Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 714891, 11 pages
doi:10.1155/2010/714891

Research Article

Strictly Increasing Solutions of Nonautonomous Difference Equations Arising in Hydrodynamics

Lukáš Rachůnek and Irena Rachůnková

Department of Mathematics, Faculty of Science, Palacký University, tř. 17. listopadu 12, 77146 Olomouc, Czech Republic

Correspondence should be addressed to Irena Rachůnková, rachunko@inf.upol.cz

Received 19 December 2009; Revised 14 February 2010; Accepted 10 March 2010

Academic Editor: Ağacik Zafer

The paper provides conditions sufficient for the existence of strictly increasing solutions of the second-order nonautonomous difference equation $x(n+1) = x(n) + (n/(n+1))^2(x(n) - x(n-1) + h^2 f(x(n)))$, $n \in \mathbb{N}$, where $h > 0$ is a parameter and f is Lipschitz continuous and has three real zeros $L_0 < 0 < L$. In particular we prove that for each sufficiently small $h > 0$ there exists a solution $\{x(n)\}_{n=0}^{\infty}$ such that $\{x(n)\}_{n=1}^{\infty}$ is increasing, $x(0) = x(1) \in (L_0, 0)$, and $\lim_{n\to\infty} x(n) > L$. The problem is motivated by some models arising in hydrodynamics.

1. Formulation of Problem

We will investigate the following second-order non-autonomous difference equation

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \left(x(n) - x(n-1) + h^2 f(x(n))\right), \quad n \in \mathbb{N}, \tag{1.1}$$

where f is supposed to fulfil

$$L_0 < 0 < L, \quad f \in \mathrm{Lip}_{\mathrm{loc}}[L_0, \infty), \quad f(L_0) = f(0) = f(L) = 0, \tag{1.2}$$

$$x f(x) < 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \qquad f(x) \geq 0 \quad \text{for } x \in (L, \infty), \tag{1.3}$$

$$\exists \overline{B} \in (L_0, 0) \text{ such that } \int_{\overline{B}}^{L} f(z)\,\mathrm{d}z = 0. \tag{1.4}$$

Let us note that $f \in \mathrm{Lip}_{\mathrm{loc}}[L_0, \infty)$ means that for each $[L_0, A] \subset [L_0, \infty)$ there exists $K_A > 0$ such that $|f(x) - f(y)| \le K_A|x - y|$ for all $x, y \in [L_0, A]$. A simple example of a function f satisfying (1.2)–(1.4) is $f(x) = c(x - L_0)x(x - L)$, where c is a positive constant.

A sequence $\{x(n)\}_{n=0}^{\infty}$ which satisfies (1.1) is called a solution of (1.1). For each values $B, B_1 \in [L_0, \infty)$ there exists a unique solution $\{x(n)\}_{n=0}^{\infty}$ of (1.1) satisfying the initial conditions

$$x(0) = B, \qquad x(1) = B_1. \tag{1.5}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called a solution of problem (1.1), (1.5).

In [1] we have shown that (1.1) is a discretization of differential equations which generalize some models arising in hydrodynamics or in the nonlinear field theory; see [2–6]. Increasing solutions of (1.1), (1.5) with $B = B_1 \in (L_0, 0)$ has a fundamental role in these models. Therefore, in [1], we have described the set of all solutions of problem (1.1), (1.6), where

$$x(0) = B, \quad x(1) = B, \quad B \in (L_0, 0). \tag{1.6}$$

In this paper, using [1], we will prove that for each sufficiently small $h > 0$ there exists at least one $B \in (L_0, 0)$ such that the corresponding solution of problem (1.1), (1.6) fulfils

$$x(0) = x(1), \quad \lim_{n\to\infty} x(n) > L, \quad \{x(n)\}_{n=1}^{\infty} \text{ is increasing.} \tag{1.7}$$

Note that an autonomous case of (1.1) was studied in [7]. We would like to point out that recently there has been a huge interest in studying the existence of monotonous and nontrivial solutions of nonlinear difference equations. For papers during last three years see, for example, [8–22]. A lot of other interesting references can be found therein.

2. Four Types of Solutions

Here we present some results of [1] which we need in next sections. In particular, we will use the following definitions and lemmas.

Definition 2.1. Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6) such that

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n\to\infty} x(n) = 0. \tag{2.1}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a damped solution*.

