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Research Article

A Discrete Equivalent of the Logistic Equation

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A discrete equivalent and not analogue of the well-known logistic differential equation is proposed. This discrete equivalent logistic equation is of the Volterra convolution type, is obtained by use of a functional-analytic method, and is explicitly solved using the z-transform method. The connection of the solution of the discrete equivalent logistic equation with the solution of the logistic differential equation is discussed. Also, some differences of the discrete equivalent logistic equation and the well-known discrete analogue of the logistic equation are mentioned. It is hoped that this discrete equivalent of the logistic equation could be a better choice for the modelling of various problems, where different versions of known discrete logistic equations are used until nowadays.

1. Introduction

The well-known logistic differential equation was originally proposed by the Belgian mathematician Pierre-François Verhulst (1804–1849) in 1838, in order to describe the growth of a population $P(t)$ under the assumptions that the rate of growth of the population was proportional to

(A1) the existing population and

(A2) the amount of available resources.

When this problem is "translated" into mathematics, results to the differential equation

$$\frac{dP(t)}{dt} = rP(t)\left[1 - \frac{P(t)}{K}\right], \quad P(0) = P_0, \tag{1.1}$$

where t denotes time, P_0 is the initial population, and r, K are constants associated with the growth rate and the carrying capacity of the population. A more general form of (1.1), which will be used in this paper, is

$$y'(t) = \beta y(t) - \gamma [y(t)]^2, \quad y(0) = a, \tag{1.2}$$

where $t \in \mathbb{R}$ and a, β, γ are real constants with $\gamma, \beta \neq 0$ (in order to exclude trivial cases).

Equation (1.2) can be regarded as a Bernoulli differential equation or it can be solved by applying the simplest method of separation of variables. In any case, the solution of the initial value problem (1.2) is given by

$$y(t) = \frac{a\beta}{a\gamma + (\beta - a\gamma)e^{-\beta t}}. \tag{1.3}$$

Although, (1.2) can be considered as a simple differential equation, in the sense that it is completely solvable by use of elementary techniques of the theory of differential equations, it has tremendous and numerous applications in various fields. The first application of (1.2) was already mentioned, and it is connected with population problems, and more generally, problems in ecology. Other applications of (1.2) appear in problems of chemistry, medicine (especially in modelling the growth of tumors), pharmacology (especially in the production of antibiotic medicines) [1], epidemiology [2, 3], atmospheric pollution, flow in a river [4], and so forth.

Nowadays, the logistic differential equation can be found in many biology textbooks and can be considered as a cornerstone of ecology. However, it has also received much criticism by several ecologists. One may find the basis of these criticisms and several paradoxes in [5].

However, as it often happens in applications, when modelling a realistic problem, one may decide to describe the problem in terms of differential equations or in terms of difference equations. Thus, the initial value problem (1.2) which describes the population problem studied by Verhulst, could be formulated instead as an initial value problem of a difference equation. Also, there is a great literature on topics regarding discrete analogues of the differential calculus. In this context, the general difference equation

$$x_{n+1} = \lambda x_n - \mu x_n^2, \quad x_1 = a \quad (\text{or } x_0 = a) \tag{1.4}$$

has been known as the discrete logistic equation and it serves as an analogue to the initial value problem (1.2) (see, e.g., [6]).

There are several ways to "end up" with (1.4) starting from (1.1) or (1.2) as:

(a) by iterating the function $F(x) = \mu x(1-x)$, $x \in [0,1]$, $\mu > 0$ which gives rise to the difference equation [7, page 43]

$$x_{n+1} = \mu x_n (1 - x_n), \tag{1.5}$$

(b) by discretizing (1.1) using a forward difference scheme for the derivative, which gives rise to the difference equation

$$x_{n+1} = (1 + rh)x_n - \frac{rh}{K}x_n^2, \quad x_0 = a, \tag{1.6}$$

where $x_n \simeq P(nh)$, h being the step size of the scheme [8], or

(c) by "translating" the population problem studied by Verhulst in terms of differences: if p_n is the population under study at time $n \in \mathbb{N}$, its growth is indicated by $\Delta p_n = p_{n+1} - p_n$. Thus, according to the assumptions (A1) and (A2), the following initial value problem appears:

$$\Delta p_n = rp_n\left(1 - \frac{p_n}{K}\right), \quad p_0 = P_0 \Longrightarrow p_{n+1} = (1 + r)p_n - \frac{r}{K}p_n^2, \quad p_0 = P_0. \tag{1.7}$$

Notice of course that all three equations (1.5)–(1.7) are special cases of (1.4).

