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Research Article

Singular Cauchy Initial Value Problem for Certain Classes of Integro-Differential Equations

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The existence and uniqueness of solutions and asymptotic estimate of solution formulas are studied for the following initial value problem: $g(t)y'(t) = ay(t)[1 + f(t, y(t), \int_{0^+}^{t} K(t, s, y(t), y(s))ds)]$, $y(0^+) = 0$, $t \in (0, t_0]$, where $a > 0$ is a constant and $t_0 > 0$. An approach which combines topological method of T. Ważewski and Schauder's fixed point theorem is used.

1. Introduction and Preliminaries

The singular Cauchy problem for first-order differential and integro-differential equations resolved (or unresolved) with respect to the derivatives of unknowns is fairly well studied (see, e.g., [1–16]), but the asymptotic properties of the solutions of such equations are only partially understood. Although the singular Cauchy problems were widely considered by using various methods (see, e.g., [1–13, 16–18]), the method used here is based on a different approach. In particular, we use a combination of the topological method of T. Ważewski (see, e.g., [19, 20]) and Schauder's fixed point theorem [21]. Our technique leads to the existence and uniqueness of solutions with asymptotic estimates in the right neighbourhood of a singular point.

Consider the following problem:

$$g(t)y'(t) = ay(t)\left[1 + f\left(t, y(t), \int_{0^+}^{t} K\big(t, s, y(t), y(s)\big)ds\right)\right], \qquad (1.1)$$
$$y(0^+) = 0,$$

where $f \in C^0(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $K \in C^0(J \times J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $J = (0, t_0]$, $t_0 > 0$. Denote

$f(t) = o(g(t))$ as $t \to 0^+$ if there is valid $\lim_{t \to 0^+}(f(t)/g(t)) = 0$,

$f(t) \sim g(t)$ as $t \to 0^+$ if there is valid $\lim_{t \to 0^+}(f(t)/g(t)) = 1$.

The functions g, f, K will be assumed to satisfy the following.

(i) $a > 0$ is a constant, $g(t) \in C^1(J)$, $g(t) > 0$, $g(0^+) = 0$, $g'(t) \sim \psi(t) g^{\lambda}(t)$ as $t \to 0^+$, $\lambda > 0$, $\psi(t) g^{\tau}(t) = o(1)$ as $t \to 0^+$ for each $\tau > 0$, $\psi \in C(J, \mathbb{R}^+)$.

(ii) $|f(t,u,v)| \le |u| + |v|$, $|\int_{0^+}^{t} K(t,s,y(t),y(s))ds| \le r(t)|y|, 0 < r(t) \in C(J)$, $r(t) = \phi(t,C) o(1)$ as $t \to 0^+$, where $\phi(t,C) = C \exp(\int_{t_0}^{t}(a/g(s))ds)$ is the general solution of the equation $g(t)y'(t) = ay(t)$.

In the text we will apply the topological method of Ważewski and Schauder's theorem. Therefore, we give a short summary of them.

Let $f(t,y)$ be a continuous function defined on an open (t,y)-set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, Ω^0 an open set of Ω, $\partial\Omega^0$ the boundary of Ω^0 with respect to Ω, and $\overline{\Omega}^0$ the closure of Ω^0 with respect to Ω. Consider the system of ordinary differential equations

$$y' = f(t,y). \tag{1.2}$$

Definition 1.1 (see [19]). The point $(t_0, y_0) \in \Omega \cap \partial\Omega^0$ is called an egress (or an ingress point) of Ω^0 with respect to system (1.2) if for every fixed solution of system (1.2), $y(t_0) = y_0$, there exists an $\epsilon > 0$ such that $(t, y(t)) \in \Omega^0$ for $t_0 - \epsilon \le t < t_0$ $(t_0 < t \le t_0 + \epsilon)$. An egress point (ingress point) (t_0, y_0) of Ω^0 is called a strict egress point (strict ingress point) of Ω^0 if $(t, y(t)) \notin \overline{\Omega}^0$ on interval $t_0 < t \le t_0 + \epsilon_1$ $(t_0 - \epsilon_1 \le t < t_0)$ for an ϵ_1.