Definition 2.2. Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6) which fulfils

$$\{x(n)\}_{n=1}^{\infty} \text{ is increasing}, \quad \lim_{n\to\infty} x(n) = L. \tag{2.2}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a homoclinic solution*.

Definition 2.3. Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6). Assume that there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and

$$x(b) \leq L < x(b+1). \tag{2.3}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *an escape solution*.

Definition 2.4. Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem (1.1), (1.6). Assume that there exists $b \in \mathbb{N}$, $b > 1$, such that $\{x(n)\}_{n=1}^{b}$ is increasing and

$$0 < x(b) < L, \quad x(b+1) \leq x(b). \tag{2.4}$$

Then $\{x(n)\}_{n=0}^{\infty}$ is called *a non-monotonous solution*.

Lemma 2.5 (see [1] (on four types of solutions)). *Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem* (1.1), (1.6). *Then $\{x(n)\}_{n=0}^{\infty}$ is just one of the following four types:*

(I) *$\{x(n)\}_{n=0}^{\infty}$ is an escape solution;*

(II) *$\{x(n)\}_{n=0}^{\infty}$ is a homoclinic solution;*

(III) *$\{x(n)\}_{n=0}^{\infty}$ is a damped solution;*

(IV) *$\{x(n)\}_{n=0}^{\infty}$ is a non-monotonous solution.*

Lemma 2.6 (see [1] (estimates of solutions)). *Let $\{x(n)\}_{n=0}^{\infty}$ be a solution of problem* (1.1), (1.6). *Then there exists a maximal $b \in \mathbb{N} \cup \{\infty\}$ satisfying*

$$\begin{gathered} x(n) \in [B, L) \quad \text{for } n = 1, \ldots, b, \text{ if } b \in \mathbb{N}, \\ x(n) \in [B, L) \quad \text{for } n \in \mathbb{N}, \text{ if } b = \infty. \end{gathered} \tag{2.5}$$

Further, if $b > 1$, then moreover

$$\{x(n)\}_{n=1}^{b} \text{ is increasing}, \tag{2.6}$$

$$\Delta x(n) < h\sqrt{(L - 2L_0)M_0} + h^2 M_0 \tag{2.7}$$

for $n = 1, \ldots, b-1$ if $b \in \mathbb{N}$, and for $n \in \mathbb{N}$ if $b = \infty$, where

$$M_0 = \max\{|f(x)| : x \in [L_0, L]\}. \tag{2.8}$$

In [1] we have proved that the set consisting of damped and non-monotonous solutions of problem (1.1), (1.6) is nonempty for each sufficiently small $h > 0$. This is contained in the next lemma.

Lemma 2.7 (see [1] (on the existence of non-monotonous or damped solutions)). *Let $B \in (\overline{B}, 0)$, where $\overline{B}$ is defined by* (1.4). *There exists $h_B > 0$ such that if $h \in (0, h_B]$, then the corresponding solution $\{x(n)\}_{n=0}^{\infty}$ of problem* (1.1), (1.6) *is non-monotonous or damped.*

In Section 4 of this paper we prove that also the set of escape solutions of problem (1.1), (1.6) is nonempty for each sufficiently small $h > 0$. Note that in our next paper [23] we prove this assertion for the set of homoclinic solutions.

3. Properties of Solutions

Now, we provide other properties of solutions important in the investigation of escape solutions.

Lemma 3.1. *Let* $\{x(n)\}_{n=0}^{\infty}$ *be an escape solution of problem* (1.1), (1.6). *Then* $\{x(n)\}_{n=1}^{\infty}$ *is increasing.*

Proof. Due to (1.1), $\{x(n)\}_{n=0}^{\infty}$ fulfils

$$\Delta x(n) = \left(\frac{n}{n+1}\right)^2 \left(\Delta x(n-1) + h^2 f(x(n))\right), \quad n \in \mathbb{N}. \tag{3.1}$$

According to Definition 2.3 there exists $b \in \mathbb{N}$, such that $\{x(n)\}_{n=1}^{b+1}$ is increasing and (2.3) holds. By (1.3) we get $f(x(b+1)) \geq 0$. Consequently, by (3.1) and (2.3), $\Delta x(b+1) \geq (b+1)^2/(b+2)^2 \Delta x(b) > 0$ and $f(x(b+2)) \geq 0$. Similarly $\Delta x(b+j) \geq (b+j)^2/(b+1+j)^2 \Delta x(b+j-1)$ and

$$\Delta x(b+j) \geq \left(\frac{b+1}{b+1+j}\right)^2 \Delta x(b), \quad j \in \mathbb{N}. \tag{3.2}$$