The similarities between (1.2) and (1.4) are obvious even at a first glance. However, these similarities are only superficial, since there are many qualitative differences between their solutions. Perhaps the most important difference between (1.2) and (1.4) is that in contrast to (1.2), (the solution of which is given explicitly in (1.3)) (1.4) (or even its simplest form (1.5)) cannot be solved explicitly so as to obtain its solution in closed form (except for certain values of the parameters) (see, e.g., [6, page 120] and [7, page 14]).

Also, (1.4) is one of the simplest examples of discrete autonomous equations leading to chaos, whereas the solution (1.3) of (1.2) guarantees the regularity of (1.2). Finally, it worths mentioning that the numerical scheme (1.6) or other nonlinear difference equations approximations of (1.2) given for example in [6, page 120] or in [8, pages 297–303] gives rise to approximate solutions of (1.2), which are qualitatively different from the true solution (1.3). These solutions are many times referred to as spurious solutions. These spurious solutions "disappear" when better approximations are used, for example, by applying nonstandard difference schemes (see, e.g., [9–11]).

Recently, in [12, 13] a nonstandard way was proposed for solving "numerically" an ordinary differential equation accompanied with initial or boundary conditions in the real or complex plane. This method was successfully applied to the Duffing equation, the Lorenz system, and the Blasius equation. The technique used is based on the equivalent transformation of the ordinary differential equation under consideration to an ordinary difference equation through an operator equation utilizing a specific isomorphism in specific Banach spaces. One of the aims of the present paper is to apply this technique to (1.2) so as to obtain the following equation:

$$ny_{n+1} - \beta_1 y_n = -\gamma_1 \sum_{k=1}^{n} y_k y_{n-k+1}, \quad y_1 = a, \tag{1.8}$$

where β_1, γ_1 are constants, which in the rest of the paper will be called *discrete equivalent logistic equation*. It should be mentioned at this point that although the application of the technique in [12] to (1.2) is interesting on each own, its side effect, that is, the derivation of (1.8) is more important, since it is proposed as the discrete equivalent of (1.2). It is also emphasized that

(1.8) is the discrete equivalent logistic equation derived by straightforward analytical means unlike the known versions of discrete logistic equation such as (1.4). Thus, the solutions of (1.8) are expected to have similar behavior with those of the differential logistic equation and not the peculiar characteristics appearing in the solutions of (1.4) discussed above. Conclusively it is the main aim of the present paper to convince the reader, that (1.8) deserves to be called discrete equivalent logistic equation. It is also hoped that (1.8) could be a better choice for the modelling of various problems, where different versions of known discrete logistic equations are used until nowadays.

Equation (1.8) is a nonlinear Volterra difference equation of convolution type. The Volterra difference equations have been thoroughly studied, and there exists an enormous literature for them. For example, there are several results concerning the boundedness, asymptotic behavior, admissibility, and periodicity of the solution of a Volterra difference equation. Although the list of papers cited in the present work is by no means exhaustive, the review papers [14, 15] on the boundedness, stability, and asymptoticity of Volterra difference equations should be mentioned (see also the references in these two papers). Indicatively, one could also mention the papers [16–32], the general results of which can also be applied to convolution-type Volterra difference equations. Also, in [33–36], linear Volterra difference equations of convolution type are exclusively studied.

In Section 2, (1.8) is fully derived. Moreover, in the same section conditions are given for the existence of a unique solution of (1.2) in the Banach space

$$H_1(\Delta) = \left\{ f : \Delta \longrightarrow \mathbb{R} \text{ where } f(x) = \sum_{n=1}^{\infty} f_n x^{n-1} \text{ analytic in } \Delta \text{ with } \sum_{n=1}^{\infty} \left| f_n \right| < +\infty \right\}, \tag{1.9}$$

where $\Delta = \{x \in \mathbb{R} : |x| < 1\} = (-1, 1)$ and of (1.8) in the Banach space

$$\ell_1 = \left\{ f_n : \mathbb{N} \longrightarrow \mathbb{R} \text{ with } \sum_{n=1}^{\infty} \left| f_n \right| < +\infty \right\}. \tag{1.10}$$

It should be mentioned at this point that the issue of the existence of a unique solution in ℓ_1 of the discrete analogue logistic equation (1.4) has been studied in [37] under the framework of a more general difference equation.

In Section 3, (1.8) is explicitly solved by applying the z-transform method. Finally, in Section 4, several differences between (1.4) and (1.8) are discussed. These differences concern their solutions (see Figure 1), their bifurcation diagrams, and their stability.