Definition 1.2 (see [19]). An open subset Ω^0 of the set Ω is called a (u,v)-subset of Ω with respect to system (1.2) if the following conditions are satisfied.

(1) There exist functions $u_i(t,y) \in C^1(\Omega, \mathbb{R})$, $i = 1, \dots, m$, and $v_j(t,y) \in C[\Omega, \mathbb{R}]$, $j = 1, \dots, n$, $m + n > 0$ such that

$$\Omega_0 = \{(t,y) \in \Omega : u_i(t,y) < 0,\ v_j(t,y) < 0\ \ \forall i,j\}. \tag{1.3}$$

(2) $\dot{u}_\alpha(t,y) < 0$ holds for the derivatives of the functions $u_\alpha(t,y)$, $\alpha = 1, \dots, m$, along trajectories of system (1.2) on the set

$$U_\alpha = \{(t,y) \in \Omega : u_\alpha(t,y) = 0,\ u_i(t,y) \le 0,\ v_j(t,y) \le 0,\ \forall j, i \ne \alpha\}. \tag{1.4}$$

(3) $\dot{v}_\beta(t,y) > 0$ holds for the derivatives of the functions $v_\beta(t,y)$, $\beta = 1, \dots, n$, along trajectories of system (1.2) on the set

$$V_\beta = \{(t,y) \in \Omega : v_\beta(t,y) = 0,\ u_i(t,y) \le 0,\ v_j(t,y) \le 0,\ \forall i, j \ne \beta\}. \tag{1.5}$$

The set of all points of egress (strict egress) is denoted by Ω_e^0 (Ω_{se}^0).

Lemma 1.3 (see [19]). *Let the set Ω_0 be a (u,v)-subset of the set Ω with respect to system* (1.2). *Then*

$$\Omega_{se}^0 = \Omega_e^0 = \bigcup_{\alpha=1}^{m} U_\alpha \setminus \bigcup_{\beta=1}^{n} V_\beta. \tag{1.6}$$

Definition 1.4 (see [19]). Let X be a topological space and $B \subset X$.

Let $A \subset B$. A function $r \in C(B,A)$ such that $r(a) = a$ for all $a \in A$ is a retraction from B to A in X.

The set $A \subset B$ is a retract of B in X if there exists a retraction from B to A in X.

Theorem 1.5 (Ważewski's theorem [19]). *Let Ω^0 be some (u,v)-subset of Ω with respect to system* (1.2). *Let S be a nonempty compact subset of $\Omega^0 \cup \Omega_e^0$ such that the set $S \cap \Omega_e^0$ is not a retract of S but is a retract Ω_e^0. Then there is at least one point $(t_0, y_0) \in S \cap \Omega_0$ such that the graph of a solution $y(t)$ of the Cauchy problem $y(t_0) = y_0$ for* (1.2) *lies in Ω_0 on its right-hand maximal interval of existence.*

Theorem 1.6 (Schauder's theorem [21]). *Let E be a Banach space and S its nonempty convex and closed subset. If P is a continuous mapping of S into itself and PS is relatively compact then the mapping P has at least one fixed point.*

2. Main Results

Theorem 2.1. *Let assumptions (i) and (ii) hold, then for each $C \neq 0$, there exists one solution $y(t,C)$ of initial problem* (1.1) *such that*

$$\left| y^{(i)}(t,C) - \phi^{(i)}(t,C) \right| \leq \delta \left(\phi^2(t,C) \right)^{(i)}, \quad i = 0,1, \tag{2.1}$$

for $t \in (0, t^\Delta]$, where $0 < t^\Delta \leq t_0$, $\delta > 1$ is a constant, and t^Δ depends on δ, C.