This yields that $\{x(n)\}_{n=1}^{\infty}$ is increasing. □

Lemma 3.2. *Assume that* $f(x) = 0$ *for* $x > L$. *Choose an arbitrary* $\varrho > 0$. *Let* $B_1, B_2 \in (L_0, 0)$ *and let* $\{x(n)\}_{n=0}^{\infty}$ *and* $\{y(n)\}_{n=0}^{\infty}$ *be a solution of problem* (1.1), (1.6) *with* $B = B_1$ *and* $B = B_2$, *respectively. Let* K_L *be the Lipschitz constant for* f *on* $[L_0, L]$. *Then*

$$\left|x(n) - y(n)\right| \leq |B_1 - B_2| \mathrm{e}^{\varrho^2 K_L}, \tag{3.3}$$

$$\left|\frac{\Delta x(n) - \Delta y(n)}{h}\right| \leq |B_1 - B_2| \varrho K_L \, \mathrm{e}^{\varrho^2 K_L}, \tag{3.4}$$

where $n \in \mathbb{N}$, $n \leq \varrho/h$.

Proof. By (3.1) we have

$$(j+1)^2 \Delta x(j) - j^2 \Delta x(j-1) = h^2 j^2 f(x(j)), \quad j \in \mathbb{N}. \tag{3.5}$$

Summing it for $j = 1, \ldots, k$, we get by (1.6)

$$\Delta x(k) = h^2 \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(x(j)), \quad k \in \mathbb{N}. \tag{3.6}$$

Summing it again for $k = 1, \ldots, n-1$, we get

$$x(n) = B_1 + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(x(j)), \quad n \in \mathbb{N}, \tag{3.7}$$

and similarly

$$y(n) = B_2 + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 f(y(j)), \quad n \in \mathbb{N}. \tag{3.8}$$

From this and by using summation by parts we easily obtain

$$\begin{aligned} |x(n) - y(n)| &\le |B_1 - B_2| + h^2 \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \sum_{j=1}^{k} j^2 |f(x(j)) - f(y(j))| \\ &\le |B_1 - B_2| + (n-1) h^2 K_L \sum_{j=1}^{n-1} |x(j) - y(j)|, \quad n \in \mathbb{N}. \end{aligned} \tag{3.9}$$

By the discrete analogue of the Gronwall-Bellman inequality (see, e.g., [24, Lemma 4.34]), we get

$$|x(n) - y(n)| \le |B_1 - B_2| e^{(n-1)^2 h^2 K_L} \quad \text{for } n \in \mathbb{N}, \tag{3.10}$$

which yields (3.3).

By (3.6) and (3.3) we have for $n \in \mathbb{N}$, $n \le \varrho/h$,

$$\begin{aligned} \left| \frac{\Delta x(n) - \Delta y(n)}{h} \right| &\le h \frac{1}{(n+1)^2} \sum_{j=1}^{n} j^2 |f(x(j)) - f(y(j))| \\ &\le h K_L \sum_{j=1}^{n} |x(j) - y(j)| \le |B_1 - B_2| \varrho K_L e^{\varrho^2 K_L}. \end{aligned} \tag{3.11}$$

□

4. Existence of Escape Solutions

Lemma 4.1. *Assume that* $C \in (L_0, \overline{B})$ *and* $\{B_k\}_{k=1}^{\infty} \subset (L_0, C)$. *Let* $\{x_k(n)\}_{n=0}^{\infty}$ *be a solution of problem* (1.1), (1.6) *with* $B = B_k$, $k \in \mathbb{N}$. *For* $k \in \mathbb{N}$ *choose a maximal* $b_k \in \mathbb{N} \cup \{\infty\}$ *such that* $x_k(n) \in [B_k, L)$ *for* $n = 1, \ldots, b_k$ *if* b_k *is finite, and for* $n \in \mathbb{N}$ *if* $b_k = \infty$, *and* $\{x_k(n)\}_{n=1}^{b_k}$ *is increasing if* $b_k > 1$. *Then there exists* $h^* > 0$ *such that for any* $h \in (0, h^*]$ *there exists a unique* $\gamma_k \in \mathbb{N}$, $\gamma_k < b_k$, *such that*

$$x_k(\gamma_k) \ge C, \qquad x_k(\gamma_k - 1) < C. \tag{4.1}$$

Moreover, if the sequence $\{\gamma_k\}_{k=1}^{\infty}$ *is unbounded, then there exists* $\ell \in \mathbb{N}$ *such that the solution* $\{x_\ell(n)\}_{n=0}^{\infty}$ *of problem* (1.1), (1.6) *with* $B = B_\ell \in (L_0, \overline{B})$ *is an escape solution.*

