

Review Article

q -Genocchi Numbers and Polynomials Associated with q -Genocchi-Type l -Functions

Yilmaz Simsek,¹ Ismail Naci Cangul,² Veli Kurt,¹ and Daeyeoul Kim³

¹Department of Mathematics, Faculty of Arts and Science, University of Akdeniz, Antalya 07058, Turkey

²Department of Mathematics, Faculty of Arts and Science, University of Uludag, Bursa 16059, Turkey

³National Institute for Mathematical Sciences, Doryong-dong, Yuseong-gu, Daejeon 305-340, South Korea

Correspondence should be addressed to Yilmaz Simsek, ysimsek@akdeniz.edu.tr

Received 19 March 2007; Accepted 14 December 2007

Recommended by Rigoberto Medina

The main purpose of this paper is to study generating functions of the q -Genocchi numbers and polynomials. We prove a new relation for the generalized q -Genocchi numbers, which is related to the q -Genocchi numbers and q -Bernoulli numbers. By applying Mellin transformation and derivative operator to the generating functions, we define q -Genocchi zeta and l -functions, which are interpolated q -Genocchi numbers and polynomials at negative integers. We also give some applications of the generalized q -Genocchi numbers.

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1. Introduction definitions and notations

In [1], Jang et al. gave new formulae on Genocchi numbers. They defined poly-Genocchi numbers to give the relation between Genocchi numbers, Euler numbers, and poly-Genocchi numbers. In [2], Kim et al. constructed new generating functions of the q -analogue Eulerian numbers and q -analogue Genocchi numbers. They gave relations between Bernoulli numbers, Euler numbers, and Genocchi numbers. They also defined Genocchi zeta functions which interpolate these numbers at negative integers. Kim [3] gave new concept of the q -extension of Genocchi numbers and gave some relations between q -Genocchi polynomials and q -Euler numbers. In this paper, by using generating function of this numbers, we study q -Genocchi zeta and l -functions. In [4], Kim constructed q -Genocchi numbers and polynomials. By using these numbers and polynomials, he proved the q -analogue of alternating sums of powers of

consecutive integers due to Euler:

$$\sum_{j=0}^{k-1} [j : q^2] (-1)^{j-1} [j]^{n-1} q^{(k-j)(n+1)/2} = \frac{G_{n,k,q} - G_{n,k,q}(k)}{(1+q)n} \quad (1.1)$$

(cf. [4]), where if $q \in \mathbb{C}, |q| < 1$,

$$[x] = [x : q] = \frac{1 - q^x}{1 - q}, \quad [j : q^2] = \frac{1 - q^{2j}}{1 - q^2}, \quad (1.2)$$

and the numbers $G_{n,k,q}$ are called q -Genocchi numbers which are defined by

$$(1+q)t \sum_{j=0}^{\infty} q^{k-j} [j : q^2] (-1)^{j-1} \exp(t[j, q^2] q^{(k-j)/2}) = \sum_{j=0}^{\infty} G_{n,k,q} \frac{t^n}{n!}. \quad (1.3)$$

Note that $\lim_{q \rightarrow 1} [x] = x$, (cf. [3, 5–9]). The Euler numbers E_n are usually defined by means of the following generating function (cf. [10–16]):

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi. \quad (1.4)$$

The Genocchi numbers G_n are usually defined by means of the following generating function (cf. [12, 13]):

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi. \quad (1.5)$$

These numbers are classical and important in number theory. In [12], Kim defined generating functions of the q -Genocchi numbers and q -Euler numbers as follows:

$$(1+q)e^{t/(1-q)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+q^{n+1})(1-q)^n} \frac{t^n}{n!} = \sum_{m=0}^{\infty} E_{m,q} \frac{t^m}{m!}, \quad (1.6)$$

where $E_{m,q}$ denotes q -Euler numbers,

$$G_q(t) = (1+q)t \sum_{m=0}^{\infty} (-1)^m q^m e^{[m]t} = \sum_{m=0}^{\infty} G_{m,q} \frac{t^m}{m!}, \quad (1.7)$$

where $G_{m,q}$ denotes q -Genocchi numbers. Genocchi zeta function is defined as follows (cf. [13, page 108]): for $s \in \mathbb{C}$,

