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Research Article

On Connection between Second-Order Delay Differential Equations and Integrodifferential Equations with Delay

Leonid Berezansky,[1] Josef Diblík,[2,3] and Zdeněk Šmarda[3]

[1] *Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel*
[2] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, 66237, Brno University of Technology, Brno, Czech Republic*
[3] *Department of Mathematics, Faculty of Electrical Engineering and Communication, 61600, Brno University of Technology, Brno, Czech Republic*

Correspondence should be addressed to Leonid Berezansky, brznsky@cs.bgu.ac.il

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The existence and uniqueness of solutions and a representation of solution formulas are studied for the following initial value problem: $\dot{x}(t) + \int_{t_0}^{t} K(t,s)x(h(s))ds = f(t)$, $t \geq t_0$, $x \in \mathbb{R}^n$, $x(t) = \varphi(t)$, $t < t_0$. Such problems are obtained by transforming second-order delay differential equations $\ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0$ to first-order differential equations.

1. Introduction and Preliminaries

The second order delay differential equation

$$\ddot{x}(t) + a(t)\dot{x}\big(g(t)\big) + b(t)x(h(t)) = 0 \tag{1.1}$$

attracts the attention of many mathematicians because of their significance in applications.

In particular, Minorsky [1] in 1962 considered the problem of stabilizing the rolling of a ship by an “activated tanks method” in which ballast water is pumped from one position to another. To solve this problem, he constructed several delay differential equations with damping described by (1.1).

Despite the obvious importance in applications, there are only few papers on delay differential equations with damping.

One of the methods used to study (1.1) is transforming the second-order delay differential equation to a first-order differential or integrodifferential equations with delay. A transformation of the type

$$x(t) = \int_{t_0}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds, \tag{1.2}$$

where $u(t)$ is a nonnegative function is used in [2]. The following result is a restriction of [2, Theorem 1] to (1.1).

Proposition 1.1. *If* $a, b : [t_0, \infty) \to [0, \infty)$ *are Lebesgue measurable and locally essentially bounded,* $g, h : [t_0, \infty) \to \mathbb{R}$ *are Lebesgue measurable functions,* $g(t) \le t$, $h(t) \le t$ *if* $t \in [t_0, \infty)$, $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = \infty$, *there exists a locally absolutely continuous function* $u : [t_0, \infty) \to [0, \infty)$ *such that the inequality*

$$\dot{u}(t) + u^2(t) + a(t)u(g(t))e^{-\int_{g(t)}^{t} u(s)ds} + b(t)e^{-\int_{h(t)}^{t} u(s)ds} \le 0, \tag{1.3}$$

is valid for all sufficiently large t, *and the equation*

$$\dot{z}(t) + u(t)z(t) + a(t)z(g(t)) = 0 \tag{1.4}$$

has a nonoscillatory solution, then (1.1) *has a nonoscillatory solution, too.*

Proposition 1.1 means that second order delay equation (1.1) is reduced to nonlinear inequality (1.3) and first order delay differential equation (1.4).

Now we will briefly describe the scheme of another transformation, different from the one used in [2] (in this explanation we omit exact assumptions related to the functions used, which are formulated later).

Consider an auxiliary equation

$$\dot{z}(t) + a(t)z(g(t)) = p(t), \quad t \ge t_0, \tag{1.5}$$

with the initial condition

$$z(t) = \varphi(t), \quad t < t_0, \; z(t_0) = z_0. \tag{1.6}$$

It is known, see [3, 4], that the unique solution of (1.5), (1.6) has a form

$$z(t) = Z(t, t_0)z(t_0) + \int_{t_0}^{t} Z(t, s)p(s)ds - \int_{t_0}^{t} Z(t, s)\varphi(g(s))ds, \tag{1.7}$$

where $Z(t, s)$ is the fundamental matrix of (1.5) and $\varphi(g(s)) = 0$ if $g(s) < t_0$.