Proof. Choose $h_0 > 0$ such that

$$h_0\sqrt{(L-2L_0)M_0} + h_0^2 M_0 < |C|. \tag{4.2}$$

For $k \in \mathbb{N}$ denote by $\{x_k(n)\}_{n=0}^{\infty}$ a solution of problem (1.1), (1.6) with $B = B_k$. The existence of b_k is guaranteed by Lemma 2.6. By Lemma 2.5, $\{x_k(n)\}_{n=0}^{\infty}$ is just one of the types (I)–(IV), and if $h \in (0, h_0]$, then the monotonicity of $\{x_k(n)\}_{n=0}^{b_k}$ yields a unique $\gamma_k \in \mathbb{N}$, $\gamma_k < b_k$, satisfying (4.1).

For $h \in (0, h_0)$, consider the sequence $\{\gamma_k\}_{k=1}^{\infty}$ and assume that it is unbounded. Then we have

$$\lim_{k\to\infty} \gamma_k = \infty \tag{4.3}$$

(otherwise we take a subsequence.) Assume on the contrary that for any $k \in \mathbb{N}$, $\{x_k(n)\}_{n=0}^{\infty}$ is not an escape solution. Choose $k \in \mathbb{N}$. If $\{x_k(n)\}_{n=0}^{\infty}$ is damped, then by Definition 2.1, we have $b_k = \infty$ and

$$x_k(b_k) := \lim_{k\to\infty} x_k(n) = 0, \qquad \Delta x_k(b_k) := \lim_{k\to\infty} \Delta x_k(n) = 0. \tag{4.4}$$

If $\{x_k(n)\}_{n=0}^{\infty}$ is homoclinic, then by Definition 2.2, we have $b_k = \infty$ and

$$x_k(b_k) := \lim_{k\to\infty} x_k(n) = L, \qquad \Delta x_k(b_k) := \lim_{k\to\infty} \Delta x_k(n) = 0. \tag{4.5}$$

If $\{x_k(n)\}_{n=0}^{\infty}$ is non-monotonous, then by Definition 2.4, we have $b_k < \infty$ and

$$x_k(b_k) \in (0, L), \quad \Delta x_k(b_k) \le 0. \tag{4.6}$$

To summarize if $\{x_k(n)\}_{n=0}^{\infty}$ is not an escape solution, then by (4.4), (4.5), and (4.6), we have

$$x_k(b_k) \in [0, L], \quad \Delta x_k(b_k) \le 0. \tag{4.7}$$

Since $\Delta x_k(0) = 0$, there exists $\overline{\gamma}_k \in \mathbb{N}$ satisfying

$$\gamma_k \le \overline{\gamma}_k < b_k, \quad \Delta x_k(\overline{\gamma}_k) = \max\{\Delta x_k(j) : \gamma_k \le j \le b_k - 1\}. \tag{4.8}$$

Consider (3.5) with $x = x_k$. By dividing it by j^2, multiplying such obtained equality by $x_k(j+1) - x_k(j-1)$ and summing in j from 1 to n we get

$$\begin{aligned}(\Delta x_k(n))^2 - h^2\sum_{j=1}^{n} f(x_k(j))(x_k(j+1) - x_k(j-1)) \\ = -\sum_{j=1}^{n}\frac{2j+1}{j^2}\Delta x_k(j)(x_k(j+1) - x_k(j-1)), \quad n \in \mathbb{N}.\end{aligned} \tag{4.9}$$

Denote

$$E_k(n+1) = (\Delta x_k(n))^2 - h^2\sum_{j=1}^{n} f(x_k(j))(x_k(j+1) - x_k(j-1)). \tag{4.10}$$

Then we get

$$E_k(n+1) = -\sum_{j=1}^{n}\frac{2j+1}{j^2}\Delta x_k(j)(x_k(j+1) - x_k(j-1)), \quad n \in \mathbb{N}. \tag{4.11}$$

