Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 508374, 9 pages
doi:10.1155/2010/508374

Research Article

Almost Automorphic Solutions to Abstract Fractional Differential Equations

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Received 24 October 2009; Accepted 12 January 2010

Academic Editor: Gaston Mandata N'Guerekata

A new and general existence and uniqueness theorem of almost automorphic solutions is obtained for the semilinear fractional differential equation $D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t))$ $(1 < \alpha < 2)$, in complex Banach spaces, with *Stepanov-like almost automorphic coefficients*. Moreover, an application to a fractional relaxation-oscillation equation is given.

1. Introduction

In this paper, we investigate the existence and uniqueness of almost automorphic solutions to the following semilinear abstract fractional differential equation:

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $1 < \alpha < 2$, $A : \mathfrak{D}(A) \subset X \to X$ is a sectorial operator of type ω in a Banach space X, and $f : \mathbb{R} \times X \to X$ is Stepanov-like almost automorphic in $t \in \mathbb{R}$ satisfying some kind of Lipschitz conditions in $x \in X$. In addition, the fractional derivative is understood in the Riemann-Liouville's sense.

Recently, fractional differential equations have attracted more and more attentions (cf. [1–8] and references therein). On the other hand, the Stepanov-like almost automorphic problems have been studied by many authors (cf., e.g., [9, 10] and references therein). Stimulated by these works, in this paper, we study the almost automorphy of solutions to the fractional differential equation (1.1) with Stepanov-like almost automorphic coefficients.

A new and general existence and uniqueness theorem of almost automorphic solutions to the equation is established. Moreover, an application to fractional relaxation-oscillation equation is given to illustrate the abstract result.

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers, and by X a complex Banach space. In addition, we assume $1 \leq p < +\infty$ if there is no special statement. Next, let us recall some definitions of almost automorphic functions and Stepanov-like almost automorphic functions (for more details, see, e.g., [9–11]).

Definition 1.1. A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t) := \lim_{n \to \infty} f(t + s_n) \tag{1.2}$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n \to \infty} g(t - s_n) = f(t) \tag{1.3}$$

for each $t \in \mathbb{R}$. Denote by $AA(X)$ the set of all such functions.

Definition 1.2. The Bochner transform $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, of a function $f(t)$ on $\mathbb{R}$, with values in X, is defined by

$$f^b(t, s) := f(t + s). \tag{1.4}$$

Definition 1.3. The space $BS^p(\mathrm{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in X such that

$$\|f\|_{S^p} := \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{1/p} < +\infty. \tag{1.5}$$

It is obvious that $L^p(\mathbb{R}; X) \subset BS^p(\mathrm{X}) \subset L^p_{\mathrm{loc}}(\mathbb{R}; \mathrm{X})$ and $BS^p(\mathrm{X}) \subset BS^q(\mathrm{X})$ whenever $p \geq q \geq 1$.

Definition 1.4. The space $AS^p(\mathrm{X})$ of S^p-almost automorphic functions (S^p-a.a. for short) consists of all $f \in BS^p(\mathrm{X})$ such that $f^b \in AA(L^p(0, 1; X))$. In other words, a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}; X)$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathrm{X})$ is almost automorphic in the sense that for every sequence of real numbers (s'_n), there exist a subsequence (s_n) and a function $g \in L^p_{\text{loc}}(\mathbb{R}; X)$ such that

$$\lim_{n\to\infty}\left(\int_0^1 \|f(t+s_n+s)-g(t+s)\|^p ds\right)^{1/p}=0,$$
$$\lim_{n\to\infty}\left(\int_0^1 \|g(t-s_n+s)-f(t+s)\|^p ds\right)^{1/p}=0, \tag{1.6}$$

for each $t \in \mathbb{R}$.

Remark 1.5. It is clear that if $1 \leq p < q < \infty$ and $f \in L^q_{\text{loc}}(\mathbb{R}; X)$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also if $f \in AA(X)$, then f is S^p-almost automorphic for any $1 \leq p < \infty$.

