \tag{5.2}$$

(cf. [17, 19, 21, 22]). From (5.1) and (5.2), we can give Witt's type formula for twisted Dirichlet's type q-Euler numbers as follows.

Theorem 5.1. *Let χ be a Dirichlet's character with conductor $d(=\text{odd}) \in \mathbb{N}$. For each $w \in T_p$, $n \in \mathbb{N} \cup \{0\}$, we have*

$$\int_X \chi(x) w^x e^{tx} d\mu_{-q}(x) = E_{n,\chi,w}^q. \tag{5.3}$$

We note that if $w = 1$, then $E_{n,\chi,1}^q = E_{n,\chi}^q$ is the generalized q-Euler numbers attached to χ (see [18, 26]). From (5.2), we also see that

$$\begin{aligned} F_{w,\chi}^q(t) &= [2]_q \sum_{i=0}^{d-1} (-1)^{d-1-i} q^i \chi(i) w^i e^{ti} \sum_{l=0}^{\infty} q^{ld} w^{ld} e^{ldt} (-1)^l \\ &= [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^n \chi(n) e^{nt}. \end{aligned} \tag{5.4}$$

By (5.2) and (5.4), we obtain that

$$E_{k,\chi,w}^q = \frac{d^k}{dt^k} F_{w,\chi}^q(t) \mid_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^n \chi(n) n^k. \tag{5.5}$$

From (5.5), we can define the $l_{w,\chi}^q$-function as follows:

$$l_{\chi,w}^q(s) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n w^n \chi(n)}{n^s} \tag{5.6}$$

for all $s \in \mathbb{C}$. We note that $l_{\chi,w}^q(s)$ is analytic function in the whole complex s-plane. From (5.5) and (5.6), we can derive the following result.

Theorem 5.2. *Let χ be a Dirichlet's character with conductor $d(=\text{odd}) \in \mathbb{N}$. For each $w \in T_p$, $n \in \mathbb{N} \cup \{0\}$, we have*

$$l_{w,\chi}^q(-n) = E_{n,\chi,w}^q. \tag{5.7}$$

Now, in view of (5.1), we can define multiple twisted Dirichlet's type q-Euler numbers by means of the generating function as follows:

$$F_{w,\chi}^{(h,q)}(t) = \left(\frac{[2]_q \sum_{i=0}^{d-1} (-1)^{d-1-i} q^i \chi(i) w^i e^{ti}}{q^d w^d e^{td} + 1}\right)^h = \left(\int_X \chi(x) w^x e^{tx} d\mu_{-q}(x)\right)^h = \sum_{n=0}^{\infty} E_{n,\chi,w}^{(h,q)} \frac{t^n}{n!}, \tag{5.8}$$

where $|t + \log(qw)| < \pi/d$. We note that if $w = 1$, then $E_{n,\chi,1}^{q}$ is a multiple generalized q-Euler number (see [22]).

By using the same method used in (2.8) and (2.9),

$$\sum_{n=0}^{\infty} \underbrace{\int_X \cdots \int_X}_{h\text{-times}} \chi(x_1 + \cdots + x_h) w^{x_1+\cdots+x_h} (x_1 + \cdots + x_h)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,w}^{(h,q)} \frac{t^n}{n!}. \tag{5.9}$$

From (5.9), we can give Witt's type formula for multiple twisted Dirichlet's type q-Euler numbers.

Theorem 5.3. *Let χ be a Dirichlet's character with conductor $d(=\text{odd}) \in \mathbb{N}$. For each $w \in T_p$, $h \in \mathbb{N}$, and $n \in \mathbb{N} \cup \{0\}$, we have*

$$\underbrace{\int_X \cdots \int_X}_{h\text{-times}} \chi(x_1 + \cdots + x_h) w^{x_1+\cdots+x_h} (x_1 + \cdots + x_h)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_h) = E_{n,\chi,w}^{(h,q)}, \tag{5.10}$$

where $\chi(x_1 + \cdots + x_h) = \chi(x_1) \cdots \chi(x_h)$ and

$$(x_1 + \cdots + x_h)^n = \sum_{\substack{l_1+\cdots+l_h=n \\ l_1,\ldots,l_h \geq 0}} \frac{n!}{l_1! \cdots l_h!} x_1^{l_1} \cdots x_h^{l_h}. \tag{5.11}$$

From (5.10), we also obtain the sums of powers of consecutive multiple twisted Dirichlet's type q-Euler numbers as follows.

