Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 584890, 6 pages
doi:10.1155/2010/584890

Research Article

On a Max-Type Difference Equation

Ali Gelisken, Cengiz Cinar, and Ibrahim Yalcinkaya

Mathematics Department, Ahmet Kelesoglu Education Faculty, Selcuk University, Meram Yeni Yol, 42090 Konya, Turkey

Correspondence should be addressed to Ali Gelisken, aligelisken@yahoo.com.tr

Received 8 December 2009; Revised 20 April 2010; Accepted 23 April 2010

Academic Editor: Gaston M. N'Guérékata

We prove that every positive solution of the max-type difference equation $x_n = \max\{A/x^{\alpha}_{n-p}, B/x^{\beta}_{n-k}\}$, $n = 0, 1, 2, \ldots$ converges to $\overline{x} = \max\{A^{1/(1+\alpha)}, B^{1/(1+\beta)}\}$ where p, k are positive integers, $0 < \alpha, \beta < 1$, and $0 < A, B$.

1. Introduction

Recently, the study of max-type difference equations attracted a considerable attention. Although max-type difference equations are relatively simple in form, it is unfortunately extremely difficult to understand thoroughly the behavior of their solutions; see, for example, [1–20] and the relevant references cited therein. The max operator arises naturally in certain models in automatic control theory (see [13, 14]). Furthermore, difference equation appear naturally as a discrete analogue and as a numerical solution of differential and delay differential equations having applications and various scientific branches, such as in ecology, economy, physics, technics, sociology, and biology.

In [20], Yang et al. proved that every positive solution of the difference equation

$$x_n = \max\left\{\frac{1}{x^{\alpha}_{n-1}}, \frac{B}{x_{n-2}}\right\}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

converges to $\overline{x} = 1$ or eventually periodic with period 4, where $0 < \alpha < 1$ and $0 < A$.

In [9], We proved that every positive solution of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-3}^{\alpha}}\right\}, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

converges to $\overline{x} = 1$ or eventually periodic with period 2, where $0 < \alpha < 1$ and $0 < A$.

In [17], Sun proved that every positive solution of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-2}^{\beta}}\right\}, \quad n = 0, 1, 2, \ldots \tag{1.3}$$

converges to $\overline{x} = \max\{A^{1/(1+\alpha)}, B^{1/(1+\beta)}\}$ where $0 < \alpha, \beta < 1$, and $0 < A, B$.

The following difference equation is more general than (1.3):

$$x_n = \max\left\{\frac{A}{x_{n-p}^{\alpha}}, \frac{B}{x_{n-k}^{\beta}}\right\}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where p, k are positive integers, $0 < \alpha, \beta < 1$, $0 < A, B$, and initial conditions are positive real numbers.

In this paper, we investigate the asymptotic behavior of the positive solutions of (1.4). We prove that every positive solution of (1.4) converges to $\overline{x} = \max\{A^{1/(1+\alpha)}, B^{1/(1+\beta)}\}$. Clearly, we can assume that $p \le k$ without loss of generality.

2. Main Results

2.1. The Case $B^{1/(1+\beta)} \le A^{1/(1+\alpha)}$

In this section, we consider the asymptotic behavior of the positive solutions of (1.4) in the case $B^{1/(1+\beta)} \le A^{1/(1+\alpha)}$.

It is easy to see that by the change

$$x_n = A^{1/(1+\alpha)} C^{y_n} \quad \text{for } 0 < C < 1,\ n \ge -k. \tag{2.1}$$

Equation (1.4) is transformed into the difference equation

$$C^{y_n} = \max\left\{C^{-\alpha y_{n-p}}, DC^{-\beta y_{n-k}}\right\}, \tag{2.2}$$

where $D = B/A^{(1+\beta)/(1+\alpha)}$ and the initial conditions are real numbers. Since $B^{1/(1+\beta)} \le A^{1/(1+\alpha)}$, we have $D \le 1$.

We need the following two lemmas in order to prove the main result of this section.

Lemma 2.1. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of* (2.2)*. If* $D = 1$*, then*

$$|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\} \quad \forall n \ge -k. \tag{2.3}$$

Proof. Clearly, (2.2) implies the following difference equation:

$$y_n = \min\{-\alpha y_{n-p}, -\beta y_{n-k}\} \quad \forall n \ge -k. \tag{2.4}$$

From (2.4), we get the following statements.

(i) If $y_{n-p} \ge 0$ and $y_{n-k} \ge 0$, then $|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\}$.

(ii) If $y_{n-p} \le 0$ and $y_{n-k} \le 0$, then $|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\}$.

(iii) If $y_{n-p} \ge 0$ and $y_{n-k} \le 0$, then $|y_n| = \alpha|y_{n-p}|$.

(iv) If $y_{n-p} \le 0$ and $y_{n-k} \ge 0$, then $|y_n| = \beta|y_{n-k}|$.