Proof. (1) Denote E the Banach space of continuous functions $h(t)$ on the interval $[0, t_0]$ with the norm

$$\|h(t)\| = \max_{t \in [0,t_0]} |h(t)|. \tag{2.2}$$

The subset S of Banach space E will be the set of all functions $h(t)$ from E satisfying the inequality

$$\left| h(t) - \phi(t,C) \right| \leq \delta \phi^2(t,C). \tag{2.3}$$

The set S is nonempty, convex and closed.

(2) Now we will construct the mapping P. Let $h_0(t) \in S$ be an arbitrary function. Substituting $h_0(t), h_0(s)$ instead of $y(t), y(s)$ into (1.1), we obtain the differential equation

$$g(t) y'(t) = a y(t) \left[1 + f\left(t, y(t), \int_{0^+}^{t} K(t,s,h_0(t),h_0(s)) ds \right) \right]. \tag{2.4}$$

Set

$$y(t) = \phi(t,C) + C\exp\left(\int_{t_0}^{t} \frac{(1-\alpha)a}{g(s)}ds\right)Y_0(t), \tag{2.5}$$

$$y'(t) = \phi'(t,C) + \frac{1}{g(t)}C\exp\left(\int_{t_0}^{t} \frac{(1-\alpha)a}{g(s)}ds\right)Y_1(t), \tag{2.6}$$

where $0 < \alpha < 1$ is a constant and new functions $Y_0(t)$ and $Y_1(t)$ satisfy the differential equation

$$g(t)Y_0'(t) = (\alpha - 1)aY_0(t) + Y_1(t). \tag{2.7}$$

From (2.3), it follows that

$$h_0(t) = \phi(t,C) + H_0(t), \qquad |H_0(t)| \le \delta\phi^2(t,C). \tag{2.8}$$

Substituting (2.5), (2.6) and (2.8) into (2.4) we get

$$\begin{aligned} Y_1(t) = aY_0(t) &+ \left(aC\exp\left(\int_{t_0}^{t} \frac{\alpha a}{g(s)}ds\right) + aY_0(t)\right) \\ &\times f\Bigg(t, \phi(t,C) + C\exp\left(\int_{t_0}^{t} \frac{(1-\alpha)a}{g(s)}ds\right)Y_0(t), \\ &\qquad \int_{0^+}^{t} K\big(t,s,\phi(t,C)+H_0(t),\phi(s,C)+H_0(s)\big)ds\Bigg). \end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.7) we get

$$\begin{aligned} g(t)Y_0'(t) = \alpha aY_0(t) &+ \left(aC\exp\left(\int_{t_0}^{t} \frac{\alpha a}{g(s)}ds\right) + aY_0(t)\right) \\ &\times f\Bigg(t, \phi(t,C) + C\exp\left(\int_{t_0}^{t} \frac{(1-\alpha)a}{g(s)}ds\right)Y_0(t), \\ &\qquad \int_{0^+}^{t} K\big(t,s,\phi(t,C)+H_0(t),\phi(s,C)+H_0(s)\big)ds\Bigg). \end{aligned} \tag{2.10}$$

In view of (2.5), (2.6) it is obvious that a solution of (2.10) determines a solution of (2.4).

Now we will use Ważewski's topological method. Consider an open set $\Omega \subset \mathbb{R}^{+} \times \mathbb{R}$. Investigate the behaviour of integral curves of (2.10) with respect to the boundary of the set

$$\Omega_0 \subset \Omega, \quad \Omega_0 = \{(t, Y_0) : \ 0 < t < t_0, \ u_0(t, Y_0) < 0\}, \tag{2.11}$$

where

$$u_0(t, Y_0) = Y_0^2 - \left(\delta C \exp\left(\int_{t_0}^{t} \frac{(1+\alpha)a}{g(s)} ds\right)\right)^2. \tag{2.12}$$