2. Derivation of the Discrete Equivalent Logistic Equation

In this section, the method proposed in [12, 13] will be applied to (1.2). As already mentioned in the introduction, the main idea is to transform (1.2) into an equivalent operator equation in an abstract Banach space and from this to deduce the equivalent difference equation (1.8). This method can be applied only when the ordinary differential equation under consideration is studied in the Banach space $H_1(\Delta)$ defined by (1.9). Moreover, the solution of (1.8), which will eventually give the solution of (1.2), belongs to the Banach space of absolutely summable sequences ℓ_1 defined by (1.10).

2.1. Basic Definitions and Propositions

First of all, define the Hilbert space $H_2(\Delta)$ by

$$H_2(\Delta) = \left\{ f : \Delta \longrightarrow \mathbb{R} \text{ where } f(x) = \sum_{n=1}^{\infty} f_n x^{n-1} \text{ analytic in } \Delta \text{ with } \sum_{n=1}^{\infty} \left|f_n\right|^2 < +\infty \right\}, \tag{2.1}$$

where $\Delta = \{x \in \mathbb{R} : |x| < 1\} = (-1, 1)$. Denote now by H an abstract separable Hilbert space over the real field, with the orthonormal base $\{e_n\}$, $n = 1, 2, 3, \ldots$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm in H, respectively. Define also in H the shift operator V and its adjoint V^*

$$\begin{gathered} Ve_n = e_{n+1}, \quad n = 1, 2, 3, \ldots, \\ V^* e_n = e_{n-1}, \quad n = 2, 3, \ldots, \qquad V^* e_1 = 0, \end{gathered} \tag{2.2}$$

as well as the diagonal operator C_0

$$C_0 e_n = n e_n, \quad n = 1, 2, 3, \ldots. \tag{2.3}$$

Proposition 2.1. *The representation*

$$\langle f_x, f \rangle = \sum_{n=1}^{\infty} f_n x^{n-1} = f(x), \quad x \in \Delta, \tag{2.4}$$

is a one-by-one mapping from H onto $H_2(\Delta)$ which preserves the norm, where $f_x = \sum_{n=1}^{\infty} x^{n-1} e_n$, $f_0 = e_1$, is the complete system in H of eigenvectors of V^* and $f = \sum_{n=1}^{\infty} f_n e_n = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n$ an element of H [38].

The unique element $f = \sum_{n=1}^{\infty} f_n e_n = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n$ appearing in (2.4) is called the *abstract form* of $f(x)$ in H. In general, if $G(f(x))$ is a function from $H_2(\Delta)$ to $H_2(\Delta)$ and $N(f)$ is the unique element in H for which

$$G(f(x)) = \langle f_x, N(f) \rangle, \tag{2.5}$$

then $N(f)$ is called the *abstract form* of $G(f(x))$ in H.

Consider now the linear manifold of all $f(x) \in H_2(\Delta)$ which satisfy the condition $\sum_{n=1}^{\infty} |f_n| < +\infty$. Define the norm $\|f(x)\|_{H_1(\Delta)} = \sum_{n=1}^{\infty} |f_n|$. Then, this manifold becomes the Banach space $H_1(\Delta)$ defined by (1.9). Denote also by H_1 the corresponding by the representation (2.4), abstract Banach space of the elements $f = \sum_{n=1}^{\infty} f_n e_n = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n \in H$ for which $\sum_{n=1}^{\infty} |f_n| < +\infty$.

The following properties hold [38–40]:

(1) H_1 is invariant under the operators V^k, $(V^*)^k$, $k = 1, 2, 3, \ldots$ as well as under every bounded diagonal operator;

(2) the abstract form of $f'(x)$ is the element $C_0 V^* f$, that is, $f'(x) = \langle f_x, C_0 V^* f \rangle$;

(3) the abstract form of $[f(x)]^2$ is the element $N(f) = f(V)f$, that is,

$$[f(x)]^2 = \langle f_x, N(f) \rangle, \quad \text{where } f(V) = \sum_{n=1}^{\infty} f_n V^{n-1}, \quad \text{and } \|f(V)\|_1 = \|f\|_1^2; \tag{2.6}$$

(4) the operator $N(f)$ is the Frechét differentiable in H_1.

Proposition 2.2. *The linear function*

$$\begin{gathered} \phi : H_1 \longrightarrow \ell_1, \\ \phi(f) = \langle f, e_n \rangle = f_n \end{gathered} \tag{2.7}$$

is an isomorphism from H_1 onto ℓ_1, that is, it is a $1-1$ mapping from H_1 onto ℓ_1 which preserves the norm [37].