From the above statements, we have $|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\}$ for all $n \ge -k$. Therefore, the proof is complete. □

Lemma 2.2. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of* (2.2)*. If* $D < 1$*, then*

$$|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\} \quad \forall n \ge -k. \tag{2.5}$$

Proof. Assume that $C = D$. Then (2.2) implies the following difference equation:

$$y_n = \min\{-\alpha y_{n-p}, 1 - \beta y_{n-k}\} \quad \forall n \ge -k. \tag{2.6}$$

From (2.6), we get the following statements.

(i) If $y_{n-p} \ge 0$ and $y_{n-k} \ge 0$, then $|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\}$.

(ii) If $y_{n-p} \le 0$ and $y_{n-k} \le 0$, then $|y_n| \le \alpha|y_{n-p}|$.

(iii) If $y_{n-p} \ge 0$ and $y_{n-k} \le 0$, then $|y_n| = \alpha|y_{n-p}|$.

(iv) If $y_{n-p} \le 0$ and $y_{n-k} \ge 0$, then $|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\}$.

From the above statements, we have $|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\}$ for all $n \ge -k$. Therefore, the proof is complete. □

Theorem 2.3. *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (1.4) *where* $B^{1/(1+\beta)} \le A^{1/(1+\alpha)}$*. Then* $\{x_n\}_{n=-k}^{\infty}$ *converges to* $\overline{x} = A^{1/(1+\alpha)}$.

Proof. Assume that $D = 1$. $\{y_n\}_{n=-k}^{\infty}$ is a solution of (2.2). If it is proved that $\{y_n\}_{n=-k}^{\infty}$ converges to zero as $n \to \infty$, then $\{x_n\}_{n=-k}^{\infty}$ converges to $\overline{x} = A^{1/(1+\alpha)}$.

From Lemma 2.1, we have that

$$|y_n| \le \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\} \quad \forall n \ge -k. \tag{2.7}$$

Let $\gamma = \max\{\alpha, \beta\}$. Immediately, we have that the following inequality

$$|y_n| \leq \gamma \max\{|y_{n-p}|, |y_{n-k}|\} \quad \forall n \geq -k. \tag{2.8}$$

From (2.8) and by induction, we get

$$|y_n| \leq \gamma^{[|n/k|]+1} \max_{1\leq j\leq k}\{y_{-j}\} \quad \forall n \geq -k. \tag{2.9}$$

From (2.9), it is clear that $\{y_n\}_{n=-k}^{\infty}$ converges to zero as $n \to \infty$.

Now, we assume that $D < 1$. From Lemma 2.2, we have that

$$|y_n| \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}| - 1\} \leq \max\{\alpha|y_{n-p}|, \beta|y_{n-k}|\} \quad \forall n \geq -k. \tag{2.10}$$

Then, the rest of proof is similar to the case $D = 1$ and will be omitted. Therefore, the proof is complete. □

2.2. *The Case* $A^{1/(1+\alpha)} < B^{1/(1+\beta)}$

In this section, we consider the asymptotic behavior of the positive solutions of (1.4) in the case $A^{1/(1+\alpha)} < B^{1/(1+\beta)}$.

It is easy to see that by the change

$$x_n = B^{1/(1+\beta)} C^{y_n} \quad \text{for } C = \frac{A}{B^{(1+\alpha)/(1+\beta)}}, \; n \geq -k. \tag{2.11}$$

Equation (1.4) is transformed into the difference equation:

$$y_n = \min\{1 - \alpha y_{n-p}, -\beta y_{n-k}\}, \tag{2.12}$$

where initial conditions are real numbers.

We need the following lemma in order to prove the main result of this section.

Lemma 2.4. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of* (2.12). *Then*

$$|y_n| \leq \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\} \quad \forall n \geq -k. \tag{2.13}$$

Proof. From (2.12), we get the following statements.

(i) If $y_{n-p} \geq 0$ and $y_{n-k} \geq 0$, then $|y_n| \leq \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\}$.

(ii) If $y_{n-p} \leq 0$ and $y_{n-k} \leq 0$, then $|y_n| \leq \beta|y_{n-k}|$.

(iii) If $y_{n-p} \geq 0$ and $y_{n-k} \leq 0$, then $|y_n| \leq \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\}$.

(iv) If $y_{n-p} \leq 0$ and $y_{n-k} \geq 0$, then $|y_n| = \beta|y_{n-k}|$.

From the above statements, we have $|y_n| \leq \max\{\alpha|y_{n-p}| - 1, \beta|y_{n-k}|\}$ for all $n \geq -k$. Therefore, the proof is complete. □

Theorem 2.5. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.4) where $A^{1/(1+\alpha)} < B^{1/(1+\beta)}$. Then $\{x_n\}_{n=-k}^{\infty}$ converges to $\overline{x} = B^{1/(1+\beta)}$.*

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of (2.12). To prove the desired result, it suffices to prove that $\{y_n\}_{n=-k}^{\infty}$ converges to zero.