Calculating the derivative $\dot{u}_0(t, Y_0)$ along the trajectories of (2.10) on the set

$$\partial\Omega_0 = \{(t, Y_0) : 0 < t < t_0, \ u_0(t, Y_0) = 0\}, \tag{2.13}$$

we obtain

$$\begin{aligned} \dot{u}_0(t, Y_0) = \frac{2a}{g(t)} \Bigg[& \alpha Y_0^2(t) + \left(Y_0(t) C \exp\left(\int_{t_0}^{t} \frac{\alpha a}{g(s)} ds\right) + Y_0^2(t)\right) \\ & \times f\Bigg(t, \phi(t, C) + C \exp\left(\int_{t_0}^{t} \frac{(1-\alpha)a}{g(s)} ds\right) Y_0(t), \\ & \qquad \int_{0^+}^{t} K(t, s, \phi(t, C) + H_0(t), \phi(s, C) + H_0(s)) ds \Bigg) \\ & - \delta^2 (1+\alpha) C^2 \exp\left(\int_{t_0}^{t} \frac{2(1+\alpha)a}{g(s)} ds\right)\Bigg]. \end{aligned} \tag{2.14}$$

Since

$$\begin{gathered} \lim_{t \to +0} \psi(t) g^{\tau}(t) = 0 \quad \text{for any } \tau > 0, \\ g'(t) \sim \psi(t) g^{\lambda}(t) \quad \text{for } t \to 0^{+}, \ \lambda > 0, \end{gathered} \tag{2.15}$$

then there exists a positive constant M such that

$$g'(t) < M, \quad t \in (0, t_0]. \tag{2.16}$$

Consequently,

$$\int_{t_0}^{t} \frac{ds}{g(s)} < \frac{1}{M}\int_{t_0}^{t} \frac{g'(s)dt}{g(s)} = \frac{1}{M}\ln\frac{g(t)}{g(t_0)} \longrightarrow -\infty \quad \text{if } t \longrightarrow 0^{+}. \tag{2.17}$$

From here $\lim_{t\to 0^+}\phi(t,C) = 0$ and by L'Hospital's rule $\phi^{\tau}(t,C)g^{\sigma}(t) = o(1)$ for $t \to 0^+$, σ is an arbitrary real number. These both identities imply that the powers of $\phi(t,C)$ affect the convergence to zero of the terms in (2.14), in decisive way.

Using the assumptions of Theorem 2.1 and the definition of $Y_0(t)$, $\phi(t,C)$, we get that the first term $\alpha Y_0^2(t)$ in (2.14) has the form

$$\alpha Y_0^2(t) = \alpha\delta^2 C^2 \exp\left(\int_{t_0}^{t} \frac{2(1+\alpha)a}{g(s)}ds\right), \tag{2.18}$$

and the second term

$$\left(Y_0(t)C\exp\left(\int_{t_0}^{t}\frac{\alpha a}{g(s)}ds\right) + Y_0^2(t)\right) \times f\left(t, \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)Y_0(t),\right.$$
$$\left.\int_{0^+}^{t} K\left(t,s,\phi(t,C)+H_0(t),\phi(s,C)+H_0(s)\right)ds\right) \tag{2.19}$$

is bounded by terms with exponents which are greater than

$$\int_{t_0}^{t}\frac{2(1+\alpha)a}{g(s)}ds. \tag{2.20}$$

From here, we obtain

$$\operatorname{sgn}\dot{u}_0(t,Y_0) = \operatorname{sgn}\left(-\delta^2C^2(1+\alpha)\exp\left(\int_{t_0}^{t}\frac{2(1+\alpha)a}{g(s)}ds\right)\right) = -1 \tag{2.21}$$

for sufficiently small t^*, depending on C, δ, $0 < t^* \le t_0$.

The relation (2.21) implies that each point of the set $\partial\Omega_0$ is a strict ingress point with respect to (2.10). Change the orientation of the axis t into opposite. Now each point of the set $\partial\Omega_0$ is a strict egress point with respect to the new system of coordinates. By Ważewski's topological method, we state that there exists at least one integral curve of (2.10) lying in Ω_0 for $t \in (0, t^*)$. It is obvious that this assertion remains true for an arbitrary function $h_0(t) \in S$.