$$\zeta_G(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}. \quad (1.8)$$

Kim [17] defined the fermionic and deformed expression of p -adic q -Volkenborn integral at $q = -1$ and $q = 1$. He constructed integral equation of the fermionic expression of p -adic q -Volkenborn integral at $q = -1$. By using this integral equation, he defined new generating functions of λ -Euler numbers and polynomials. By using derivative operator to this functions, he constructed new λ -zeta, λ - l -functions and p -adic λ - l -functions, which are interpolated λ -Euler numbers and polynomials. He also gave some applications which are the formulae of the trigonometric functions by applying fermionic and deformed expression of p -adic q -Volkenborn integral at $q = -1$ and $q = 1$. Kim and Rim [18] defined two-variable L -function. They gave main properties of this function. In [6], Kim constructed the two-variable p -adic q - L -function which interpolates the generalized q -Bernoulli polynomials attached to Dirichlet character. In [19], Simsek et al. constructed the two-variable Dirichlet q - L -function and the two-variable multiple Dirichlet-type Changhee q - L -function. In [8, 20], Simsek defined generating functions, which are interpolates twisted Bernoulli numbers and polynomials, twisted Euler numbers and polynomials. He [21] also gave new generating functions which produce q -Genocchi zeta functions and q - l -series with attached to Dirichlet character. Therefore, by using these generating functions, he constructed new q -analogue of Hardy-Berndt sums. He gave relations between these sums, q -Genocchi zeta functions and q - l -series as well,

$$\zeta_G(s)\Gamma(s) = \int_0^\infty \frac{2x^{s-1}}{e^{-x} + 1} dx \tag{1.9}$$

(cf. [21]), where $\Gamma(s)$ is Euler's gamma function and $\zeta_G(1 - n) = -G_n/n$, $n > 1$ (cf. [1], [13, page 108, equation (2.43)]). The first author defined q -analogue of the Genocchi zeta functions as follows [21].

Definition 1.1. Let $s \in \mathbb{C}$ and $\text{Re}(s) > 1$. q -analogue of the Genocchi type zeta function is expressed by the formula

$$\mathfrak{J}_{G,q}(s) = (1 + q) \sum_{n=1}^\infty \frac{(-1)^n q^{-n}}{(q^{-n}[n])^s}. \tag{1.10}$$

Remark 1.2. If $q \rightarrow 1$, then (1.10) reduces to ordinary Genocchi zeta functions (see [13, page 108]). Cenkci et al. [22], defined different type of q -Genocchi zeta functions, which are defined as follows:

$$\zeta_q^{(G)}(s) = q(1 + q) \sum_{n=1}^\infty \frac{(-1)^{n+1} q^n}{[n]^s}. \tag{1.11}$$

Simsek [21] defined q -analogue of the Hurwitz-type Genocchi zeta function by applying the Mellin transformations as follows:

$$\mathfrak{J}_q(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\sum_{n=0}^\infty (-1)^n q^{-n} e^{-(q^{-n}[n]+x)t} \right) dt. \tag{1.12}$$

Definition 1.3 (see [21]). Let $s \in \mathbb{C}$, $\operatorname{Re}(s) > 1$, and $0 < x \leq 1$. q -analogue of the Hurwitz-type Genocchi zeta function is expressed by the formula

$$\mathfrak{J}_{G,q}(s, x) := [2]\mathfrak{J}_q(s, x). \quad (1.13)$$

Observe that when $x = 1$, the $\mathfrak{J}_{G,q}(s, x)$ is reduced to $\mathfrak{J}_{G,q}(s)$ and if $q \rightarrow 1$, then $\mathfrak{J}_{G,q}(s, x) \rightarrow \mathfrak{J}_G(s, x)$. A function $\mathfrak{J}_G(s, x)$ is called an ordinary Hurwitz-type Genocchi zeta function if $\mathfrak{J}_G(s, x)$ is expressed by the formula

$$\mathfrak{J}(s, x) := 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \quad (1.14)$$

where $s \in \mathbb{C}$, $\operatorname{Re}(s) > 1$, and $0 < x \leq 1$, cf. [13].