Definition 1.6. A function $f : \mathbb{R} \times X \to X$, $(t,u) \mapsto f(t,u)$ with $f(\cdot,u) \in L^p_{\text{loc}}(\mathbb{R}, X)$ for each $u \in X$ is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly for $u \in X$, if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) and a function $g : \mathbb{R} \times X \to X$ with $g(\cdot,u) \in L^p_{\text{loc}}(\mathbb{R}, X)$ such that

$$\lim_{n\to\infty}\left(\int_0^1 \|f(t+s_n+s,u)-g(t+s,u)\|^p ds\right)^{1/p}=0,$$
$$\lim_{n\to\infty}\left(\int_0^1 \|g(t-s_n+s,u)-f(t+s,u)\|^p ds\right)^{1/p}=0, \tag{1.7}$$

for each $t \in \mathbb{R}$ and for each $u \in X$. We denote by $AS^p(\mathbb{R} \times X, X)$ the set of all such functions.

2. Almost Automorphic Solution

First, let us recall that a closed and densely defined linear operator A is called sectorial if there exist $0 < \theta < \pi/2$, $M > 0$, and $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\},$$
$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \omega + S_\theta. \tag{2.1}$$

Recently, in [3], Cuesta proved that if A is sectorial operator for some $0 < \theta < \pi(1-\alpha/2)$ $(1 < \alpha < 2)$, $M > 0$, and $\omega < 0$, then there exits $C > 0$ such that

$$\|E_\alpha(t)\| \leq \frac{CM}{1+|\omega|t^\alpha}, \quad t \geq 0, \tag{2.2}$$

where

$$E_\alpha(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda, \tag{2.3}$$

where γ is a suitable path lying outside the sector $\omega + S_\theta$.

In addition, by [2], we have the following definition.

Definition 2.1. A function $u : \mathbb{R} \to X$ is called a mild solution of (1.1) if $s \to E_\alpha(t-s)f(s,u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and

$$u(t) = \int_{-\infty}^{t} E_\alpha(t-s)f(s,u(s))ds, \quad t \in \mathbb{R}. \tag{2.4}$$

Lemma 2.2. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded and linear operators such that*

$$\|S(t)\| \leq \phi(t), \quad t \in \mathbb{R}^+, \tag{2.5}$$

where $\phi \in L^1(\mathbb{R}^+)$ is nonincreasing. Then, for each $f \in AS^1(X)$,

$$\int_{-\infty}^{t} S(t-s)f(s)ds \in AA(X). \tag{2.6}$$

Proof. For each $n \in \mathbb{N}$, let

$$f_n(t) := \int_{t-n}^{t-n+1} S(t-s)f(s)ds = \int_{n-1}^{n} S(s)f(t-s)ds, \quad t \in \mathbb{R}. \tag{2.7}$$

In addition, for each $n \in \mathbb{N}$, by the principle of uniform boundedness,

$$M_n := \sup_{n-1\leq s\leq n} \|S(s)\| < +\infty. \tag{2.8}$$

Fix $n \in \mathbb{N}$ and $t \in \mathbb{R}$. We have

$$\begin{aligned}\left\|f_n(t+h) - f_n(t)\right\| &\leq \int_{n-1}^{n} \|S(s)\| \cdot \left\|f(t+h-s) - f(t-s)\right\|ds \\ &\leq M_n \cdot \int_{t-n}^{t-n+1} \left\|f(s+h) - f(s)\right\|ds.\end{aligned} \tag{2.9}$$

In view of $f \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$, we get

$$\lim_{h\to 0}\int_{t-n}^{t-n+1} \left\|f(s+h) - f(s)\right\|ds = 0, \tag{2.10}$$

which yields that

$$\lim_{h\to 0}\left\| f_n(t+h) - f_n(t)\right\| = 0. \tag{2.11}$$

This means that $f_n(t)$ is continuous.