If we denote $z(t) = \dot{x}(t)$, then (1.1) can be rewritten in the form

$$\dot{z}(t) + a(t)z(g(t)) = -b(t)x(h(t)). \tag{1.8}$$

Applying (1.7) and equality $z(t) = \dot{x}(t)$ to (1.8), we have the following equation

$$\dot{x}(t) = Z(t,t_0)\dot{x}(t_0) - \int_{t_0}^{t} Z(t,s)b(s)x(h(s))ds - \int_{t_0}^{t} Z(t,s)\varphi(g(s))ds. \tag{1.9}$$

Define

$$\begin{gathered} K(t,s) := Z(t,s)b(s), \\ f(t) := Z(t,t_0)\dot{x}(t_0) - \int_{t_0}^{t} Z(t,s)\varphi(g(s))ds. \end{gathered} \tag{1.10}$$

Then (1.1) is transformed into the integrodifferential equation with delay

$$\dot{x}(t) + \int_{t_0}^{t} K(t,s)x(h(s))ds = f(t), \quad t \geq t_0. \tag{1.11}$$

Since (1.11) is a result of transforming (1.1), qualitative properties of (1.11) such as the existence and uniqueness of solutions, oscillation and nonoscillation, stability and asymptotic behavior can imply similar qualitative properties of (1.1).

The advantage of the suggested method in comparison with the method used in [2] is that a second order delay equation is reduced to one first-order integrodifferential delay equation while in [2] a second-order equation is reduced to a system of a nonlinear inequality and a linear delay equation.

Similar in a sense problems for delay difference equations were studied in [5, 6] as well.

This paper aims to investigate the problems of the existence, uniqueness and solution representation of (1.11). Problems related to oscillation/nonoscillation, stability and applications to second-order equations will be studied in our forthcoming papers. Throughout this paper, $|\cdot|$ will denote the matrix or vector norm used.

2. Main Results

Together with (1.11) we consider an initial condition

$$x(t) = \varphi(t), \quad t < t_0, \; x(t_0) = x_0. \tag{2.1}$$

We will assume that the following conditions hold:

(a1) For all $c > t_0$, the elements $k_{ij} : [t_0,c]\times[t_0,c] \to \mathbb{R}$, $i,j = 1,2,\ldots,n$ of the $n\times n$ matrix function K are measurable in the square $[t_0,c]\times[t_0,c]$, the elements $f_i : [t_0,c] \to \mathbb{R}$, $i = 1,2,\ldots,n$ of the vector function f are measurable in the interval $[t_0,c]$,

$$\sup_{t\in[t_0,c]} \int_{t_0}^{c} |K(t,s)|ds < \infty, \qquad \int_{t_0}^{c} |f(s)|ds < \infty. \tag{2.2}$$

(a2) $h : [t_0, \infty) \to \mathbb{R}$ is a measurable scalar function satisfying $h(t) \le t$, $\lim_{t\to\infty} h(t) = \infty$.

(a3) The initial function $\varphi : (-\infty, t_0) \to \mathbb{R}^n$ is a Borel bounded function.

A function $x : \mathbb{R} \to \mathbb{R}^n$ is called a solution of the problem (1.11), (2.1) if it is a locally absolutely continuous function on $[t_0, \infty)$, satisfies equation (1.11) on $t \ge t_0$ almost everywhere, and initial conditions (2.1) for $t \le t_0$.

Theorem 2.1. *Let conditions (a1)–(a3) hold. Then there exists a unique solution of problem* (1.11), (2.1).

Proof. It is sufficient to prove that there exists a unique solution of (1.11), (2.1) on the interval $[t_0, c]$ for any $c > t_0$.

Denote

$$x_h(t) = \begin{cases} x(h(t)) & \text{if } h(t) \ge t_0, \\ 0 & \text{if } h(t) < t_0, \end{cases}$$
$$\varphi^h(t) = \begin{cases} \varphi(h(t)) & \text{if } h(t) < t_0, \\ 0 & \text{if } h(t) \ge t_0. \end{cases} \tag{2.3}$$

Then $x(h(t)) = x_h(t) + \varphi^h(t),\ t \ge t_0$ and (1.11), (2.1) takes the form

$$\dot{x}(t) + \int_{t_0}^{t} K(t,s) x_h(s) ds = g(t), \quad t \ge t_0,\ x(t_0) = x_0, \tag{2.4}$$

where

$$g(t) = f(t) - \psi(t)$$
$$\psi(t) = \int_{t_0}^{t} K(t,q) \varphi^h(q) dq. \tag{2.5}$$

