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Research Article

Note on the Persistent Property of a Discrete Lotka-Volterra Competitive System with Delays and Feedback Controls

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A nonautonomous N-species discrete Lotka-Volterra competitive system with delays and feedback controls is considered in this work. Sufficient conditions on the coefficients are given to guarantee that all the species are permanent. It is shown that these conditions are weaker than those of Liao et al. 2008.

1. Introduction

Traditional Lotka-Volterra competitive systems have been extensively studied by many authors [1–7].The autonomous model can be expressed as follows:

$$u_i'(t) = b_i u_i(t)\left[1 - \sum_{j=1}^{N} a_{ij} u_j(t)\right], \quad i = 1, \dots, N, \tag{1.1}$$

where $b_i > 0$, $a_{ii} > 0$, $a_{ij} \geq 0$ $(i \neq j)$, $u_i(t)$denoting the density of the ith species at time t. Montes de Oca and Zeeman [6] investigated the general nonautonomous N-species Lotka-Volterra competitive system

$$u_i'(t) = u_i(t)\left[b_i(t) - \sum_{j=1}^{N} c_{ij}(t) u_j(t)\right], \quad c_{ij} \geq 0, \ i = 1, \dots, N, \tag{1.2}$$

and obtained that if the coefficients are continuous and bounded above and below by positive constants, and if for each $i = 2, \ldots, N$, there exists an integer $k_i < i$ such that

$$\frac{\overline{b}_i}{\underline{c}_{ij}} < \frac{\underline{b}_{ki}}{\overline{c}_{k_i j}}, \quad j = 1, \ldots, i, \tag{1.3}$$

then $u_i \to 0$ exponentially for $2 \leq i \leq N$, and $u_i(t) \to X^*$, where X^* is a certain solution of a logistic equation. Teng [8] and Ahmad and Stamova [9] also studied the coexistence on a nonautonomous Lotka-Volterra competitive system. They obtained the necessary or sufficient conditions for the permanence and the extinction. For more works relevant to system (1.1), one could refer to [1–9] and the references cited therein.

However, to the best of the authors' knowledge, to this day, still less scholars consider the general nonautonomous discrete Lotka-Volterra competitive system with delays and feedback controls. Recently, in [1] Liao et al. considered the following general nonautonomous discrete Lotka-Volterra competitive system with delays and feedback controls:

$$\begin{aligned} x_i(n+1) &= x_i(n)\exp\left\{ b_i(n) - \sum_{j=1}^{N} a_{ij}(n) x_j(n-\tau_{ij}) - d_i(n) u_i(n) \right\}, \\ \Delta u_i(n) &= r_i(n) - e_i(n) u_i(n) + c_i(n) x_i(n-\sigma_i), \quad i = 1, 2, \ldots, N, \\ x_i(\theta) &= \phi_i(\theta) \geq 0, \quad \theta \in \mathbb{N}[-\tau, 0] := \{-\tau, -\tau+1, \ldots, -1, 0\}, \end{aligned} \tag{1.4}$$

where $x_i(n)$ $(i = 1, 2, \ldots, N)$ is the density of competitive species; $u_i(n)$ is the control variable; $e_i(n) : \mathbb{Z} \to (0,1)$; bounded sequences $r_i(n)$, $c_i(n)$, $b_i(n)$, $a_{ij}(n)$, and $d_i(n) : \mathbb{Z} \to \mathbb{R}^+$; τ_{ij} and σ_i are positive integer; $\mathbb{Z}, \mathbb{R}^+$ denote the sets of all integers and all positive real numbers, respectively; Δ is the first-order forward difference operator $\Delta u_i(n) = u_i(n+1) - u_i(n)$; $\tau = \max\{\max_{1\leq i,j\leq N}\tau_{ij}, \max_{1\leq i\leq N}\sigma_i\} > 0$.

In [1], Liao et al. obtained sufficient conditions for permanence of the system (1.4).

They obtained what follows.