In [21], Simsek defined q -analogue (Genocchi-type) one- and two-variable l -functions as follows, respectively; let χ be a Dirichlet character; let $s \in \mathbb{C}$ and $\operatorname{Re}(s) > 1$;

$$l_{G,q}(s, \chi) = \frac{(1+q)}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{n=1}^{\infty} (-1)^n \chi(n) q^{-n} e^{-q^{-n}[n]t} \right) dt, \quad (1.15)$$

$$l_{G,q}(s, x, \chi) = \frac{(1+q)}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{n=0}^{\infty} (-1)^n \chi(n) q^{-n} e^{-(q^{-n}[n]+x)t} \right) dt. \quad (1.16)$$

A function $l_G(s, \chi)$ is called an ordinary Genocchi-type l -function if $l_G(s, \chi)$ is expressed by the formula

$$l(s, x) := 2 \sum_{n=0}^{\infty} \frac{(-1)^n \chi(n)}{(n+x)^s}, \quad (1.17)$$

where $s \in \mathbb{C}$, $\operatorname{Re}(s) > 1$ and $0 < x \leq 1$, cf. [13].

Observe that when $\chi \equiv 1$, (1.15) reduces to (1.10):

$$l_q(s, 1) = \mathfrak{J}_q(s). \quad (1.18)$$

We summarize our work as follows. In Section 2, we study generating functions of the q -Genocchi numbers and polynomials. By using infinite and finite series, we give some definitions of the q -Genocchi numbers and polynomials. We find new relations between generalized q -Genocchi numbers with attached to χ , q -Genocchi numbers and Barnes' type Changhee q -Bernoulli numbers. In Section 3, by applying Mellin transformation and derivative operator to the generating functions of the q -Genocchi numbers, we construct q -Genocchi zeta and l -functions, which are interpolated q -Genocchi numbers and polynomials at negative integers. We also give some new relations related to these numbers and polynomials.

2. q -Genocchi number and polynomials

In this section, we give some new relations and identities related to q -Genocchi numbers and polynomials. Firstly we give some generating functions of the q -Genocchi numbers, which were defined by Kim [3, 10, 11]:

$$F_q(t) = e^{t/(1-q)} \sum_{j=0}^{\infty} \frac{(1+q)}{[2 : q^{j+1}]} \left(\frac{1}{q-1} \right)^j \frac{t^j}{j!} = (1+q) \sum_{l=0}^{\infty} (-q)^l e^{[l]t}, \quad (2.1)$$

and let

$$F_q^*(t) = t(1+q) \sum_{l=0}^{\infty} (-q)^l e^{[l]t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!} \tag{2.2}$$

(cf. [3, 10, 11, 23]), where $G_{n,q}$ denotes q -Genocchi numbers.

We note that q -Genocchi numbers, $G_{n,q}$, were defined by Kim [3, 10, 11].

By using the above generating functions, q -Genocchi polynomials, $G_{n,q}(x)$, are defined by means of the following generating function:

$$F_q^*(t, x) = F_q^*(t)e^{tx} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{2.3}$$

Our generating function of $G_{n,q}(x)$ is similar to that of [3, 12, 21, 23]. By using Cauchy product in (2.3), we easily obtain

$$\sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{t^n x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n G_{k,q} \frac{x^{n-k}}{k!(n-k)!} t^n. \tag{2.4}$$

Then by comparing coefficients of t^n on both sides of the above equation, for $n \geq 2$, we obtain the following result.

Theorem 2.1. *Let n be an integer with $n \geq 2$. Then one has*

$$G_{n,q}(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} G_{k,q}. \tag{2.5}$$

By using the same method in [3, 12, 21] in (2.3), we have

$$\sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = (1+q)t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]t+xt} = (1+q)t \sum_{n=0}^{\infty} (-1)^n q^n \sum_{k=0}^{\infty} \frac{([n]+x)^k t^k}{k!}, \tag{2.6}$$

and after some elementary calculations, we have

$$\sum_{k=0}^{\infty} G_{k,q}(x) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \left((1+q) \sum_{n=0}^{\infty} (-1)^n q^n ([n]+x)^{k-1} k \right) \frac{t^k}{k!}. \tag{2.7}$$

By comparing coefficients of $t^k/k!$ on both sides of the above equation, we arrive at the following corollary.