Let us put $n = \gamma_k - 1$ and $n = b_k - 1$ to (4.11) and subtract. By (4.7) and (4.8) we get

$$\begin{aligned}E_k(\gamma_k) - E_k(b_k) &= \sum_{j=\gamma_k}^{b_k-1}\frac{2j+1}{j^2}\Delta x_k(j)(x_k(j+1) - x_k(j-1)) \\ &\leq 2\frac{2\gamma_k+1}{\gamma_k^2}\Delta x_k(\overline{\gamma}_k)(L - L_0).\end{aligned} \tag{4.12}$$

Let us put $n = \gamma_k - 1$ and $n = b_k - 1$ to (4.10) and subtract. We get

$$\begin{aligned}E_k(\gamma_k) - E_k(b_k) &= (\Delta x_k(\gamma_k - 1))^2 - (\Delta x_k(b_k - 1))^2 \\ &\quad + 2h^2\sum_{j=\gamma_k}^{b_k-1} f(x_k(j))\frac{x_k(j+1) - x_k(j-1)}{2}.\end{aligned} \tag{4.13}$$

Choose $\varepsilon > 0$ and $h_1 > 0$ such that

$$\varepsilon < \frac{1}{2}\int_{C}^{L} f(z)\mathrm{d}z, \quad h_1 M_0 < \sqrt{\varepsilon}. \tag{4.14}$$

Let $b_k < \infty$. Then (4.6) holds. Since $\Delta x_k(b_k - 1) > 0$, $f(x_k(b_k)) < 0$ and $\Delta x_k(b_k) \leq 0$, (3.1) yields

$$\left(\frac{b_k+1}{b_k}\right)^2 |\Delta x_k(b_k)| + \Delta x_k(b_k - 1) = h^2 |f(x_k(b_k))|, \tag{4.15}$$

and hence

$$0 < \Delta x_k(b_k - 1) \le -h^2 f(x_k(b_k)) < h^2 M_0 < h\sqrt{\varepsilon} \quad \text{for } h \in (0, h_1]. \tag{4.16}$$

Clearly, if $b_k = \infty$, then by (4.4) and (4.5), inequality (4.16) holds, as well. By (1.2), f is integrable on $[L_0, L]$. So, having in mind (4.1), we can find $\delta > 0$ such that if

$$\frac{x_k(j+1) - x_k(j-1)}{2} < \delta, \quad j = \gamma_k, \dots, b_k - 1, \tag{4.17}$$

then

$$\left| \sum_{j=\gamma_k}^{b_k-1} f(x_k(j)) \frac{x_k(j+1) - x_k(j-1)}{2} - \int_C^{b_k} f(z)\mathrm{d}z \right| < \varepsilon. \tag{4.18}$$

Therefore, due to (1.3) and (4.7),

$$\sum_{j=\gamma_k}^{b_k-1} f(x_k(j)) \frac{x_k(j+1) - x_k(j-1)}{2} > \int_C^{b_k} f(z)\ \mathrm{d}z - \varepsilon \ge \int_C^{L} f(z)\mathrm{d}z - \varepsilon. \tag{4.19}$$

Let $h_2 > 0$ be such that

$$h_2\left(\sqrt{(L - 2L_0)M_0} + h_2 M_0\right) < \delta. \tag{4.20}$$

If $h \in (0, h_2]$, then (2.7) implies (4.17) and hence (4.19) holds.

Now, let us put $h^* = \min\{h_0, h_1, h_2\}$ and choose $h \in (0, h^*]$. Then, (4.2), (4.14), (4.20), and (4.13)–(4.19) yield

$$\begin{aligned} E_k(\gamma_k) - E_k(b_k) &> -h^2\varepsilon + 2h^2\left(\int_C^L f(z)\mathrm{d}z - \varepsilon\right) \\ &= 2h^2\left(\int_C^L f(z)\mathrm{d}z - \frac{3}{2}\varepsilon\right) > h^2\varepsilon > 0. \end{aligned} \tag{4.21}$$

Finally, (4.12) and (4.21) imply

$$\begin{gathered} 0 < h^2\varepsilon < E_k(\gamma_k) - E_k(b_k) \le 2\frac{2\gamma_k + 1}{\gamma_k^2} \Delta x_k(\overline{\gamma}_k)(L - L_0), \\ \frac{h^2\varepsilon}{2(L - L_0)} \cdot \frac{\gamma_k^2}{2\gamma_k + 1} < \Delta x_k(\overline{\gamma}_k). \end{gathered} \tag{4.22}$$

Letting $k \to \infty$, we obtain, by (4.3), that $\lim_{k\to\infty}\Delta x_k(\overline{\gamma}_k) = \infty$, contrary to (4.17). Therefore an escape solution $\{x_\ell(n)\}_{n=0}^{\infty}$ of problem (1.1), (1.6) with $B = B_\ell \in (L_0, \overline{B})$ must exist. □

Now, we are in a position to prove the next main result.