Fix $n \in \mathbb{N}$. By the definition of $AS^1(\mathbb{X})$, for every sequence of real numbers (s'_m), there exist a subsequence (s_m) and a function $g \in L^1_{\text{loc}}(\mathbb{R};\mathbb{X})$ such that

$$\lim_{m\to\infty}\int_0^1 \left\| f(t+s_m+s) - g(t+s)\right\| ds = \lim_{m\to\infty}\int_0^1 \left\| g(t-s_m+s) - f(t+s)\right\| ds = 0, \tag{2.12}$$

for each $t \in \mathbb{R}$. Combining this with

$$\begin{aligned}\left\| f_n(t+s_m) - \int_{n-1}^{n} S(s)g(t-s)ds\right\| &\le M_n \cdot \int_{n-1}^{n} \left\| f(t+s_m-s) - g(t-s)\right\| ds \\ &= M_n \cdot \int_0^1 \left\| f(t-n+s_m+s) - g(t-n+s)\right\| ds,\end{aligned} \tag{2.13}$$

we get

$$\lim_{m\to\infty} f_n(t+s_m) = \int_{n-1}^{n} S(s)g(t-s)ds \tag{2.14}$$

for each $t \in \mathbb{R}$. Similar to the above proof, one can show that

$$\lim_{m\to\infty}\int_{n-1}^{n} S(s)g(t-s_m-s)ds = f_n(t) \tag{2.15}$$

for each $t \in \mathbb{R}$. Therefore, $f_n \in AA(\mathbb{X})$ for each $n \in \mathbb{N}$.

Noticing that

$$\begin{aligned}\left\| f_n(t)\right\| &\le \int_{n-1}^{n} \phi(s)\cdot \left\| f(t-s)\right\| ds \le \phi(n-1)\cdot \|f\|_{S^1}, \\ \sum_{n=1}^{\infty}\phi(n-1)\cdot \|f\|_{S^1} &\le \left(\phi(0) + \sum_{n=2}^{\infty}\int_{n-2}^{n-1}\phi(t)dt\right)\cdot \|f\|_{S^1} \\ &\le \left(\phi(0) + \|\phi\|_{L^1(\mathbb{R}^+)}\right)\cdot \|f\|_{S^1} < +\infty,\end{aligned} \tag{2.16}$$

we know that $\sum_{n=1}^{\infty} f_n(t)$ is uniformly convergent on $\mathbb{R}$. Thus

$$\int_{-\infty}^{t} S(t-s)f(s)ds = \sum_{n=1}^{\infty} f_n(t) \in AA(\mathbb{X}). \tag{2.17}$$

□

Remark 2.3. For the case of $f \in AA(X)$, the conclusion of Lemma 2.2 was given in [1, Lemma 3.1].

The following theorem will play a key role in the proof of our existence and uniqueness theorem.

Theorem 2.4 (see [11]). *Assume that*

(i) $f \in AS^p(\mathbb{R} \times X, X)$ *with* $p > 1$;

(ii) *there exists a nonnegative function* $L \in AS^r(\mathbb{R})$ *with* $r \geq \max\{p, p/(p-1)\}$ *such that for all* $u, v \in X$ *and* $t \in \mathbb{R}$,

$$\|f(t,u) - f(t,v)\| \leq L(t)\|u - v\|; \tag{2.18}$$

(iii) $x \in AS^p(X)$ *and* $K = \overline{\{x(t) : t \in \mathbb{R}\}}$ *is compact in* X.

Then there exists $q \in [1, p)$ *such that* $f(\cdot, x(\cdot)) \in AS^q(X)$.

Now, we are ready to present the existence and uniqueness theorem of almost automorphic solutions to (1.1).