Corollary 2.2. *Let $k \in \mathbb{N}$. Then one has*

$$G_{k,q}(x) = k(q+1) \sum_{n=0}^{\infty} \sum_{j=0}^{k-1} \sum_{d=0}^j \binom{k-1}{j} \binom{j}{d} \frac{(-1)^{n+d} q^{d(n+1)} x^{k-j-1}}{(1-q)^j}. \tag{2.8}$$

We give some of q -Genocchi polynomials as follows: $G_{0,q}(x) = 0$, $G_{1,q}(x) = 1$, $G_{2,q}(x) = 2x - 2q/(1 + q^2), \dots$

From the generating function $F_q^*(t)$, we have the following.

Corollary 2.3. *Let $k \in \mathbb{N}$. Then one has*

$$G_{k,q} = k(1+q) \sum_{n=0}^{\infty} (-1)^n q^n [n]^{k-1} = \frac{k(1-q^2)}{(1-q)^k} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j} (-1)^j}{1+q^{j+1}}. \quad (2.9)$$

Proof of the Corollary 2.3 was given by Kim [3, 12]. We give some of q -Genocchi numbers as follows: $G_{0,q} = 0$, $G_{1,q} = 1$, $G_{2,q} = -2q/(1 + q^2), \dots$

Observe that if $q \rightarrow 1$, then $G_{2,1} = -1$.

By using derivative operator to (2.6), we have

$$\frac{d}{dx} \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = \frac{d}{dx} \left((1+q)t \sum_{n=0}^{\infty} (-1)^n q^n e^{([n]+x)t} \right) = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^{n+1}}{n!}. \quad (2.10)$$

After some elementary calculations, we arrive at the following corollary.

Corollary 2.4. *Let n be a positive integer. Then one has*

$$\frac{d}{dx} G_{n,q}(x) = n G_{n-1,q}(x). \quad (2.11)$$

Corollary 2.5. *Let n be a positive integer. Then one has*

$$G_{q,n}(x+y) = \sum_{k=0}^n \binom{n}{k} G_{k,q}(x) y^{n-k}. \quad (2.12)$$

Proof. Proof of this corollary is easily obtained from (2.4). \square

Generalized q -Genocchi numbers are defined by means of the following generating function (this generating function is similar to that of [3, 12, 21–24]):

$$F_{q,\chi}(t) = (1+q)t \sum_{n=0}^{\infty} \chi(n) q^n (-1)^n e^{[n]t} = \sum_{n=0}^{\infty} G_{n,\chi,q} \frac{t^n}{n!}, \quad (2.13)$$

where χ denotes the Dirichlet character with conductor $d \in \mathbb{Z}^+$, the set of positive integers.

Observe that when $\chi \equiv 1$, (2.13) reduces to (2.3).

By (2.13), we have

$$\sum_{m=0}^{\infty} G_{m,\chi,q} \frac{t^m}{m!} = (1+q) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\chi(n) q^n (-1)^n [n]^m t^{m+1}}{m!}. \quad (2.14)$$

After some elementary calculations and by comparing coefficients t^m on both sides of the above equation, we get

$$G_{m,\chi,q} = (1+q)m \sum_{n=0}^{\infty} (-1)^n q^n \chi(n) [n]^{m-1}. \quad (2.15)$$

By setting $n = a + dj$, where $(j = 0, 1, 2, \dots, \infty; a = 1, 2, \dots, d)$, and $\chi(a + jd) = \chi(a)$, in the above equation, we obtain

$$\begin{aligned} G_{m,\chi,q} &= (1+q)m \sum_{j=0}^{\infty} \sum_{a=1}^d (-1)^{a+jd} q^{a+jd} \chi(a+jd) [a+jd]^{m-1} \\ &= (1+q)m \sum_{a=1}^d \sum_{i=0}^{m-1} (-1)^a \binom{m-1}{i} q^{a(m-i)} \chi(a) [a]^i [d]^{m-i-1} \sum_{j=0}^{\infty} (-1)^{dj} q^{dj} [j, q^d]^{m-i-1}. \end{aligned} \quad (2.16)$$

In [15], Srivastava et al. defined the following generalized Barnes-type Changhee q -Bernoulli numbers.