Theorem 5.4. *Let χ be a Dirichlet's character with conductor $d(=\text{odd}) \in \mathbb{N}$. For each $w \in T_p$, $h \in \mathbb{N}$, and $n \in \mathbb{N} \cup \{0\}$, we have*

$$E_{k,\chi,w}^{(h,q)} = \sum_{\substack{l_1+\cdots+l_h=k \\ l_1,\ldots,l_h \geq 0}} \frac{k!}{l_1! \cdots l_h!} E_{l_1,\chi,w}^{q} \cdots E_{l_h,\chi,w}^{q}. \tag{5.12}$$

Finally, we consider multiple twisted Dirichlet's type q-Euler polynomials defined by means of the generating functions as follows:

$$F_{w,\chi}^{q}(t,x) = \left(\frac{[2]_q \sum_{i=0}^{d-1} (-1)^{d-1-i} q^i \chi(i) w^i e^{ti}}{q^d w^d e^{td} + 1}\right)^h e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,w}^{(h,q)}(x) \frac{t^n}{n!}, \tag{5.13}$$

where $|t+\log(qw)| < \pi/d$ and $\mathrm{Re}(x) > 0$. From (5.13), we note that

$$\sum_{n=0}^{\infty}\underbrace{\int_X\cdots\int_X}_{h\text{-times}}\chi(x_1+\cdots+x_h)w^{x_1+\cdots+x_h}(x_1+\cdots+x_h+x)^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_h)\frac{t^n}{n!}=\sum_{n=0}^{\infty}E_{n,\chi,w}^{(h,q)}(x)\frac{t^n}{n!}. \tag{5.14}$$

Clearly, we obtain the following two theorems.

Theorem 5.5. *Let χ be a Dirichlet's character with conductor $d(=\text{odd})\in\mathbb{N}$. For each $w\in T_p$, $h\in\mathbb{N}$, $n\in\mathbb{N}\cup\{0\}$, and $\mathrm{Re}(x)>0$, we have*

$$\underbrace{\int_X\cdots\int_X}_{h\text{-times}}\chi(x_1+\cdots+x_h)w^{x_1+\cdots+x_h}(x_1+\cdots+x_h+x)^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_h)=E_{n,\chi,w}^{(h,q)}(x), \tag{5.15}$$

where

$$(x_1+\cdots+x_h+x)^n=\sum_{\substack{l_1+\cdots+l_h=n\\ l_1,\ldots,l_h\geq 0}}\frac{n!}{l_1!\cdots l_h!}x_1^{l_1}\cdots(x_h+x)^{l_h}. \tag{5.16}$$

Theorem 5.6. *Let χ be a Dirichlet's character with conductor $d(=\text{odd})\in\mathbb{N}$. For each $w\in T_p$, $h\in\mathbb{N}$, $n\in\mathbb{N}\cup\{0\}$, and $\mathrm{Re}(x)>0$, we have*

$$E_{k,\chi,w}^{(h,q)}(x)=\sum_{\substack{l_1+\cdots+l_h=k\\ l_1,\ldots,l_h\geq 0}}\frac{k!}{l_1!\cdots l_h!}E_{l_1,\chi,w}^{q}\cdots E_{l_{h-1},\chi,w}^{q}\cdot E_{l_h,\chi,w}^{q}(x). \tag{5.17}$$

References

[1] T. Kim, "Analytic continuation of multiple q-zeta functions and their values at negative integers," *Russian Journal of Mathematical Physics*, vol. 11, no. 1, pp. 71–76, 2004.

[2] T. Kim, "On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 779–792, 2007.

[3] T. Kim, "A note on some formulae for the q-Euler numbers and polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 9, no. 2, pp. 227–232, 2006.