Now we will prove the uniqueness of a solution of (2.10). Let $\overline{Y_0}(t)$ be also the solution of (2.10). Putting $Z_0 = Y_0 - \overline{Y_0}$ and substituting into (2.10), we obtain

$$
\begin{aligned}
g(t)Z_0' = \alpha a Z_0 + &\left(aC\exp\left(\int_{t_0}^{t}\frac{\alpha a}{g(s)}ds\right) + a(t)Z_0(t)\right) \\
&\times\Bigg[f\Bigg(t,\phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)\left(Z_0(t)+\overline{Y_0}(t)\right), \\
&\qquad\int_{0^+}^{t}K\left(t,s,\phi(t,C)+H_0(t),\phi(s,C)+H_0(s)\right)ds\Bigg) \\
&\quad - f\Bigg(t,\ \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)\overline{Y_0}(t), \\
&\qquad\int_{0^+}^{t}K\left(t,s,\phi(t,C)+H_0(t),\phi(s,C)+H_0(s)\right)ds\Bigg)\Bigg].
\end{aligned}
\tag{2.22}
$$

Let

$$
\Omega_1(\delta) = \left\{(t, Z_0) :\ 0 < t < t^*,\ u_1(t, Z_0) < 0\right\},
\tag{2.23}
$$

where

$$
u_1(t, Z_0) = Z_0^2 - \left(\delta C\exp\left(\int_{t_0}^{t}\frac{(1+\alpha-\mu)a}{g(s)}ds\right)\right)^2, \quad 0 < \mu < \alpha.
\tag{2.24}
$$

Using the same method as above, we have

$$
\operatorname{sgn}\dot{u}_1(t, Z_0) = -1
\tag{2.25}
$$

for $t \in (0, t^*]$. It is obvious that $\Omega_0 \subset \Omega_1(\delta)$ for $t \in (0, t^*)$. Let $\overline{Z_0}(t)$ be any nonzero solution of (2.14) such that $(t_1, \overline{Z_0}(t_1)) \in \Omega_1$ for $0 < t_1 < t^*$. Let $\overline{\delta} \in (0, \delta)$ be such a constant that $(t_1, \overline{Z_0}(t_1)) \in \partial\Omega_1(\overline{\delta})$. If the curve $\overline{Z_0}(t)$ lays in $\Omega_1(\overline{\delta})$ for $0 < t < t_1$, then $(t_1, \overline{Z_0}(t_1))$ would have to be a strict egress point of $\partial\Omega_1(\overline{\delta})$ with respect to the original system of coordinates. This contradicts the relation (2.25). Therefore, there exists only the trivial solution $Z_0(t) \equiv 0$ of (2.22), so $Y_0 = \overline{Y_0}(t)$ is the unique solution of (2.10).

From (2.5), we obtain

$$
\left|y_0(t, C) - \phi(t, C)\right| \le \delta\phi^2(t, C),
\tag{2.26}
$$

where $y_0(t,C)$ is the solution of (2.4) for $t \in (0,t^*]$. Similarly, from (2.6), (2.9) we have

$$\begin{aligned}
\left|y_0'(t,C)-\phi'(t,C)\right| &= \left|\frac{1}{g(t)}C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)Y_1(t)\right| \\
&\le \left|\frac{1}{g(t)}C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)\right| \\
&\quad\times\left(\left|a\delta C\exp\left(\int_{t_0}^{t}\frac{(1+\alpha)a}{g(s)}ds\right)\right|+\left|a\delta C\exp\left(\int_{t_0}^{t}\frac{(1+\alpha)a}{g(s)}ds\right)\right|\right) \\
&\le \frac{2\delta a}{g(t)}C^2\exp\left(\int_{t_0}^{t}\frac{2a}{g(s)}ds\right)=\delta\left(\phi^2(t,C)\right)'.
\end{aligned} \tag{2.27}$$

It is obvious (after a continuous extension of $y_0(t)$ for $t=0$ and $y(0^+)=0$) that $P: h_0 \to y_0$ maps S into itself and $PS \subset S$.