Let χ be the Dirichlet character with conductor d . Then the generalized Barnes-type Changhee q -Bernoulli numbers with attached to χ are defined as follows:

$$F_{q,\chi}(t | w_1) = -w_1 t \sum_{n=0}^{\infty} \chi(n) q^{w_1 n} e^{[w_1 n]t} = \sum_{n=0}^{\infty} \frac{\beta_{n,\chi,q}(w_1) t^n}{n!}, \quad |t| < 2\pi \quad (2.17)$$

(cf. [15]). Substituting $\chi \equiv 1$ and $w_1 = 1$ into the above equation, we have

$$F_{q,1}(t | 1) = -t \sum_{n=0}^{\infty} q^n e^{[n]t} = \sum_{n=0}^{\infty} \frac{\beta_{n,q} t^n}{n!}. \quad (2.18)$$

By using derivative operator to the above, we obtain

$$\frac{d^m}{dt^m} F_{q,1}(t | 1)|_{t=0} = \beta_{m,q} = -m \sum_{n=0}^{\infty} q^n [n]^{m-1}. \quad (2.19)$$

By substituting (2.9) and (2.19) into (2.16), after some calculations, we arrive at the following theorem.

Theorem 2.6. *Let χ be the Dirichlet character with conductor d . If d is odd, then one has*

$$G_{m,\chi}(q) = \sum_{a=1}^d \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^a q^{a(m-i)} \chi(a) [a]^i [d]^{m-i-1} G_{m-i}(q^d), \quad (2.20)$$

if d is even, then one has

$$G_{m,\chi}(q) = \sum_{a=1}^d \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^{a+1} \frac{m}{m-i} q^{a(m-i)} \chi(a) [a]^i [d]^{m-i-1} \beta_{m-i,q^d}, \quad (2.21)$$

where β_{m-i,q^d} is defined in (2.19).

Remark 2.7. In Theorem 2.6, we give new relations between generalized q -Genocchi numbers, $G_{m,\chi}(q)$ with attached to χ , q -Genocchi numbers, $G_m(q)$, and Barnes-type Changhee q -Bernoulli numbers. For detailed information about generalized Barnes-type Changhee q -Bernoulli numbers with attached to χ see [15].

Generalized Genocchi polynomials are defined by means of the following generating function:

$$F_{q,\chi}(t, x) = F_{q,\chi}(t) e^{tx} = \sum_{n=0}^{\infty} G_{n,\chi,q}(x) \frac{t^n}{n!}. \quad (2.22)$$

Theorem 2.8. Let χ be the Dirichlet character with conductor d . Then one has

$$G_{n,\chi,q}(x) = \sum_{k=0}^{\infty} \binom{n}{k} G_{n,\chi,q} x^{n-k}. \quad (2.23)$$

Remark 2.9. Generating functions of $G_{n,q}(x)$ and $G_{n,\chi,q}(x)$ are different from those of [3, 12, 22, 23]. Kim defined generating function of $G_{n,q}(x)$, as follows [12]:

$$F_q(t, x) = (1+q)t \sum_{m=0}^{\infty} q^m (-1)^m e^{[m+x]t} = \sum_{m=0}^{\infty} G_{m,q}(x) \frac{t^m}{m!}. \quad (2.24)$$

In [21], Simsek defined generating function of $G_{n,q}(x)$ by

$$F_q(t, x) = \sum_{n=0}^{\infty} (-1)^n q^{-n} \exp(-(q^{-n}[n] + x)t). \quad (2.25)$$

3. q -Genocchi zeta and l -functions

In recent years, many mathematicians and physicians have investigated zeta functions, multiple zeta functions, l -series, q -Genocchi zeta, and l -functions, and q -Bernoulli, Euler, and Genocchi numbers and polynomials mainly because of their interest and importance. These functions and numbers are not only used in complex analysis, but also used in p -adic analysis and other areas. In particular, multiple zeta functions occur within the context of Knot theory, quantum field theory, applied analysis and number theory, (cf. [15]). In this section, we define q -Genocchi zeta and l -functions, which are interpolated q -Genocchi polynomials and generalized q -Genocchi numbers at negative integers. By applying the Mellin transformation to (2.3), we obtain

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} F_q^*(-t, x) dt = \frac{-(1+q)}{\Gamma(s)} \int_0^{\infty} t^{s-1} \sum_{n=0}^{\infty} (-1)^n q^n e^{-([n]+x)t} dt = (1+q) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^n}{([n]+x)^s}, \quad (3.1)$$

where $\operatorname{Re} s > 1$, $0 < x \leq 1$, and $|q| < 1$.