Lemma 1.1. *Assume that*

$$\min_{1\leq i\leq N} M_i \Delta_i > 1 \tag{1.5}$$

hold, then system (1.4) *is permanent, where*

$$\begin{aligned} M_i\Delta_i &= \frac{\exp\{b_i^u - 1\}}{a_{ii}^l \exp\{-b_i^u \tau_{ii}\}} \cdot \frac{a_{ii}^u \exp\left\{\tau_{ii}\left(\sum_{j=1}^{N} a_{ij}^u M_j + W_i d_i^u - b_i^l\right)\right\}}{b_i^l - \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j - d_i^u W_i}, \\ W_i &= \frac{r_i^u + c_i^u M_i}{e_i^l}, \qquad M_i = \frac{\exp(b_i^u - 1)}{a_{ii}^l \exp(-b_i^u \tau_{ii})}. \end{aligned} \tag{1.6}$$

Since

$$\exp\{b_i^u - 1\} > 0, \qquad a_{ii}^l \exp\{-b_i^u \tau_{ii}\} > 0, \qquad a_{ii}^u \exp\left\{\tau_{ii}\left(\sum_{j=1}^{N} a_{ij}^u M_j + W_i d_i^u - b_i^l\right)\right\} > 0. \tag{1.7}$$

Hence, the above inequality (1.5) *implies*

$$b_i^l - \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j - d_i^u W_i > 0. \tag{1.8}$$

That is

$$\begin{aligned} b_i^l &> \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j + d_i^u W_i \\ &= \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j + d_i^u \frac{r_i^u + c_i^u M_i}{e_i^l} \\ &= \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j + \frac{d_i^u r_i^u}{e_i^l} + \frac{d_i^u c_i^u M_i}{e_i^l}. \end{aligned} \tag{1.9}$$

It was shown that in [1] Liao et al. considered system (1.4) *where all coefficients* $r_i(n)$, $c_i(n)$, $d_i(n)$, $a_{ij}(n)$, $e_i(n)$, *and* $b_i(n)$ *were assumed to satisfy conditions* (1.9).

In this work, we shall study system (1.4) and get the same results as [1] do under the weaker assumption that

$$b_i^l > \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j + \frac{d_i^u r_i^u}{e_i^l}. \tag{1.10}$$

Our main results are the following Theorem 1.2.

Theorem 1.2. *Assume that* (1.10) *holds, then system* (1.4) *is permanent.*

Remark 1.3. The inequality (1.9) implies (1.10), but not conversely, for

$$\sum_{j=1, j\neq i}^{N} a_{ij}^u M_j + \frac{d_i^u r_i^u}{e_i^l} \leq \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j + \frac{d_i^u r_i^u}{e_i^l} + \frac{d_i^u c_i^u M_i}{e_i^l}. \tag{1.11}$$

Therefore, we have improved the permanence conditions of [1] for system (1.4).

Theorem 1.2 will be proved in Section 2. In Section 3, an example will be given to illustrate that (1.10) does not imply (1.9); that is, the condition (1.10) is better than (1.9).

2. Proof of Theorem 1.2

The following lemma can be found in [10].

Lemma 2.1. *Assume that* $A > 0$ *and* $y(0) > 0$*, and further suppose that*

(1)

$$y(n+1) \le Ay(n) + B(n), \quad n = 1, 2, \ldots. \tag{2.1}$$

Then for any integer $k \le n$*,*

$$y(n) \le A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1). \tag{2.2}$$

Especially, if $A < 1$ *and* B *is bounded above with respect to* M*, then*

$$\limsup_{n\to\infty} y(n) \le \frac{M}{1-A}. \tag{2.3}$$

(2)

$$y(n+1) \ge Ay(n) + B(n), \quad n = 1, 2, \ldots. \tag{2.4}$$

Then for any integer $k \le n$*,*

$$y(n) \ge A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1). \tag{2.5}$$

Especially, if $A < 1$ *and* B *is bounded below with respect to* m^**, then*

$$\liminf_{n\to\infty} y(n) \ge \frac{m^*}{1-A}. \tag{2.6}$$

Following comparison theorem of difference equation is Theorem 2.1 *of [11, page 241].*

Lemma 2.2. *Let* $n \in N_{n_0}^+ = \{n_0, n_0+1, \ldots, n_0+l, \ldots\}$*,* $r \ge 0$*. For any fixed* $n, g(n,r)$ *is a nondecreasing function with respect to* r*, and for* $n \ge n_0$*, following inequalities hold:* $y(n+1) \le g(n, y(n))$*,* $u(n+1) \ge g(n, u(n))$*. If* $g(n_0) \le u(n_0)$*, then* $y(n) \le u(n)$ *for all* $n \ge n_0$*.*