Remark 2.3. The basic Propositions 2.1 and 2.2 were originally proved for complex valued sequences and functions (z also in $\mathbb{C}$), as well as for H, H_1 defined over the complex field. However, in the present paper a restriction to the real plane is made due to the physical applications of the logistic equation.

2.2. Derivation of (1.8)

In order to apply the method of [12, 13] to the logistic differential equation (1.2), it is considered that $|t| < T$, $T > 0$ finite and (1.2) is restricted to $\Delta = (-1, 1)$ by using the simple transformation $x = t/T$, $y(t) = y(xT) = Y(x)$. Then, (1.2) becomes

$$Y'(x) - \beta T Y(x) = -\gamma T [Y(x)]^2, \quad Y(x = 0) = a, \ \gamma \neq 0. \tag{2.8}$$

Using Proposition 2.1 and what mentioned in Section 2.1, (2.8) is rewritten as

$$\begin{gathered} \langle f_x, C_0 V^* Y \rangle - \beta T \langle f_x, Y \rangle = -\gamma T \langle f_x, N(Y) \rangle \\ \Longleftrightarrow \langle f_x, C_0 V^* Y - \beta T Y + \gamma T N(Y) \rangle = 0, \quad Y = \sum_{n=1}^{\infty} Y_n e_n = \sum_{n=1}^{\infty} \langle Y, e_n \rangle e_n, \end{gathered} \tag{2.9}$$

which holds for all f_x, $x \in \Delta$. But f_x is the complete system in H of eigenvectors of V^*, which gives the following equivalent operator equation:

$$C_0 V^* Y - \beta T Y = -\gamma T N(Y). \tag{2.10}$$

By taking the inner product of both parts of (2.10) with e_n and taking into consideration Proposition 2.2 one obtains

$$\begin{aligned}
&\langle C_0 V^* Y, e_n\rangle - \beta T\langle Y, e_n\rangle = -\gamma T\langle N(Y), e_n\rangle \\
&\Longrightarrow \langle V^* Y, C_0 e_n\rangle - \beta T\langle Y, e_n\rangle = -\gamma T\left\langle \sum_{k=1}^{\infty} Y_k V^{k-1} Y, e_n\right\rangle \\
&\Longrightarrow n\langle V^* Y, e_n\rangle - \beta T\langle Y, e_n\rangle = -\gamma T\sum_{k=1}^{\infty} Y_k \left\langle V^{k-1} Y, e_n\right\rangle \\
&\Longrightarrow n\langle Y, V e_n\rangle - \beta T\langle Y, e_n\rangle = -\gamma T\sum_{k=1}^{\infty} Y_k \left\langle Y, (V^*)^{k-1} e_n\right\rangle \\
&\Longrightarrow n\langle Y, e_{n+1}\rangle - \beta T\langle Y, e_n\rangle = -\gamma T\sum_{k=1}^{\infty} Y_k \langle Y, e_{n-k+1}\rangle \\
&\Longrightarrow nY_{n+1} - \beta_1 Y_n = -\gamma_1 \sum_{k=1}^{n} Y_k Y_{n-k+1},
\end{aligned} \tag{2.11}$$

where $\beta_1 = \beta T$, $\gamma_1 = \gamma T$, which is (1.8), *the discrete equivalent logistic equation*. It is obvious that in (2.11), it is $n = 1, 2, 3, \ldots$ and that $Y_1 = a$, since $Y(x = 0) = \sum_{n=1}^{\infty} Y_n x^{n-1}|_{x=0} = Y_1 = a$ and $\langle Y, e_1\rangle = Y_1$.

Of course, for all the above to hold, one has to assure that $Y(x) \in H_1(\Delta)$ and $Y_n \in \ell_1$. This is guaranteed by the theorems presented in the next section.

2.3. Existence and Uniqueness Theorems

As mentioned in Section 2.2, conditions must be found so that $Y(x) \in H_1(\Delta)$ and $Y_n \in \ell_1$. In order to do so, it is helpful to work with the operator equation (2.10), which is equivalent to both (2.8) and (2.11). Equation (2.10) can be rewritten as

$$V^* Y - \beta T B_0 Y = -\gamma T B_0 N(Y), \tag{2.12}$$

where B_0 is the bounded operator $B_0 e_n = (1/n) e_n$, $n = 1, 2, 3, \ldots$ or as