(3) We will prove that PS is relatively compact and P is a continuous mapping.

It is easy to see, by (2.26) and (2.27), that PS is the set of uniformly bounded and equicontinuous functions for $t \in [0,t^*]$. By Ascoli's theorem, PS is relatively compact.

Let $\{h_r(t)\}$ be an arbitrary sequence functions in S such that

$$\|h_r(t)-h_0(t)\|=\epsilon_r, \quad \lim_{r\to\infty}\epsilon_r=0, \quad h_0(t)\in S. \tag{2.28}$$

The solution $\overline{Y_k}(t)$ of the equation

$$\begin{aligned}
g(t)Y_0'(t) &= \alpha a Y_0(t)+\left(aC\exp\left(\int_{t_0}^{t}\frac{\alpha a}{g(s)}ds\right)+aY_0(t)\right) \\
&\quad\times f\left(t,\phi(t,C)+C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)Y_0(t),\right. \\
&\qquad\left.\int_{0^+}^{t}K\left(t,s,\phi(t,C)+H_k(t),\phi(s,C)+H_k(s)\right)ds\right)
\end{aligned} \tag{2.29}$$

corresponds to the function $h_k(t)$ and $\overline{Y_k}(t) \in \Omega_0$ for $t \in (0,t^*)$. Similarly, the solution $\overline{Y_0}(t)$ of (2.10) corresponds to the function $h_0(t)$. We will show that $|\overline{Y_k}(t)-\overline{Y_0}(t)| \to 0$ uniformly on $[0,t^{\Delta}]$, where $0<t^{\Delta}\le t^*$, t^{Δ} is a sufficiently small constant which will be specified later. Consider the region

$$\Omega_{0k}=\{(t,Y_0):\ 0<t<t^*,\ u_{0k}(t,Y_0)<0\}, \tag{2.30}$$

where

$$u_{0k}(t, Y_0) = \left(Y(t) - \overline{Y_0}(t)\right)^2 - \left(\epsilon_k C \exp\left(\left(\int_{t_0}^{t} \frac{(1+\alpha-\nu)a}{g(s)} ds\right)\right)\right)^2, \quad 0 < \nu < \alpha,\ k \geq 1. \tag{2.31}$$

There exists sufficiently small constant $t^{\Delta} \leq t^*$ such that $\Omega_0 \subset \Omega_{0k}$ for any k, $t \in (0, t^{\Delta})$. Investigate the behaviour of integral curves of (2.29) with respect to the boundary $\partial\Omega_{0k}$, $t \in (0, t^{\Delta}]$. Using the same method as above, we obtain for trajectory derivatives

$$\operatorname{sgn} \dot{u}_{0k}(t, Y_0) = -1 \tag{2.32}$$

for $t \in (0, t^{\Delta}]$ and any k. By Ważewski's topological method, there exists at least one solution $\overline{Y_k}(t)$ lying in Ω_{0k}, $0 < t < t^{\Delta}$. Hence, it follows that

$$\left|\overline{Y_k}(t) - \overline{Y_0}(t)\right| \leq \epsilon_k C \exp\left(\int_{t_0}^{t} \frac{(1+\alpha-\nu)a}{g(s)} ds\right) \leq M\epsilon_k, \tag{2.33}$$

and $M > 0$ is a constant depending on C, t^{Δ}. From (2.5), we obtain

$$\left|y_k(t) - y_0(t)\right| = C \exp\left(\int_{t_0}^{t} \frac{(1-\alpha)a}{g(s)} ds\right) \left|\overline{Y_k}(t) - \overline{Y_0}(t)\right| \leq m\epsilon_k, \tag{2.34}$$

where $m > 0$ is a constant depending on $t^{\Delta}, C,\ M$. This estimate implies that P is continuous.

We have thus proved that the mapping P satisfies the assumptions of Schauder's fixed point theorem and hence there exists a function $h(t) \in S$ with $h(t) = P(h(t))$. The proof of existence of a solution of (1.1) is complete.