Theorem 4.2 (On the existence of escape solutions). *There exists $h^* > 0$ such that for any $h \in (0, h^*]$ the initial value problem* (1.1), (1.6) *has an escape solution for some $B \in (L_0, \overline{B})$.*

Proof. We have the following steps.

Step 1. Let us define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \le L, \\ 0 & \text{for } x > L, \end{cases} \tag{4.23}$$

and consider an auxiliary equation

$$x(n+1) = x(n) + \left(\frac{n}{n+1}\right)^2 \left(x(n) - x(n-1) + h^2 \tilde{f}(x(n))\right), \quad n \in \mathbb{N}. \tag{4.24}$$

Let $h^* > 0$ be the constant of Lemma 4.1 for problem (4.24), (1.6). Choose $h \in (0, h^*]$, $C \in (L_0, \overline{B})$ and let K_L be the Lipschitz constant for $\tilde{f}$ on $[L_0, \infty)$. Consider a sequence $\{B_k\}_{k=1}^{\infty} \subset (L_0, C)$ such that $\lim_{k\to\infty} B_k = L_0$. Then, for each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that

$$|B_{k_m} - L_0| < e^{-m^2 K_L}(C - L_0). \tag{4.25}$$

Let $\tilde{x}_0(0) = \tilde{x}_0(n) = L_0$ for $n \in \mathbb{N}$. Then the sequence $\{\tilde{x}_0(n)\}_{n=0}^{\infty}$ is the unique solution of problem (4.24), (1.6) with $B = L_0$. Let $\{\tilde{x}_k(n)\}_{n=0}^{\infty}$ be a solution of problem (4.24), (1.6) with $B = B_k$, $k \in \mathbb{N}$, and let $\{\gamma_k\}_{k=1}^{\infty}$ be the sequence corresponding to $\{\tilde{x}_k(n)\}_{n=0}^{\infty}$ by Lemma 4.1. We prove that $\{\gamma_k\}_{k=1}^{\infty}$ is unbounded. According to Lemma 3.2, for each $m \in \mathbb{N}$,

$$|\tilde{x}_{k_m}(n) - \tilde{x}_0(n)| \le |B_{k_m} - L_0| e^{m^2 K_L}, \quad n \le \frac{m}{h}. \tag{4.26}$$

Consequently, (4.25) and (4.26) give

$$|\tilde{x}_{k_m}(n) - \tilde{x}_0(n)| \le C - L_0, \quad n \le \frac{m}{h}, \tag{4.27}$$

and hence

$$\tilde{x}_{k_m}(n) \le C, \quad n \le \frac{m}{h}. \tag{4.28}$$

Therefore

$$\gamma_{k_m}(n) \ge \frac{m}{h}, \quad m \in \mathbb{N}, \tag{4.29}$$

which yields that $\{\gamma_k\}_{k=1}^{\infty}$ is unbounded. By Lemma 4.1, the auxiliary initial value problem (4.24), (1.6) has an escape solution for some $B = B_\ell \in (L_0, \widetilde{B})$. Denote this solution by $\{\widetilde{x}_\ell(n)\}_{n=0}^{\infty}$.

Step 2. By Definition 2.3, there exists $b \in \mathbb{N}$ such that

$$\{\widetilde{x}(n)\}_{n=1}^{b+1} \text{ is increasing}, \qquad \widetilde{x}_\ell(b) \le L < \widetilde{x}_\ell(b+1). \tag{4.30}$$

Now, consider the solution $\{x_\ell(n)\}_{n=0}^{\infty}$ of our original problem (1.1), (1.6) with $B = B_\ell$. Due to (4.23), $x_\ell(n) = \widetilde{x}_\ell(n)$ for $n = 0, 1, \ldots, b+1$. Using (4.30) and Definition 2.3, we get that $\{x_\ell(n)\}_{n=0}^{\infty}$ is an escape solution of problem (1.1), (1.6). □

Acknowledgments

The paper was supported by the Council of Czech Government MSM 6198959214. The authors thank the referees for valuable comments.