$$(I - \beta T V B_0) Y = -\gamma T V B_0 N(Y) + c e_1, \tag{2.13}$$

due to the definition of V^*, where c is a constant which can be defined by taking the inner product of both parts of (2.13) with the element e_1. Indeed, this gives

$$\begin{aligned}
&\langle Y, e_1\rangle - \beta T\langle V B_0 Y, e_1\rangle = -\gamma T\langle V B_0 N(Y), e_1\rangle + c\langle e_1, e_1\rangle \\
&\Longrightarrow \left\langle \sum_{n=1}^{\infty} Y_n e_n, e_1\right\rangle - \beta T\langle B_0 Y, V^* e_1\rangle = -\gamma T\langle B_0 N(Y), V^* e_1\rangle + c\langle e_1, e_1\rangle \\
&\Longrightarrow Y_1 - \beta T\langle B_0 Y, 0\rangle = -\gamma T\langle B_0 N(Y), 0\rangle + c \Longrightarrow c = Y_1 = a,
\end{aligned} \tag{2.14}$$

since $Y(z=0)=a$. Thus (2.13) becomes

$$(I-\beta TVB_0)Y=-\gamma TVB_0N(Y)+ae_1. \tag{2.15}$$

In order to assure the existence of a unique solution of the nonlinear operator equation (2.15) in H_1, some conditions must be imposed on the parameters appearing in the equation. Moreover, since it is a non linear equation, a fixed-point theorem would be useful. Indeed, the following well-known theorems concerning the inversion of linear operators and the existence of a unique fixed point of an equation will be used.

Theorem 2.4. *If T is a linear bounded operator of a Hilbert space H or a Banach space B, with $\|T\|<1$, then $I-T$ is invertible with $\|(I-T)^{-1}\|\le 1/(1-\|T\|)$ and is defined on all H or B (see, e.g., [41, pages 70-71]).*

Theorem 2.5. *If $f: X\to X$ is holomorphic, that is, its Fréchet derivative exists, and $f(X)$ lies strictly inside X, then f has a unique fixed point in X, where X is a bounded, connected, and open subset of a Banach space E. (By saying that a subset X′ of X lies strictly inside X, it is meant that there exists an $\epsilon_1>0$ such that $\|x'-y\|>\epsilon_1$ for all $x'\in X'$ and $y\in E-X$) [42].*

If it is assumed that

$$|\beta|T<1, \tag{2.16}$$

then $\|-\beta TVB_0\|_1<1$ and due to Theorem 2.4, the operator $(I-\beta TVB_0)^{-1}$ is defined on all H_1 and is bounded by $1/(1-|\beta|T)$. Thus, (2.15) takes the form

$$Y=(I-\beta TVB_0)^{-1}\left[-\gamma TVB_0N(Y)+ae_1\right]=g(Y), \tag{2.17}$$

from which one finds that

$$\|g(Y)\|_1\le\frac{1}{1-|\beta|T}\left[|\gamma|T\|Y\|_1^2+|a|\right]. \tag{2.18}$$

Suppose that $\|Y\|_1\le R$. Then, from (2.18) it is obvious that

$$\|g(Y)\|_1\le\frac{1}{1-|\beta|T}\left[|\gamma|TR^2+|a|\right]. \tag{2.19}$$

Define the function $P(R)=R-(|\gamma|T/(1-|\beta|T))R^2$, which attains its maximum $P_0=(1-|\beta|T)/4|\gamma|T$ at the point $R_0=(1-|\beta|T)/2|\gamma|T$. Then, for $\|Y\|_1\le R_0-\epsilon<R_0$, $\epsilon>0$, it follows that if

$$\frac{|a|}{1-|\beta|T}\le P_0-\epsilon<P_0, \tag{2.20}$$

or if

$$|a| < \frac{(1-|\beta|T)^2}{4|\gamma|T}, \tag{2.21}$$

then (2.19) gives $\|g(Y)\|_1 \le P_0 - \epsilon + R_0 - P_0 = R_0 - \epsilon < R_0$, which means that Theorem 2.5 is applied to (2.17). Thus, the following has just been proved.

Theorem 2.6. *If conditions* (2.16) *and* (2.21) *hold, then the abstract operator equation* (2.10) *has a unique solution in* H_1 *bounded by* $R_0 = (1-|\beta|T)/2|\gamma|T$.

Equivalently, this theorem can be "translated" to the following two.

Theorem 2.7. *If conditions* (2.16) *and* (2.21) *hold, then the discrete equivalent logistic equation* (2.11)*, has a unique solution in* ℓ_1 *bounded by* R_0.