[4] T. Kim, M.-S. Kim, L.-C. Jang, and S.-H. Rim, "New q-Euler numbers and polynomials associated with p-adic q-integrals," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 243–252, 2007.

[5] L.-C. Jang, "On a q-analogue of the p-adic generalized twisted L-functions and p-adic q-integrals," *Journal of the Korean Mathematical Society*, vol. 44, no. 1, pp. 1–10, 2007.

[6] L.-C. Jang, S.-D. Kim, D.-W. Park, and Y.-S. Ro, "A note on Euler number and polynomials," *Journal of Inequalities and Applications*, vol. 2006, Article ID 34602, 5 pages, 2006.

[7] H. Ozden and Y. Simsek, "A new extension of q-Euler numbers and polynomials related to their interpolation functions," *Applied mathematics Letters*. In press.

[8] H. Ozden, Y. Simsek, and I. N. Cangul, "Remarks on sum of products of (h,q)-twisted Euler polynomials and numbers," *Journal of Inequalities and Applications*, vol. 2008, Article ID 816129, 8 pages, 2008.

[9] H. Ozden, Y. Simsek, and I. N. Cangul, "Multivariate interpolation functions of higher order q-Euler numbers and their applications," *Abstract and Applied Analysis*. In press.

[10] Y. Simsek, "Twisted (h,q)-Bernoulli numbers and polynomials related to twisted (h,q)-zeta function and L-function," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 790–804, 2006.

[11] Y. Simsek, "On p-adic twisted q-L-functions related to generalized twisted Bernoulli numbers," *Russian Journal of Mathematical Physics*, vol. 13, no. 3, pp. 340–348, 2006.
[12] Y. Simsek, "On twisted q-Hurwitz zeta function and q-two-variable L-function," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 466–473, 2007.
[13] Y. Simsek, "The behavior of the twisted p-adic (h, q)-L-functions at $s = 0$," *Journal of the Korean Mathematical Society*, vol. 44, no. 4, pp. 915–929, 2007.
[14] Y. Simsek, V. Kurt, and D. Kim, "New approach to the complete sum of products of the twisted (h, q)-Bernoulli numbers and polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 44–56, 2007.
[15] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, no. 4, pp. 987–1000, 1948.
[16] M. Cenkci, "The p-adic generalized twisted (h, q)-Euler-L-function and its applications," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 1, pp. 37–47, 2007.
[17] M. Cenkci and M. Can, "Some results on q-analogue of the Lerch zeta function," *Advanced Studies in Contemporary Mathematics*, vol. 12, no. 2, pp. 213–223, 2006.
[18] M. Cenkci, M. Can, and V. Kurt, "p-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 9, no. 2, pp. 203–216, 2004.
[19] M. Cenkci, Y. Simsek, and V. Kurt, "Further remarks on multiple p-adic q-L-function of two variables," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 1, pp. 49–68, 2007.
[20] T. Kim, "On Euler-Barnes multiple zeta functions," *Russian Journal of Mathematical Physics*, vol. 10, no. 3, pp. 261–267, 2003.
[21] T. Kim, "Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 186–196, 2005.
[22] T. Kim, "Multiple p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 13, no. 2, pp. 151–157, 2006.
[23] T. Kim, "A note on p-adic q-integral on $\mathbb{Z}_p$ associated with q-Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 133–137, 2007.
[24] T. Kim, "q-Euler numbers and polynomials associated with p-adic q-integrals," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 15–27, 2007.
[25] T. Kim, "An invariant p-adic q-integral on $\mathbb{Z}_p$," *Applied Mathematics Letters*, vol. 21, no. 2, pp. 105–108, 2008.
[26] T. Kim, "On p-adic interpolating function for q-Euler numbers and its derivatives," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 598–608, 2008.
[27] T. Kim and S.-H. Rim, "New Changhee q-Euler numbers and polynomials associated with p-adic q-integrals," *Computers & Mathematics with Applications*, vol. 54, no. 4, pp. 484–489, 2007.
[28] T. Kim, J. Y. Choi, and J. Y. Sug, "Extended q-Euler numbers and polynomials associated with fermionic p-adic q-integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 14, no. 2, pp. 160–163, 2007.