Thus, Hurwitz-type q -Genocchi zeta function is defined by the following definition.

Definition 3.1. Let $s \in \mathbb{C}$ with $\text{Res} > 1$ and let $q \in \mathbb{C}$ with $|q| < 1$. Then one defines

$$\zeta_{G,q}(s, x) = (1 + q) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^n}{([n] + x)^s}. \quad (3.2)$$

Observe that when $x = 1$ in (3.2), then we obtain Riemann-type q -Genocchi zeta function:

$$\zeta_{G,q}(s) = (1 + q) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^n}{[n]^s}. \quad (3.3)$$

Hurwitz-type q -Genocchi zeta function interpolates q -Genocchi polynomials at negative integers. For $s = 1 - k$, $k \in \mathbb{Z}^+$, and by applying Cauchy residue theorem to (3.1), we can obtain the following theorem.

Theorem 3.2. For $s = 1 - k$, $k > 0$, then one has

$$\zeta_{G,q}(1 - k, x) = -\frac{G_{k,q}(x)}{k}. \quad (3.4)$$

Remark 3.3. The second proof of Theorem 3.2 can be obtained by using $(d^k / dt^k)|_{t=0}$ derivative operator to (2.3) as follows:

$$\begin{aligned} \left. \frac{d^k}{dt^k} F_q^*(t, x) \right|_{t=0} &= (1 + q) \left. \frac{d^k}{dt^k} \left(t \sum_{n=0}^{\infty} (-1)^n q^n e^{([n]+x)t} \right) \right|_{t=0}, \\ \frac{-G_{k,q}(x)}{k} &= (1 + q) \sum_{n=0}^{\infty} (-1)^{n+1} q^n ([n] + x)^{k-1}. \end{aligned} \quad (3.5)$$

Thus we obtained the desired result.

By applying Mellin transformation to (2.13), we obtain

$$l_{q,G}(s, \chi) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} F_{q,x}(-t) dt = (1 + q) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \chi(n) q^n}{[n]^s}. \quad (3.6)$$

Thus we can define Dirichlet-type q -Genocchi l -function as follows.

Definition 3.4. Let χ be the Dirichlet character with conductor d . Let $s \in \mathbb{C}$ with $\text{Re } s > 1$. One defines

$$l_{q,G}(s, \chi) = (1 + q) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \chi(n) q^n}{[n]^s}. \quad (3.7)$$

Relation between $l_{q,G}(s, \chi)$ and $\zeta_{q,G}(s, x)$ is given by the following theorem.

Theorem 3.5. Let χ be the Dirichlet character with conductor d . Then one has

$$l_{q,G}(s, \chi) = \frac{(1+q)}{(1+q^d)[d]^s} \sum_{a=0}^{d-1} \chi(a) q^{a(1-s)} (-1)^a \zeta_{q^d, G} \left(s, \frac{q^{-a}[a]}{[d]} \right). \quad (3.8)$$

Proof. By setting $n = a + dk$, where $(k = 0, 1, 2, \dots, \infty; a = 1, 2, 3, \dots, d)$ in (3.7), we obtain,

$$\begin{aligned} l_{q,G}(s, \chi) &= (1+q) \sum_{a=0}^d \sum_{k=1}^{\infty} \frac{\chi(a+kd) q^{a+kd} (-1)^{a+kd+1}}{[a+kd]^s} \\ &= (1+q) \sum_{a=0}^d \sum_{k=0}^{\infty} \frac{\chi(a+kd) q^{a+kd} (-1)^{a+kd+1}}{([a] + q^a [d][k : q^d])^s} \\ &= \frac{(1+q)}{(1+q^d)} \sum_{a=0}^{d-1} \frac{\chi(a) q^{a(1-s)} (-1)^a}{[d]^s} \sum_{k=0}^{\infty} \frac{(1+q^d) q^{kd} (-1)^{kd+1}}{([k : q^d] + q^{-a}[a]/[d])^s} \end{aligned} \quad (3.9)$$

After some elementary calculations, we arrive at the desired result of the theorem. \square

The function $l_{q,G}(s, \chi)$ interpolates generalized q -Genocchi numbers, which are given by the following theorem.