Theorem 2.8. *If conditions* (2.16) *and* (2.21) *hold, then the logistic differential equation* (1.2) *has a unique analytic solution of the form* $y(t) = \sum_{n=1}^{\infty} Y_n(t^{n-1}/T^{n-1})$ *bounded by* R_0*, which together with its first derivative converges absolutely for* $|t| < T$*. (The coefficients* Y_n *are defined of course by* (2.11)*).*

Remark 2.9. Following the same technique as the one applied for the proof of Theorems 2.6 and 2.7, conditions were given in [37], so that the difference equation (1.4) is to have a unique solution in ℓ_1 or $\ell_1 + \{(\lambda-1)/\mu\}$, $\mu \ne 0$. Indeed, it was proved that

(a) if $|\lambda| < 1$ and $|a| < (1-|\lambda|)/4|\mu|$, then (1.4) has a unique solution in ℓ_1 and

(b) if $|2-\lambda| < 1$ and $|a-((\lambda-1)/\mu)| < (1-|2-\lambda|)/4|\mu|$, then (1.4) has a unique solution in $\ell_1 + \{(\lambda-1)/\mu\}$, $\mu \ne 0$.

It is obvious that conditions (2.16) and (2.21) are very similar to the conditions derived in [37].

3. Solution of the Discrete Equivalent Logistic Equation

In this section, the discrete equivalent logistic equation (2.11), that is, equation

$$nY_{n+1} - \beta_1 Y_n = -\gamma_1 \sum_{k=1}^{n} Y_k Y_{n-k+1}, \quad n = 1,2,3,\ldots, \ Y_1 = a \tag{3.1}$$

will be solved by applying the well-known z-transform method (see, e.g., [6, pages 77–82], [7, Chapter 6], and [8, pages 159–172]). Suppose $Z(Y_n) = \sum_{j=0}^{\infty} Y_j z^{-j} = \widetilde{Y}(z)$ is the z-transform of the unknown sequence Y_n. It is obvious that Y_0 is required. However, since n starts from 1, an "overstepping" should be made, by defining arbitrarily Y_0 in such a way so that (3.1) is consistent. Indeed, by setting $n = 0$ to (3.1), one obtains $Y_0 = 0$.

Equation (3.1) is of convolution type, and it can be rewritten as

$$nY_{n+1} - \beta_1 Y_n = -\gamma_1 Y_n * Y_{n+1}. \tag{3.2}$$

Taking the z-transform of both sides of (3.2), one obtains

$$\begin{aligned}
&Z(nY_{n+1}) - \beta_1 Z(Y_n) = -\gamma_1 Z(Y_n * Y_{n+1}) \\
&\Longrightarrow -z\frac{d}{dz}[Z(Y_{n+1})] - \beta_1 Z(Y_n) = -\gamma_1 Z(Y_n) Z(Y_{n+1}) \\
&\Longrightarrow -z\frac{d}{dz}\left[z\widetilde{Y}(z) - zY_0\right] - \beta_1 \widetilde{Y}(z) = -\gamma_1 \widetilde{Y}(z)\left[z\widetilde{Y}(z) - zY_0\right] \\
&\Longrightarrow \widetilde{Y}'(z) + \frac{z+\beta_1}{z^2}\widetilde{Y}(z) = \frac{\gamma_1}{z}\left[\widetilde{Y}(z)\right]^2,
\end{aligned} \tag{3.3}$$

which is a Bernoulli differential equation with respect to $\widetilde{Y}(z)$. (Remember that the original differential equation (1.2) was also of Bernoulli type!)

The solution of (3.3) is

$$\widetilde{Y}(z) = \frac{1}{((\gamma/\beta) + ce^{-\beta_1/z})z}, \tag{3.4}$$

where c is the arbitrary constant of integration. This constant c can be determined by using the following property of this z-transform (since $Y_0 = 0$):

$$\lim_{z\to\infty} z\widetilde{Y}(z) = Y_1, \tag{3.5}$$

from which it is easily obtained that $c = (\beta - a\gamma)/\beta a$. Thus, (3.4) becomes

$$\widetilde{Y}(z) = \frac{a\beta}{\left[a\gamma + (\beta - a\gamma)e^{-\beta T/z}\right]z}. \tag{3.6}$$

It should be mentioned at this point that since $Y_n \in \ell_1$ according to Theorem 2.7, the function $\widetilde{Y}(z)$ defined by (3.6) is analytic for $|z| \geq 1$ (see [7, Theorem 6.14, page 292]). By expanding $\widetilde{Y}(z)$, it is found that

$$Y_n = \frac{1}{n!}\left.\frac{d^n f}{d\omega^n}\right|_{\omega=0}, \qquad f(\omega) = \frac{a\beta\omega}{a\gamma + (\beta - a\gamma)e^{-\beta T\omega}}, \tag{3.7}$$

which is the solution of (3.1).