Now we will prove the uniqueness of a solution of (1.1). Substituting (2.5), (2.6) into (1.1), we get

$$\begin{aligned} Y_1(t) = aY_0(t) + &\left(aC \exp\left(\int_{t_0}^{t} \frac{\alpha a}{g(s)} ds\right) + a(t)Y_0(t)\right) \\ &\times f\left(t, \phi(t, C) + C \exp\left(\int_{t_0}^{t} \frac{(1-\alpha)a}{g(s)} ds\right) Y_0(t),\right. \\ &\quad \int_{0^+}^{t} K\left(t, s, \phi(t, C) + C \exp\left(\int_{t_0}^{t} \frac{(1-\alpha)a}{g(u)} du\right) Y_0(t),\right. \\ &\quad\quad \left.\left.\phi(s, C) + C \exp\left(\int_{t_0}^{s} \frac{(1-\alpha)a}{g(u)} du\right) Y_0(s)\right) ds\right). \end{aligned} \tag{2.35}$$

Equation (2.7) may be written in the following form:

$$\begin{aligned} g(t)Y_0'(t) = \alpha a Y_0(t) + &\left(aC\exp\left(\int_{t_0}^{t}\frac{\alpha a}{g(s)}ds\right) + aY_0(t)\right) \\ &\times f\Bigg(t, \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)Y_0(t), \\ &\int_{0^+}^{t} K\Bigg(t,s,\phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(u)}du\right)Y_0(t), \\ &\phi(s,C) + C\exp\left(\int_{t_0}^{s}\frac{(1-\alpha)a}{g(u)}du\right)Y_0(s)\Bigg)ds\Bigg). \end{aligned} \tag{2.36}$$

Now we know that there exists the solution $y_0(t,C)$ of (1.1) satisfying (2.1) such that

$$y_0(t,C) = \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)U_0(t), \tag{2.37}$$

where $U_0(t)$ is the solution of (2.36).

Denote $W_0(t) = Y_0(t) - U_0(t)$ and substituting it into (2.36), we obtain

$$\begin{aligned} g(t)W_0'(t) = \alpha a W_0(t) + a&\left(C\exp\left(\int_{t_0}^{t}\frac{\alpha a}{g(s)}ds\right) + W_0(t)\right) \\ &\times \Bigg[f\Bigg(t, \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)(W_0(t)+U_0(t)), \\ &\int_{0^+}^{t} K\Bigg(t,s,\phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(u)}du\right)(W_0(t)+U_0(t)), \\ &\phi(s,C) + C\exp\left(\int_{t_0}^{s}\frac{(1-\alpha)a}{g(u)}du\right)(W_0(s)+U_0(s))\Bigg)ds\Bigg) \\ &- f\Bigg(t, \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)U_0(t), \\ &\int_{0^+}^{t} K\Bigg(t,s,\phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(u)}du\right)U_0(t), \\ &\phi(s,C) + C\exp\left(\int_{t_0}^{s}\frac{(1-\alpha)a}{g(u)}du\right)U_0(s)\Bigg)ds\Bigg)\Bigg]. \end{aligned} \tag{2.38}$$

Let

$$\Omega_{00} = \left\{(t, W_0) : \ 0 < t < t^{\Delta}, \, u_{00}(t, W_0) < 0\right\}, \tag{2.39}$$

where

$$u_{00}(t, W_0) = W_0^2 - \left(\delta C \exp\left\{\int_{t_0}^{t} \frac{(1+\alpha-\mu)a}{g(s)} ds\right\}\right)^2, \quad 0 < \mu < \alpha. \tag{2.40}$$

If (2.38) had only the trivial solution lying in Ω_{00}, then $Y_0(t) = U_0(t)$ would be the only solution of (2.38) and from here, by (2.36), $y_0(t, C)$ would be the only solution of (1.1) satisfying (2.1) for $t \in (0, t^{\Delta}]$.