Theorem 2.5. *Assume that* A *is sectorial operator for some* $0 < \theta < \pi(1 - \alpha/2)$, $M > 0$ *and* $\omega < 0$; *and the assumptions (i) and (ii) of Theorem 2.4 hold. Then* (1.1) *has a unique almost automorphic mild solution provided that*

$$\|L\|_{S^1} < \frac{\alpha \sin(\pi/\alpha)}{CM\left(\alpha \sin(\pi/\alpha) + |\omega|^{-1/\alpha}\pi\right)}. \tag{2.19}$$

Proof. For each $\varphi \in AA(X)$, let

$$\mathfrak{F}(\varphi)(t) := \int_{-\infty}^{t} E_\alpha(t-s) f(s, \varphi(s)) ds, \quad t \in \mathbb{R}. \tag{2.20}$$

In view of $\overline{\{\varphi(t) : t \in \mathbb{R}\}}$ which is compact in X, by Theorem 2.4, there exists $q \in [1, p)$ such that $f(\cdot, \varphi(\cdot)) \in AS^q(X)$. On the other hand, by (2.2), we have

$$\|E_\alpha(t)\| \leq \frac{CM}{1 + |\omega| t^\alpha}, \quad t \geq 0. \tag{2.21}$$

Since $1 < \alpha < 2$, $CM/(1 + |\omega| t^\alpha) \in L^1(\mathbb{R}^+)$ and is nonincreasing. So Lemma 2.2 yields that $\mathfrak{F}(\varphi) \in AA(X)$. This means that $\mathfrak{F}$ maps $AA(X)$ into $AA(X)$.

For each $\varphi, \psi \in AA(X)$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}
\|\mathfrak{F}(\varphi)(t) - \mathfrak{F}(\psi)(t)\| &\le \int_{-\infty}^{t} \|E_\alpha(t-s)\| \cdot \|f(s,\varphi(s)) - f(s,\psi(s))\| ds \\
&\le \int_{-\infty}^{t} \frac{CM}{1+|\omega|(t-s)^\alpha} L(s) ds \cdot \|\varphi - \psi\| \\
&\le \int_{0}^{+\infty} \frac{CM}{1+|\omega|s^\alpha} L(t-s) ds \cdot \|\varphi - \psi\| \\
&= \sum_{k=0}^{\infty} \int_{k}^{k+1} \frac{CM}{1+|\omega|s^\alpha} L(t-s) ds \cdot \|\varphi - \psi\| \\
&\le \sum_{k=0}^{\infty} \frac{CM}{1+|\omega|k^\alpha} \int_{k}^{k+1} L(t-s) ds \cdot \|\varphi - \psi\| \\
&\le \sum_{k=0}^{\infty} \frac{CM}{1+|\omega|k^\alpha} \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\| \\
&\le \left(CM + \sum_{k=1}^{\infty} \int_{k-1}^{k} \frac{CM}{1+|\omega|s^\alpha} ds \right) \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\| \\
&= \left(CM + \int_{0}^{+\infty} \frac{CM}{1+|\omega|s^\alpha} ds \right) \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\| \\
&= CM \left(1 + \frac{|\omega|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \right) \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\|,
\end{aligned} \tag{2.22}$$

which gives

$$\|\mathfrak{F}(\varphi) - \mathfrak{F}(\psi)\| \le CM \left(1 + \frac{|\omega|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \right) \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\|. \tag{2.23}$$

In view of (2.19), $\mathfrak{F}$ is a contraction mapping. On the other hand, it is well known that $AA(X)$ is a Banach space under the supremum norm. Thus, $\mathfrak{F}$ has a unique fixed point $u \in AA(X)$, which satisfies

$$u(t) = \int_{-\infty}^{t} E_\alpha(t-s) f(s,u(s)) ds, \tag{2.24}$$

for all $t \in \mathbb{R}$. Thus (1.1) has a unique almost automorphic mild solution. □

In the case of $L(t) \equiv L$, by following the proof of Theorem 2.5 and using the standard contraction principle, one can get the following conclusion.