Remark 3.1. The well-known properties of the z-transform

$$\lim_{z\to\infty} \widetilde{Y}(z) = Y_0 (= 0), \qquad \lim_{z\to 1}(z-1)\widetilde{Y}(z) = \lim_{n\to\infty} Y_n (= 0 \text{ since } Y_n \in \ell_1), \tag{3.8}$$

are of course satisfied.

Remark 3.2. The solution (3.7) of (3.1) is expected, since the analytic solution (1.3) of (1.2) is known. However, the same technique can be applied to other nonlinear ordinary differential equations of interest for which their analytic solutions are not available. In this sense, a connection between specific ordinary differential and difference equations can be established.

4. Discussion

In this section, a comparison between the solutions of (1.4) and (1.8) will be given. However, since the most commonly used form of the discrete logistic equation is (1.5)

$$x_{n+1} = \mu x_n (1 - x_n), \tag{4.1}$$

a comparison between the preceding equation and (1.8) for $\beta_1 = \gamma_1 = \mu$, that is,

$$n y_{n+1} - \mu y_n = -\mu \sum_{k=1}^{n} y_k y_{n-k+1}, \tag{4.2}$$

will be given.

First of all, something should be mentioned regarding the stability of the equilibrium points of (4.1) and (4.2). At Remark 2.9, it was mentioned that if

$$|\mu| < 1, \quad |x_1| < \frac{1 - |\mu|}{4|\mu|}, \quad \mu \neq 0, \tag{4.3}$$

equation (4.1) has a unique solution in ℓ_1, whereas if

$$|2 - \mu| < 1, \quad \left| x_1 - \frac{\mu - 1}{\mu} \right| < \frac{1 - |2 - \mu|}{4|\mu|}, \quad \mu \neq 0, \tag{4.4}$$

equation (4.1) has a unique solution in $\ell_1 + \{(\mu - 1)/\mu\}$.

This means that 0 and $(\mu - 1)/\mu$ are locally asymptotically stable equilibrium points of (4.1) with regions of attraction given by (4.3) and (4.4), respectively. Actually, these results hold for a, μ, and x_n complex and not only real as regarded in the present paper (see [37]). However, when restricted to $\mathbb{R}$ and x_n is not necessary a sequence in ℓ_1 the following more general result holds [7, pages 43–45]:

"The equilibrium point 0 of (4.1) is asymptotically stable for $0 < \mu < 1$ and unstable for $\mu > 1$, and the equilibrium point $(\mu - 1)/\mu$ of (4.1) is asymptotically stable for $1 < \mu \leq 3$ and unstable for $\mu > 3$."