Denote $\chi_{[\alpha,\beta]}(t)$ the characteristic function of the interval $[\alpha, \beta]$. We will assume that $\chi_{[\alpha,\beta]}(t) \equiv 0$ if $\alpha \ge \beta$. Since

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(s) ds, \tag{2.6}$$

we have

$$x_h(t) = \left(x(t_0) + \int_{t_0}^{\max\{h(t), t_0\}} \dot{x}(s) ds \right) \chi_{[t_0,c]}(h(t)). \tag{2.7}$$

Hence problem (2.4) can be transformed into

$$\begin{aligned}&\dot{x}(t)+\int_{t_0}^{t}K(t,s)\left(\int_{t_0}^{\max\{h(s),t_0\}}\dot{x}(\tau)d\tau\right)\chi_{[t_0,c]}(h(s))ds\\&\quad+\left(\int_{t_0}^{t}K(t,s)\chi_{[t_0,c]}(h(s))ds\right)x(t_0)=g(t),\quad t\geq t_0,\ x(t_0)=t_0.\end{aligned}\tag{2.8}$$

We have

$$\begin{aligned}&\int_{t_0}^{t}K(t,s)\left(\int_{t_0}^{\max\{h(s),t_0\}}\dot{x}(\tau)d\tau\right)\chi_{[t_0,c]}(h(s))ds\\&\quad=\int_{t_0}^{t}K(t,s)\left(\int_{t_0}^{s}\dot{x}(\tau)\chi_{[t_0,h(s)]}(\tau)d\tau\right)\chi_{[t_0,c]}(h(s))ds\\&\quad=\int_{t_0}^{t}\left(\int_{\tau}^{t}K(t,s)\chi_{[t_0,h(s)]}(\tau)\chi_{[t_0,c]}(h(s))ds\right)\dot{x}(\tau)d\tau\\&\quad=\int_{t_0}^{t}\left(\int_{s}^{t}K(t,\tau)\chi_{[t_0,h(\tau)]}(s)\chi_{[t_0,c]}(h(\tau))d\tau\right)\dot{x}(s)ds.\end{aligned}\tag{2.9}$$

Denote

$$\begin{aligned}B(t,s)&:=-\int_{s}^{t}K(t,\tau)\chi_{[t_0,h(\tau)]}(s)\chi_{[t_0,c]}(h(\tau))d\tau,\\A(t)&:=-\int_{t_0}^{t}K(t,s)\chi_{[t_0,c]}(h(s))ds.\end{aligned}\tag{2.10}$$

Finally, problem (2.4) has the form

$$y(t)=\int_{t_0}^{t}B(t,s)y(s)ds+r(t),\quad t\geq t_0,\tag{2.11}$$

where $y(t)=\dot{x}(t),\ r(t)=A(t)x(t_0)+g(t)$. Consider the linear integral operator

$$(Ty)(t)=\int_{t_0}^{t}B(t,s)y(s)ds\tag{2.12}$$

in the space of all Lebesgue integrable functions $y:[t_0,c]\to\mathbb{R}^n$ with the norm $\|y\|=\int_{t_0}^{c}|y(s)|ds$.

We have

$$\begin{aligned}\sup_{t_0\le t\le c}\int_{t_0}^{c}|B(t,s)|ds &\le \sup_{t_0\le t\le c}\int_{t_0}^{c}\int_{s}^{t}|K(t,\tau)|d\tau ds\\ &\le (c-t_0)\cdot\sup_{t_0\le t\le c}\int_{t_0}^{c}|K(t,\tau)|d\tau<\infty.\end{aligned}\tag{2.13}$$

Hence the integral operator T is a compact Volterra operator and its spectral radius is equal to zero [4, 7, 8]. Then the integral equation (2.11) has a unique solution $y(t),\ t\ge t_0$. Consequently

$$x(t)=\begin{cases}x(t_0)+\displaystyle\int_{t_0}^{t}y(s)ds, & t\ge t_0,\\ \varphi(t), & t<t_0\end{cases}\tag{2.14}$$

is a unique solution of (1.11), (2.1).