Theorem 2.6. *Assume that A is sectorial operator for some $0 < \theta < \pi(1-\alpha/2)$, $M > 0$ and $\omega < 0$; and the assumptions (i) and (ii) of Theorem 2.4 hold with $L(t) \equiv L$, then (1.1) has a unique almost automorphic mild solution provided that*

$$L < \frac{\alpha \sin(\pi/\alpha)}{CM|\omega|^{-1/\alpha}\pi}. \tag{2.25}$$

Remark 2.7. Theorem 2.6 is due to [2, Theroem 3.4] in the case of $f(t,u)$ being almost automorphic in t. Thus, Theorem 2.6 is a generalization of [2, Theroem 3.4].

At last, we give an application to illustrate the abstract result.

Example 2.8. Let us consider the following fractional relaxation-oscillation equation given by

$$\partial_t^{\alpha} u(t,x) = \partial_x^2 u(t,x) - \mu u(t,x) + \partial_t^{\alpha-1}[a(t)\sin(u(t,x))], \quad t \in \mathbb{R}, x \in [0,\pi], \tag{2.26}$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}, \tag{2.27}$$

where $1 < \alpha < 2$, $\mu > 0$, and

$$a(t) = \begin{cases} \sin \dfrac{1}{2+\cos n + \cos \pi n}, & t \in (n-\varepsilon, n+\varepsilon), n \in \mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.28}$$

for some $\varepsilon \in (0, 1/2)$.

Let $X = L^2[0,\pi]$, $Au = u'' - \mu u$ with

$$\mathfrak{D}(A) = \left\{ u \in L^2[0,\pi] : u'' \in L^2[0,\pi], u(0) = u(\pi) = 0 \right\}, \tag{2.29}$$

and $f(t,\varphi)(s) = a(t)\sin(\varphi(s))$ for $\varphi \in X$ and $s \in [0,\pi]$. Then (2.26) is transformed into (1.1). It is well known that A is a sectorial operator for some $0 < \theta < \pi/2$, $M > 0$ and $\omega < 0$. By [10, Example 2.3], $a(t) \in AS^2(\mathbb{R})$. Then $f \in AS^2(\mathbb{R} \times X, X)$. In addition, for each $t \in \mathbb{R}$ and $u, v \in X$,

$$\|f(t,u) - f(t,v)\| = \left(\int_0^{\pi} |a(t)\sin(u(s)) - a(t)\sin(v(s))|^2 ds \right)^{1/2} \le |a(t)| \cdot \|u - v\|. \tag{2.30}$$

Since

$$\||a(\cdot)|\|_{S^1} = \sup_{t \in \mathbb{R}} \int_t^{t+1} |a(s)| ds \le 2\varepsilon, \tag{2.31}$$

by Theorem 2.5, there exists a unique almost automorphic mild solution to (2.26) provided that $1 < \alpha < 2(1 - \theta/\pi)$ and ε is sufficiently small.

Remark 2.9. In the above example, for any $\varepsilon > 0$, $f(t, u)$ is Lipschitz continuous about u uniformly in t with Lipschitz constant $L \equiv 1$, this means that $f(t, u)$ has a better Lipschitz continuity than (2.30). However, one cannot ensure the unique existence of almost automorphic mild solution to (2.26) when

$$\frac{\alpha \sin(\pi/\alpha)}{CM|\omega|^{-1/\alpha}\pi} \leq 1, \tag{2.32}$$

by using Theorem 2.6. On the other hand, it is interesting to note that one can use Theorem 2.5 to obtain the existence in many cases under this restriction.

Acknowledgments

The authors are grateful to the referee for valuable suggestions and comments, which improved the quality of this paper. H. Ding acknowledges the support from the NSF of China, the NSF of Jiangxi Province of China (2008GQS0057), and the Youth Foundation of Jiangxi Provincial Education Department(GJJ09456). J. Liang and T. Xiao acknowledge the support from the NSF of China (10771202), the Research Fund for Shanghai Key Laboratory of Modern Applied Mathematics (08DZ2271900), and the Specialized Research Fund for the Doctoral Program of Higher Education of China (2007035805).

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