Now let us consider the following single species discrete model:

$$N(n+1) = N(n)\exp\{a(n) - b(n)N(n)\}, \tag{2.7}$$

where $\{a(n)\}$ and $\{b(n)\}$ are strictly positive sequences of real numbers defined for $n \in N = \{0, 1, 2, \ldots\}$ and $0 < a^l \le a^u$, $0 < b^l \le b^u$. Similarly to the proof of Propositions 1 and 3 in [12], we can obtain the following.

Lemma 2.3. *Any solution of system* (2.7) *with initial condition* $N(0) > 0$ *satisfies*

$$m \le \lim_{n\to\infty} \inf N(n) \le \lim_{n\to\infty} \sup N(n) \le M, \tag{2.8}$$

where

$$M = \frac{1}{b^l} \exp\{a^u - 1\}, \qquad m = \frac{a^l}{b^u} \exp\left\{a^l - b^u M\right\}. \tag{2.9}$$

The following lemma is direct conclusion of [1].

Lemma 2.4. *Let* $x(n) = (x_1(n), x_2(n), \ldots, x_N(n), u_1(n), u_2(n), \ldots, u_N(n))$ *denote any positive solution of system* (1.4)*.Then there exist positive constants* $M_i, W_i (i = 1, 2, \ldots, N)$ *such that*

$$\lim_{n\to\infty} \sup x_i(n) \le M_i, \qquad \lim_{n\to\infty} \sup u_i(n) \le W_i, \quad i = 1, 2, \ldots, N, \tag{2.10}$$

where

$$M_i = \frac{\exp\left(b_i^u - 1\right)}{a_{ii}^l \exp\left(-b_i^u \tau_{ii}\right)}, \qquad W_i = \frac{r_i^u + c_i^u M_i}{e_i^l} \quad (i = 1, 2, \ldots, N). \tag{2.11}$$

Proposition 2.5. *Suppose assumption* (1.10) *holds, then there exist positive constant* m_i *and* w_i *such that*

$$\lim_{n\to\infty} \inf x_i(n) \ge m_i, \qquad \lim_{n\to\infty} \inf u_i(n) \ge w_i. \tag{2.12}$$

Proof. We first prove $\lim_{n\to\infty} \inf x_i(n) \ge m_i$.

By Lemma 2.4 and by the first equation of system (1.4), we have

$$\begin{aligned} x_i(n+1) &= x_i(n) \exp\left\{ b_i(n) - \sum_{j=1}^{N} a_{ij}(n) x_j\left(n - \tau_{ij}\right) - d_i(n) u_i(n) \right\} \\ &\ge x_i(n) \exp\left\{ b_i(n) - \sum_{j=1}^{N} a_{\,ij}\ \left(M_j + \varepsilon\right) - d_i(n)(W_i + \varepsilon) \right\} \end{aligned} \tag{2.13}$$

for n sufficiently large, then

$$\prod_{s=n-\tau_{ii}}^{n-1} \frac{x_i(s+1)}{x_i(s)} \ge \exp\left\{ \sum_{s=n-\tau_{ii}}^{n-1} \left(b_i(s) - \sum_{j=1}^{N} a_{ij}(s)\left(M_j + \varepsilon\right) - d_i(s)(W_i + \varepsilon) \right) \right\}. \tag{2.14}$$

Thus

$$x_i(n-\tau_{ii}) \le x_i(n)\exp\left\{\sum_{s=n-\tau_{ii}}^{n-1} D_i(s)\right\}, \tag{2.15}$$

where

$$D_i(s) = \sum_{j=1}^{N} a_{ij}(s)\left(M_j+\varepsilon\right) + d_i(s)(W_i+\varepsilon) - b_i(s). \tag{2.16}$$

From the second equation of system (1.4), we have

$$\begin{aligned} u_i(n) &= (1-e_i(n))u_i(n) + c_i(n)x_i(n-\sigma_i) + r_i(n) \\ &\le \left(1-e_i^l\right)u_i(n) + c_i(n)x_i(n-\sigma_i) + r_i(n) \\ &:= A_i u_i(n) + B_i(n). \end{aligned} \tag{2.17}$$