Let Θ be the $n\times n$ zero matrix and I the identity $n\times n$ matrix. □

Definition 2.2. For each $s\ge t_0$, the solution $X=X(\,\cdot\,,s)$ of the problem

$$\begin{gathered}\dot{X}(t,s)+\int_{s}^{t}K(t,\tau)X(h(\tau),s)d\tau=\Theta,\quad t\ge s,\\ X(t,s)=\Theta,\quad t<s,\ X(s,s)=I,\end{gathered}\tag{2.15}$$

is called the fundamental matrix of (1.11). (Here $\dot{X}(t,s)$ is the partial derivative of $X(t,s)$ with respect to its first argument.)

Theorem 2.3. *Let conditions (a1)–(a3) hold. Then the unique solution of* (1.11), (2.1) *can be represented in the form*

$$x(t)=X(t,t_0)x_0+\int_{t_0}^{t}X(t,s)f(s)ds-\int_{t_0}^{t}X(t,s)\psi(s)ds\tag{2.16}$$

for $t\ge t_0$ where ψ is defined by (2.5).

Proof. In the proof we will use notation defined in the proof of Theorem 2.1. The existence and uniqueness of a solution of (1.11), (2.1) is a consequence of Theorem 2.1. Thus, we will only prove the solution representation formula (2.16). Problem (1.11), (2.1) is equivalent to (2.4). We need to show that the function

$$x(t)=X(t,t_0)x_0+\int_{t_0}^{t}X(t,s)g(s)ds,\tag{2.17}$$

where X is the fundamental matrix of (1.11) is the solution of problem (2.4). For convenience, we will write $x(h(t))$ instead of $x_h(t)$ assuming that $x(h(t))=0$, if $h(t)<t_0$.

Equality (2.17) implies

$$
\begin{aligned}
x(h(t)) &= X(h(t),t_0)x(t_0) + \int_{t_0}^{\max\{h(t),t_0\}} X(h(t),s)g(s)ds \\
&= X(h(t),t_0)x(t_0) + \int_{t_0}^{t} X(h(t),s)g(s)ds \\
&\quad - \int_{\max\{h(t),t_0\}}^{t} X(h(t),s)g(s)ds \\
&= X(h(t),t_0)x(t_0) + \int_{t_0}^{t} X(h(t),s)g(s)ds.
\end{aligned}
\tag{2.18}
$$

We consider the left-hand side of (2.4) if assuming x to have the form (2.17). With the help of the last relation, we have

$$
\begin{aligned}
&\dot{x}(t) + \int_{t_0}^{t} K(t,s)x(h(s))ds \\
&\quad = \dot{X}(t,t_0)x(t_0) + g(t) + \int_{t_0}^{t} \dot{X}(t,s)g(s)ds \\
&\qquad + \int_{t_0}^{t} K(t,s)\left(X(h(s),t_0)x(t_0) + \int_{t_0}^{s} X(h(s),\xi)g(\xi)d\xi\right)ds \\
&\quad = g(t) + \left(\dot{X}(t,t_0) + \int_{t_0}^{t} K(t,s)X(h(s),t_0)ds\right)x(t_0) + \int_{t_0}^{t} \dot{X}(t,s)g(s)ds \\
&\qquad + \int_{t_0}^{t} K(t,s)\int_{t_0}^{s} X(h(s),\xi)g(\xi)d\xi ds \\
&\quad = g(t) + \int_{t_0}^{t} \dot{X}(t,s)g(s)ds + \int_{t_0}^{t} K(t,s)\int_{t_0}^{s} X(h(s),\xi)g(\xi)d\xi ds.
\end{aligned}
\tag{2.19}
$$

Since

$$
\begin{aligned}
\int_{t_0}^{t}\left(K(t,s)\int_{t_0}^{s} X(h(s),\xi)g(\xi)d\xi\right)ds &= \int_{t_0}^{t}\left(\int_{\xi}^{t} K(t,s)X(h(s),\xi)ds\right)g(\xi)d\xi \\
&= \int_{t_0}^{t}\left(\int_{s}^{t} K(t,\xi)X(h(\xi),s)d\xi\right)g(s)ds,
\end{aligned}
\tag{2.20}
$$

we obtain

$$\dot{x}(t) + \int_{t_0}^{t} K(t,s)x(h(s))ds = g(t) + \int_{t_0}^{t} \left(\dot{X}(t,s) + \int_{s}^{t} K(t,\xi)X(h(\xi),s)d\xi \right) g(s)ds = g(t). \tag{2.21}$$

□

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