Theorem 3.6. Let $n \in \mathbb{Z}^+$. Let χ be the Dirichlet character with conductor d . Then one has

$$l_{q,G}(1-n, \chi) = -\frac{G_{n,\chi}(q)}{n}. \quad (3.10)$$

Proof. Proof of this theorem is similar to that of Theorem 3.2. So we omit the proof. \square

We give some applications. Setting $s = 1 - n, n \in \mathbb{Z}^+$ and using Theorem 3.2 in Theorem 3.5, we get

$$l_{q,G}(1-n, \chi) = \frac{(1+q)[d]^{n-1}}{n(1+q^d)} \sum_{a=1}^d (-1)^{a+1} \chi(a) q^a G_{n,q^d} \left(\frac{q^{-a}[a]}{[d]} \right). \quad (3.11)$$

By comparing both sides of the above equation and Theorem 3.6, we obtain distributions relation of the generalized Genocchi numbers as follows.

Corollary 3.7. Let χ be the Dirichlet character with conductor d . Then one has

$$G_{n,\chi}(q) = \frac{(1+q)[d]^{n-1}}{(1+q^d)} \sum_{a=1}^d (-1)^{a+1} \chi(a) q^a G_{n,q^d} \left(\frac{q^{-a}[a]}{[d]} \right), \quad (3.12)$$

where $n \geq 0$, and $G_{n,q^d}(q^{-a}[a]/[d])$ is the q -Genocchi polynomial.

By substituting (2.5) into (3.12), we have the following corollary.

Corollary 3.8. *Let χ be the Dirichlet character with conductor d . Then one has*

$$\begin{aligned} G_{n,\chi}(q) &= \frac{(1+q)[d]^{n-1}}{(1+q^d)} \sum_{a=1}^d (-1)^a \chi(a) q^a \sum_{k=0}^n \binom{n}{k} \left(\frac{q^{-a}[a]}{[d]}\right)^{n-k} G_{k,q^d} \\ &= \frac{(1+q)}{(1+q^d)[d]} \sum_{a=1}^d (-1)^a \chi(a) q^a [a]^n \sum_{k=0}^n \binom{n}{k} \left(\frac{q^a[d]}{[a]}\right)^k G_{k,q^d}. \end{aligned} \tag{3.13}$$

If we substitute (2.7) into (3.12), we get a new relation for the distribution relation of q -Genocchi numbers:

$$\begin{aligned} G_{n,\chi}(q) &= \frac{n(1+q)[d]^{n-1}}{(1+q^d)} \sum_{a=1}^d (-1)^{a+1} \chi(a) q^a \sum_{j=0}^{\infty} (-1)^j q^j \left([j] + \frac{q^{-a}[a]}{[d]}\right)^{n-1} \\ &= \frac{n(1+q)[d]^{n-1}}{(1+q^d)} \sum_{j=0}^{\infty} \sum_{a=1}^d (-1)^{a+j+1} \chi(a) q^{a+j} \sum_{m=0}^{n-1} \binom{n-1}{m} \left(\frac{q^{-a}[a]}{[d]}\right)^m [j]^{n-1-m} \\ &= \frac{n(1+q)[d]^{n-1}}{(1+q^d)} \sum_{j=0}^{\infty} \sum_{a=1}^d \sum_{m=0}^{n-1} (-1)^{a+j+1} \binom{n-1}{m} \chi(a) q^{a+j} \left(\frac{q^{-a}[a]}{[d]}\right)^m [j]^{n-m-1}. \end{aligned} \tag{3.14}$$

Thus we arrive at the following corollary.

Corollary 3.9. *Let χ be the Dirichlet character with conductor d . Then one has*

$$G_{n,\chi}(q) = \frac{n(1+q)[d]^{n-1}}{(1+q^d)} \sum_{j=0}^{\infty} \sum_{a=1}^d \sum_{m=0}^{n-1} (-1)^{a+j+1} \binom{n-1}{m} \chi(a) q^{a+j} \left(\frac{q^{-a}[a]}{[d]}\right)^m [j]^{n-m-1}. \tag{3.15}$$

Acknowledgments

The first and third authors have been supported by the Scientific Research Project, Administration Akdeniz University. The second author has been supported by Uludag University Research Fund, Projects no. F2004/40 and F2008-31. The fourth author has been supported by National Institute for Mathematical Sciences Doryong-dong, Yuseong-gu, Daejeon. The authors express their sincere gratitude to referees for their suggestions and comments.

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