Then, Lemma 2.1 implies that for any $k \le n-\tau_{ii}$,

$$\begin{aligned} u_i(n) &\le A_i^k u_i(n-k) + \sum_{j=0}^{k-1} A_i^j B_i(n-j-1) \\ &= A_i^k u_i(n-k) + \sum_{j=0}^{k-1} A_i^j\left[r_i(n-j-1) + c_i(n-j-1)x_i(n-j-1-\sigma_i)\right] \\ &\le A_i^k u_i(n-k) + \sum_{j=0}^{k-1} A_i^j\left[r_i(n-j-1) + c_i^u \exp\left\{(j+1+\sigma_i)D_i^u\right\}x_i(n)\right] \\ &\le A_i^k u_i(n-k) + \sum_{j=0}^{k-1} A_i^j r_i^u + \sum_{j=0}^{k-1} A_i^j c_i^u c_i^u \exp\left\{(j+1+\sigma_i)D_i^u\right\}x_i(n) \\ &\le A_i^k W_i + \frac{1-A_i^k}{1-A_i} r_i^u + H_i x_i(n), \end{aligned} \tag{2.18}$$

where

$$H_i = \left[\sum_{j=0}^{k-1} A_i^j c_i^u c_i^u \exp\left\{(j+1+\sigma_i)D_i^u\right\}\right]^u. \tag{2.19}$$

For any small positive constant $\varepsilon > 0$, there exists a $K > 0$ such that

$$\left(d_i^u W_i - \frac{r_i^u d_i^u}{1-A_i}\right)A_i^k < \varepsilon \quad \forall k > K. \tag{2.20}$$

From the first equation of system (1.4), (2.18), and (2.20), we have

$$\begin{aligned}
&x_i(n+1)\\
&\geq x_i(n)\exp\Bigg\{b_i(n)-\sum_{j=1,j\neq i}^{N}a_{ij}(n)M_j-a_{ii}^u\exp\{\tau_{ii}D_i^u\}x_i(n)\\
&\qquad\qquad -d_i^uW_iA_i^k-\frac{1-A_i^k}{1-A_i}r_i^ud_i^u-d_i^uH_ix_i(n)\Bigg\}\\
&=x_i(n)\exp\Bigg\{b_i(n)-\sum_{j=1,j\neq i}^{N}a_{ij}(n)M_j-\frac{r_i^ud_i^u}{1-A_i}-\left(d_i^uW_i-\frac{r_i^ud_i^u}{1-A_i}\right)A_i^k\\
&\qquad\qquad -\left(a_{ii}^u\exp\{\tau_{ii}D_i^u\}+d_i^uH_i\right)x_i(n)\Bigg\}\\
&\geq x_i(n)\exp\Bigg\{b_i(n)-\sum_{j=1,j\neq i}^{N}a_{ij}(n)M_j-\frac{r_i^ud_i^u}{1-A_i}-\varepsilon-\left(a_{ii}^u\exp\{\tau_{ii}D_i^u\}+d_i^uH_i\right)x_i(n)\Bigg\}.
\end{aligned}\tag{2.21}$$

By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
\liminf_{n\to\infty}x_i(n)\geq{}&\frac{b_i^l-\sum_{j=1,j\neq i}^{N}a_{ij}^uM_j-\left(r_i^ud_i^u/e_i^l\right)-\varepsilon}{a_{ii}^u\exp\{\tau_{ii}D_i^u\}+d_i^uH_i}\\
&\cdot\exp\left\{b_i^l-\sum_{j=1,j\neq i}^{N}a_{ij}^uM_j-\frac{r_i^ud_i^u}{e_i^l}-\varepsilon-\left(a_{ii}^u\exp\{\tau_{ii}D_i^u\}+d_i^uH_i\right)M_i\right\}.
\end{aligned}\tag{2.22}$$