From Lemma 2.4, we have that

$$\left|y_n\right| \leq \max\{\alpha\left|y_{n-p}\right| - 1, \beta\left|y_{n-k}\right|\} \leq \max\{\alpha\left|y_{n-p}\right|, \beta\left|y_{n-k}\right|\} \quad \forall n \geq -k. \tag{2.14}$$

From (2.14) and by induction, we get

$$\left|y_n\right| \leq \gamma^{[|n/k|]+1} \max_{1\leq j\leq k}\{y_{-j}\} \quad \forall n \geq -k. \tag{2.15}$$

From (2.15), it is clear that $\{y_n\}_{n=-k}^{\infty}$ converges to zero as $n \to \infty$. Therefore, the proof is complete. □

Acknowledgment

The authors are grateful to the anonymous referees for their valuable suggestions that improved the quality of this study.

References

[1] R. M. Abu-Saris and F. M. Allan, "Periodic and nonperiodic solutions of the difference equation $x_{n+1} = \max\{x_n^2, A\}/x_n x_{n-1}$," in *Advances in Difference Equations (Veszprém, 1995)*, pp. 9–17, Gordon and Breach, Amsterdam, The Netherlands, 1997.

[2] A. M. Amleh, J. Hoag, and G. Ladas, "A difference equation with eventually periodic solutions," *Computers & Mathematics with Applications*, vol. 36, no. 10–12, pp. 401–404, 1998.

[3] K. S. Berenhaut, J. D. Foley, and S. Stević, "Boundedness character of positive solutions of a max difference equation," *Journal of Difference Equations and Applications*, vol. 12, no. 12, pp. 1193–1199, 2006.

[4] W. J. Briden, E. A. Grove, G. Ladas, and C. M. Kent, "Eventually periodic solutions of $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$," *Communications on Applied Nonlinear Analysis*, vol. 6, no. 4, pp. 31–43, 1999.

[5] W. J. Briden, E. A. Grove, G. Ladas, and L. C. McGrath, "On the nonautonomous equation $x_{n+1} = \max\{A_n/x_n, B_n, /x_{n-1}\}$," in *New Developments in Difference Equations and Applications (Taipei, 1997)*, pp. 49–73, Gordon and Breach, Amsterdam, The Netherlands, 1999.

[6] C. Çinar, S. Stević, and I. Yalçinkaya, "On positive solutions of a reciprocal difference equation with minimum," *Journal of Applied Mathematics & Computing*, vol. 17, no. 1-2, pp. 307–314, 2005.

[7] A. Gelişken, C. Çinar, and R. Karataş, "A note on the periodicity of the Lyness max equation," *Advances in Difference Equations*, vol. 2008, Article ID 651747, 5 pages, 2008.

[8] A. Gelişken, C. Çinar, and I. Yalçinkaya, "On the periodicity of a difference equation with maximum," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 820629, 11 pages, 2008.

[9] A. Gelişken and C. Çinar, "On the global attractivity of a max-type difference equation," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 812674, 5 pages, 2009.

[10] E. A. Grove, C. Kent, G. Ladas, and M. A. Radin, "On the $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$ with a period 3 parameter," in *Fields Institute Communications*, vol. 29, pp. 161–180, American Mathematical Society, Providence, RI, USA, 2001.

[11] G. Ladas, "On the recursive sequence $x_n = \max\{A_1/x_{n-1}, A_2/x_{n-2}, \dots, A_p/x_{n-p}\}$," *Journal of Difference Equations and Applications*, vol. 2, no. 3, pp. 339–341, 1996.

[12] D. P. Mishev, W. T. Patula, and H. D. Voulov, "A reciprocal difference equation with maximum," *Computers & Mathematics with Applications*, vol. 43, no. 8-9, pp. 1021–1026, 2002.

[13] A. D. Myškis, "Some problems in the theory of differential equations with deviating argument," *Uspekhi Matematicheskikh Nauk*, vol. 32, no. 2(194), pp. 173–202, 1977.

[14] E. P. Popov, *Automatic Regulation and Control*, Nauka, Moscow, Russia, 1966.

[15] I. Szalkai, "On the periodicity of the sequence $x_{n+1} = \max\{A_0/x_0, A_1/x_{n-1}, \dots, A_k/x_{n-k}\}$," *Journal of Difference Equations and Applications*, vol. 5, no. 1, pp. 25–29, 1999.

[16] S. Stević, "On the recursive sequence $x_{n+1} = \max\{c, x_n^p/x_{n-1}^p\}$," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 791–796, 2008.

[17] F. Sun, "On the asymptotic behavior of a difference equation with maximum," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 243291, 6 pages, 2008.

[18] H. D. Voulov, "On the periodic character of some difference equations," *Journal of Difference Equations and Applications*, vol. 8, no. 9, pp. 799–810, 2002.

[19] I. Yalçinkaya, B. D. Iričanin, and C. Çinar, "On a max-type difference equation," *Discrete Dynamics in Nature and Society*, vol. 2007, no. 1, Article ID 47264, 10 pages, 2007.

[20] X. Yang, X. Liao, and C. Li, "On a difference equation with maximum," *Applied Mathematics and Computation*, vol. 181, no. 1, pp. 1–5, 2006.