In a similar way, due to Theorem 2.7, if

$$|\mu| < 1, \quad |y_1| < \frac{(1 - |\mu|)^2}{4|\mu|}, \tag{4.5}$$

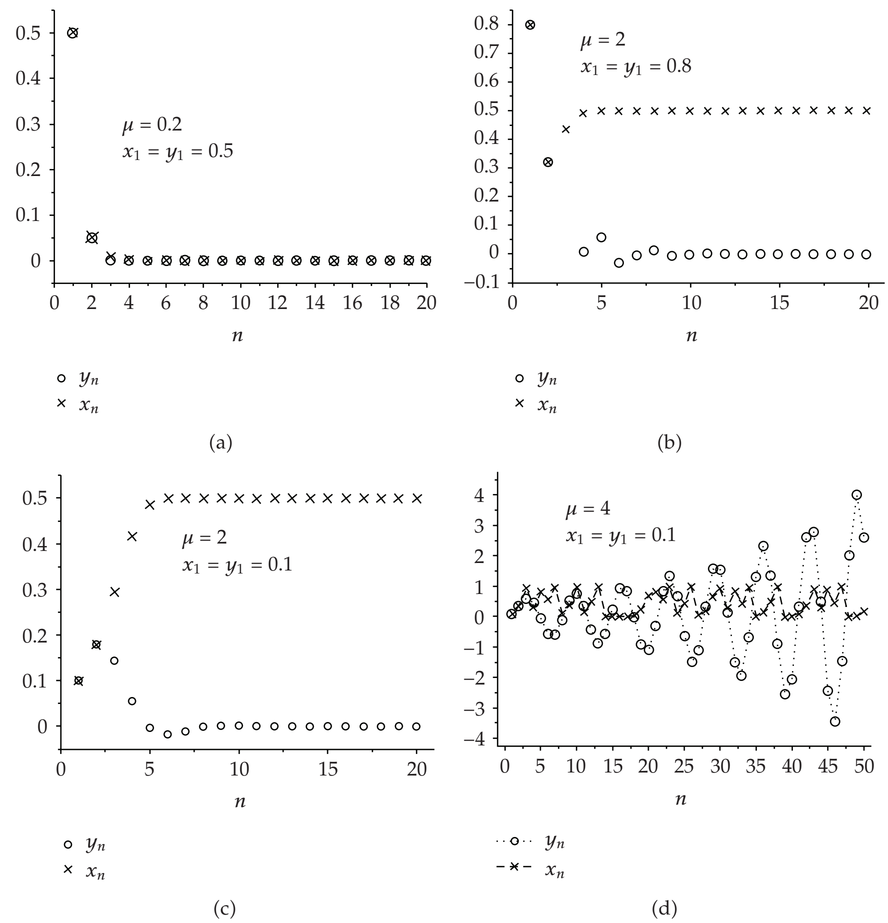

Figure 1: Solutions of (4.1) and (4.2).

equation (4.2) has a unique solution in ℓ_1, and, thus, 0 is a locally asymptotically stable equilibrium point of the nonautonomous equation (4.2) with region of attraction given by (4.5).

In Figure 1, the solutions of (4.1) and (4.2) are graphically represented for some representative values of the parameters. More precisely in Figure 1(a), the solutions of (4.1) and (4.2) are given for $\mu = 0.2$ and initial conditions $x_1 = y_1 = 0.5$. For these values, both conditions (4.3) and (4.5) are satisfied and thus these solutions of (4.1) and (4.2) both belong in ℓ_1. Moreover, it is obvious from Figure 1(a) that both x_n and y_n exhibit a very similar behavior and of course 0 is an asymptotically stable equilibrium point.

In Figures 1(b) and 1(c), the solutions of (4.1) and (4.2) are given for $\mu = 2$ and initial conditions $x_1 = y_1 = 0.8$ and $x_1 = y_1 = 0.1$, respectively. For these value of μ, the first condition of both (4.3) and (4.5) is violated. However, it is known that for this value of μ, the point $(\mu - 1)/\mu = 0.5$ is an asymptotically stable equilibrium point of (4.1) [7, pages 43–45]. In these cases, x_n and y_n do not exhibit a similar behavior, but both of them tend to a specific point as n tends to infinity, x_n to the equilibrium point 0.5 and y_n to 0. This observation is a quite promising fact that the equilibrium point 0 of (4.2) may remain asymptotically stable even for values of the parameters that do not satisfy (4.5) (since this condition is not necessary and sufficient). However, this needs further study.

Finally, in Figure 1(d), the solutions of (4.1) and (4.2) are given for $\mu = 4$ and initial conditions $x_1 = y_1 = 0.1$. For this value of μ, the first condition of both (4.3) and (4.5) is again violated. In this case, however, x_n and y_n exhibit again a very similar behavior, as they both seem to oscillate with continuously growing amplitude to infinity.

Last, but not least, it should be mentioned that as is well known, (4.1) exhibits chaotic behavior and this can be deduced from its period doubling bifurcation diagram [7, page 47]. However, numerical results for (4.2) indicate no chaotic behavior. Actually, its bifurcation diagram is a straight line at 0. For $\mu > 3.45$ (which is a region of values for which condition (4.5) is violated), y_n starts for some initial conditions to "blow up."

5. Conclusions

In this paper, a discrete equivalent to the well-known logistic differential equation is proposed. This discrete equivalent equation is of the Volterra convolution type and is obtained using a functional-analytic technique. From what mentioned in Sections 3 and 4, it seems that this discrete equivalent logistic equation better resembles the behaviour of the corresponding logistic differential equation in the sense that (a) it can be solved explicitly and (b) it does not seem to present chaotic behaviour. This author believes that although the discrete equivalent logistic equation was derived in a very specific way under the conditions of Theorem 2.6, it would be interesting to further study this equation on its own regardless of conditions with respect to the appearing parameters. In other words, the study of the discrete equivalent logistic equation, not only in the space ℓ_1 but also in several other spaces of sequences, could give rise to interesting results. It would also be interesting to investigate the possibility, (1.8) being useful in the study of biology or physics problems.

Dedication

This work is dedicated to the memory of the author's Professor, P. D. Siafarikas, who left so early at the age of 57.

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