Setting $\varepsilon\to 0$ in (2.22) leads to

$$\begin{aligned}
\liminf_{n\to\infty}x_i(n)\geq{}&\frac{b_i^l-\sum_{j=1,j\neq i}^{N}a_{ij}^uM_j-\left(r_i^ud_i^u/e_i^l\right)}{a_{ii}^u\exp\{\tau_{ii}D_i^u\}+d_i^uH_i}\\
&\cdot\exp\left\{b_i^l-\sum_{j=1,j\neq i}^{N}a_{ij}^uM_j-\frac{r_i^ud_i^u}{e_i^l}-\left(a_{ii}^u\exp\{\tau_{ii}D_i^u\}+d_i^uH_i\right)M_i\right\}.
\end{aligned}\tag{2.23}$$

Thus,

$$\liminf_{n\to\infty}x_i(n)\geq m_i,\tag{2.24}$$

where

$$m_i = \frac{b_i^l - \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j - \left(r_i^u d_i^u / e_i^l\right)}{a_{ii}^u \exp\{\tau_{ii} D_i^u\} + d_i^u H_i} \cdot \exp\left\{ b_i^l - \sum_{j=1, j\neq i}^{N} a_{ij}^u M_j - \frac{r_i^u d_i^u}{e_i^l} - \left(a_{ii}^u \exp\{\tau_{ii} D_i^u\} + d_i^u H_i\right) M_i \right\}. \tag{2.25}$$

Second, we prove $\lim_{n\to\infty}\inf u_i(n) \geq w_i$. For enough small $\varepsilon > 0$, from the second equation of system (1.4), we have

$$u_i(n+1) = (1 - e_i(n))u_i(n) + r_i(n) + c_i(n)x_i(n - \sigma_i) \geq r_i^l + c_i^l(m_i - \varepsilon) + \left(1 - e_i^u\right)u_i(n) \tag{2.26}$$

for sufficient large n. Hence

$$u_i(n) \geq \left(1 - e_i^u\right)^n u_i(0) + \frac{1 - \left(1 - e_i^u\right)}{e_i^u}\left(r_i^l + c_i^l(m_i - \varepsilon)\right). \tag{2.27}$$

Thus, we obtain

$$\lim_{n\to\infty}\inf u_i(n) \geq w_i. \tag{2.28}$$

This completes the proof. □

3. An Example

In this section, we give an example to illustrate that (1.10) does not imply (1.9). Consider the two-species system with delays and feedback controls for $t \in (-\infty, +\infty)$

$$\begin{aligned}
x_1(n+1) &= x_1(n)\exp\left\{\frac{1}{2} - 2x_1(n-1) - \frac{1}{2}x_2(n-3) - \frac{1}{2}u_1(n)\right\}, \\
x_2(n+1) &= x_2(n)\exp\left\{\frac{1}{2} - \frac{1}{2}x_1(n-3) - 2x_2(n-1) - \frac{1}{2}u_2(n)\right\}, \\
\Delta u_1(n+1) &= \frac{1}{8} - \frac{1}{2}u_1(n) + x_1(n-4), \\
\Delta u_2(n+1) &= \frac{1}{8} - \frac{1}{2}u_2(n) + x_2(n-8).
\end{aligned} \tag{3.1}$$

We have

$$b_1^l = b_2^l = \frac{1}{2}, \qquad M_1 = M_2 = \frac{1}{2}, \qquad a_{12}^u M_2 + d_1^u \frac{r_1^u}{e_1^l} = \frac{3}{8}, \qquad a_{21}^u M_1 + d_2^u \frac{r_2^u}{e_2^l} = \frac{3}{8}. \tag{3.2}$$

So

$$b_1^l > a_{12}^u M_2 + d_1^u \frac{r_1^u}{e_1^l}, \qquad b_2^l > a_{21}^u M_1 + d_2^u \frac{r_2^u}{e_2^l}. \tag{3.3}$$

Therefore (1.10) holds.
But

$$\frac{1}{2} = b_1^l < a_{12}^u M_2 + d_1^u \frac{r_1^u + c_1^u M_1}{e_1^l} = \frac{7}{8}, \qquad \frac{1}{2} = b_2^l < a_{21}^u M_1 + d_2^u \frac{r_2^u + c_2^u M_2}{e_2^l} = \frac{7}{8}. \tag{3.4}$$

Thus (1.9) does not hold.

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