We will suppose that there exists a nontrivial solution $\overline{W_0}(t)$ of (2.38) lying in Ω_{00}. Substitute $\overline{W_0}(s)$ instead of $W_0(t)$ into (2.38), we obtain the differential equation

$$\begin{aligned}
g(t)W_0'(t) = \alpha a W_0(t) + a\left(C\exp\left(\int_{t_0}^{t}\frac{\alpha a}{g(s)}ds\right) + W_0(t)\right)& \\
\times\left[f\left(t, \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)(W_0(t) + U_0(t)),\right.\right.& \\
\int_{0^+}^{t} K\left(t, s, \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(u)}du\right)\left(\overline{W_0}(t) + U_0(t)\right),\right.& \\
\left.\left.\phi(s,C) + C\exp\left(\int_{t_0}^{u}\frac{(1-\alpha)a}{g(u)}du\right)\left(\overline{W_0}(s) + U_0(s)\right)\right)ds\right)& \qquad (2.41) \\
- f\left(t, \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(s)}ds\right)U_0(t),\right.& \\
\int_{0^+}^{t} K(t, s, \phi(t,C) + C\exp\left(\int_{t_0}^{t}\frac{(1-\alpha)a}{g(u)}du\right)U_0(t),& \\
\left.\left.\phi(s,C) + C\exp\left(\int_{t_0}^{s}\frac{(1-\alpha)a}{g(u)}du\right)U_0(s)\right)ds\right)\Bigg].&
\end{aligned}$$

Calculating the derivative $\dot{u}_{00}(t, W_0)$ along the trajectories of (2.41) on the set $\partial\Omega_{00}$, we get $\operatorname{sgn} \dot{u}_{00}(t, W_0) = -1$ for $t \in (0, t^{\Delta}]$.

By the same method as in the case of the existence of a solution of (1.1), we obtain that in Ω_{00} there is only the trivial solution of (2.41). The proof is complete. □

Example 2.2. Consider the following initial value problem:

$$t^2 y'(t) = 3y(t)\left(1 + \frac{t}{1+t^2}y(t) + \int_0^t \frac{2e^{-s^{-2}}y(t)}{s^3(1+y^2(t)y^2(s))}ds\right), \quad y(0^+) = 0. \tag{2.42}$$

In our case a general solution of the equation

$$t^2 y'(t) = 3y(t) \tag{2.43}$$

has the form $\phi(t, C) = C\exp(3t_0{}^{-1} - 3t^{-1})$ and $g(t) = t^2$, $a = 3$, $\psi(t) = 2$, $\lambda = 1/2$, $\psi(t)g^{\tau}(t) = 2t^{2\tau} = o(1)$ as $t \to 0^+$.

Further

$$\begin{aligned} |f(t,u,v)| &= \left|\frac{t}{1+t^2}y(t) + \int_0^t \frac{2e^{-s^{-2}}y(t)}{s^3(1+y^2(t)y^2(s))}ds\right| \\ &\le |y(t)| + \left|\int_0^t \frac{2e^{-s^{-2}}y(t)}{s^3(1+y^2(t)y^2(s))}ds\right|, \end{aligned} \tag{2.44}$$

$r(t) = \exp(-t^{-2})$, $\exp(-t^{-2}) = C\exp(3t_0^{-1} - 3t^{-1})o(1)$ as $t \to 0^+$ and

$$\left|\int_0^t \frac{2e^{-s^{-2}}y(t)}{s^3(1+y^2(t)y^2(s))}ds\right| \le \left(\exp\left(-t^{-2}\right)\right)|y(t)|. \tag{2.45}$$

According to Theorem 2.1, there exists for every constant $C \neq 0$ the unique solution $y(t, C)$ of (2.42) such that

$$\left|y^{(i)}(t,C) - \left(C\exp\left(\frac{3}{t_0} - \frac{3}{t}\right)\right)^{(i)}\right| \le \delta\left[\left(C\exp\left(\frac{3}{t_0} - \frac{3}{t}\right)\right)^2\right]^{(i)}, \quad i = 0, 1, \tag{2.46}$$

for $t \in (0, t^{\triangle}]$.

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