References

[1] L. Rachůnek, "On four types of solutions," submitted, http://phoenix.inf.upol.cz/~rachunekl/mathair/rr7.pdf.
[2] H. Berestycki, P.-L. Lions, and L. A. Peletier, "An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$," *Indiana University Mathematics Journal*, vol. 30, no. 1, pp. 141–157, 1981.
[3] G. H. Derrick, "Comments on nonlinear wave equations as models for elementary particles," *Journal of Mathematical Physics*, vol. 5, pp. 1252–1254, 1964.
[4] F. Dell'Isola, H. Gouin, and G. Rotoli, "Nucleation of spherical shell-like interfaces by second gradient theory: numerical simulations," *European Journal of Mechanics—B/Fluids*, vol. 15, pp. 545–568, 1996.
[5] H. Gouin and G. Rotoli, "An analytical approximation of density profile and surface tension of microscopic bubbles for Van der Waals fluids," *Mechanics Research Communications*, vol. 24, pp. 255–260, 1997.
[6] G. Kitzhofer, O. Koch, P. Lima, and E. Weinmüller, "Efficient numerical solution of the density profile equation in hydrodynamics," *Journal of Scientific Computing*, vol. 32, no. 3, pp. 411–424, 2007.
[7] L. Rachůnek and I. Rachůnková, "On a homoclinic point of some autonomous second-order difference equation," submitted to *Journal of Difference Equations and Applications*.
[8] A. M. Amleh, E. Camouzis, and G. Ladas, "On second-order rational difference equation. I," *Journal of Difference Equations and Applications*, vol. 13, no. 11, pp. 969–1004, 2007.
[9] K. S. Berenhaut and S. Stević, "The difference equation $x_{n+1} = \alpha + x_{n-k}/(\sum_{i=0}^{k-1} c_i x_{n-i})$ has solutions converging to zero," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1466–1471, 2007.
[10] L. Berg, "On the asymptotics of the difference equation $x_{n-3} = x_n(1 + x_{n-1}x_{n-2})$," *Journal of Difference Equations and Applications*, vol. 14, no. 1, pp. 105–108108, 2008.
[11] L. Gutnik and S. Stević, "On the behaviour of the solutions of a second-order difference equation," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 27562, 14 pages, 2007.
[12] L.-X. Hu, W.-T. Li, and S. Stević, "Global asymptotic stability of a second order rational difference equation," *Journal of Difference Equations and Applications*, vol. 14, no. 8, pp. 779–797, 2008.
[13] B. Iričanin and S. Stević, "Eventually constant solutions of a rational difference equation," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 854–856, 2009.
[14] I. Rachůnková and L. Rachůnek, "Singular discrete problem arising in the theory of shallow membrane caps," *Journal of Difference Equations and Applications*, vol. 14, no. 7, pp. 747–767, 2008.
[15] I. Rachůnková and L. Rachůnek, "Singular discrete and continuous mixed boundary value problems," *Mathematical and Computer Modelling*, vol. 49, no. 3-4, pp. 413–422, 2009.

[16] L. Rachůnek and I. Rachůnková, “Approximation of differential problems with singularities and time discontinuities,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, pp. e1448–e1460, 2009.
[17] B. D. Rouhani and H. Khatibzadeh, “A note on the asymptotic behavior of solutions to a second order difference equation,” *Journal of Difference Equations and Applications*, vol. 14, no. 4, pp. 429–432, 2008.
[18] S. Stević, “Asymptotics of some classes of higher-order difference equations,” *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 56813, 20 pages, 2007.
[19] S. Stević, “Asymptotic periodicity of a higher-order difference equation,” *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 13737, 9 pages, 2007.
[20] S. Stević, “Existence of nontrivial solutions of a rational difference equation,” *Applied Mathematics Letters*, vol. 20, no. 1, pp. 28–31, 2007.
[21] S. Stević, “Nontrivial solutions of higher-order rational difference equations,” *Matematicheskie Zametki*, vol. 84, no. 5, pp. 772–780, 2008.
[22] T. Sun, H. Xi, and W. Quan, “Existence of monotone solutions of a difference equation,” *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 917560, 8 pages, 2008.
[23] L. Rachůnek and I. Rachůnková, “Homoclinic solutions of non-autonomous difference equations arising in hydrodynamics,” in preparation.
[24] S. N. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 2nd edition, 1999.