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Research Article

Stability of Difference Equations and Applications to Robustness Problems

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The aim of this paper is to obtain new necessary and sufficient conditions for the uniform exponential stability of variational difference equations with applications to robustness problems. We prove characterizations for exponential stability of variational difference equations using translation invariant sequence spaces and emphasize the importance of each hypothesis. We introduce a new concept of stability radius $r_{\mathrm{stab}}(A;B,C)$ for a variational system of difference equations (A) with respect to a perturbation structure (B,C) and deduce a very general estimate for the lower bound of $r_{\mathrm{stab}}(A;B,C)$. All the results are obtained without any restriction concerning the coefficients, being applicable for any system of variational difference equations.

1. Introduction

In the last decades an increasing interest was focused on the asymptotic properties of the most general class of evolution equations—the variational systems and a number of open questions were answered, increasing the applicability area not only to partial differential equations but also to systems arising from the linearization of nonlinear equations (see [1–10] and the references therein). In this context a special attention was devoted to the general case of variational systems of difference equations of the form:

$$x(\theta)(n+1) = A(\sigma(\theta,n))x(\theta)(n), \quad \forall(\theta,n) \in \Theta \times \mathbb{N}, \tag{A}$$

where $\{A(\theta)\}_{\theta\in\Theta}$ is a family of bounded linear operators on a Banach space X and σ is a flow on a metric space Θ. The interest is motivated by several notable advantages related with the system (A): the obtained results are applicable to a large class of systems; since no measurability or continuity conditions are needed, this system often models the families of equations proceeding from the linearization of nonlinear equations and also extends the nonautonomous case in infinite-dimensional spaces (see [1, 2, 4–10] and the references therein).

In recent years a notable progress was made in the study of the qualitative properties of various classes of difference equations (see [4–8, 11–32]). The input-output methods or the so-called "theorems of Perron type" have proved to be important tools in the study of the asymptotic behavior of difference equations like stability (see [27, 29, 30]), expansiveness (see [6]), dichotomy (see [1, 5, 8, 15, 17, 19, 20, 32]), and trichotomy as well (see [7, 18, 20]). A distinct method for the study of exponential stability relies on the convergence of some associated series and this was used in [28] for difference equations and in [4] for variational difference equations. The exponential stability of difference equations with several variable delays and variable coefficients was studied in [16], where the authors obtained interesting conditions for global exponential stability using new computational formulas with respect to the coefficients. The uniform asymptotic stability of positive Volterra difference equations was recently studied in [25], the authors proving the equivalence between the uniform asymptotic stability of the zero solution, the summability of the fundamental solution, and the invertibility of an associated operator outside the unit disk. A very efficient method in the study of the stability of difference equations is represented by the so-called "freezing technique," which was used in [22] for the study of absolute stability of discrete-time systems with delay and also in [23] in order to deduce explicit conditions for global feedback exponential stabilizability of discrete-time control systems with multiple state delays. In the study of the asymptotic behavior of discrete-time systems, there is an increasing interest in finding methods arising from control theory (see [5–8, 21–27, 29–32]). This is motivated by the fact that besides their large applicability area, the control-type techniques can be also applied to the analysis of the robustness of diverse properties in the presence of perturbations (see [5, 25, 26, 30, 32, 33]). In this context it is natural to extend the study to the variational case. Thus, two main questions arise: which is the most general framework for the study of the stability of variational difference equations using control type methods and how one may apply the new techniques in order to determine the behavior of the initial system in the presence of perturbations. In what follows, our attention will focus in order to provide complete answers to these open questions.

In this paper we propose a new study concerning the stability of variational difference equations and the robustness of this property. We associate with the system (A) a family of control systems $(S_A) = \{S_\theta\}_{\theta\in\Theta}$ and we attack the subject from the perspective of the solvability of (S_A) between two Banach sequence spaces invariant under translations. We split the class of Banach sequence spaces which are invariant under translations into two central subclasses and deduce necessary and sufficient conditions for uniform and exponential stability with respect to the solvability of the control system (S_A) when the input sequences belong to a space from a subclass or the solution lies in a space from the other subclass (see Theorem 3.8). By an example we show that the stability result is the most general in the topic and that the assumptions on the underlying sequence spaces cannot be removed. As particular cases of the stability results we deduce many interesting situations; among them we mention some direct generalizations at the variational case of the theorems from [27, 29, 30]. We also mention that the associated control system is distinct compared with those considered in the study of dichotomy and trichotomy (see [7, 8, 32, 34]), the input-output conditions are different, the Banach sequence norm is more flexible, and the underlying classes of sequence spaces are the largest and with more permissive properties.

Next, we apply the stability results and we propose a new approach for stability robustness of variational difference equations. We study the stable behavior of the system (A) in the presence of a general perturbation structure (B, C) with $B \in \ell^\infty(\Theta, \mathcal{L}(U, X))$, $C \in \ell^\infty(\Theta, \mathcal{L}(X, Y))$, by means of the stability radius $r_{\text{stab}}(A; B, C)$ (see Definition 4.3). Our target is to obtain a lower bound for the stability radius of variational systems of difference equations as well as to determine the largest class of Banach sequence spaces within the robustness properties hold. With this purpose we associate with the system (A) an input-output control system $(\mathcal{S}_A) = \{\mathcal{S}_\theta\}_{\theta\in\Theta}$ and consider $\mathcal{W}(\mathbb{N})$ the general class of all Banach sequence spaces W with the property that if there is $M > 0$ such that $|s \cdot \chi_{\{0,\ldots,n\}}|_W \leq M$, for all $n \in \mathbb{N}$, then $s \in W$ and $|s|_W \leq M$. For every Banach sequence space $W \in \mathcal{W}(\mathbb{N})$, we introduce the index $\lambda_W(A; B, C) := \sup_{\theta\in\Theta}\|\Lambda_W^\theta\|$, where $\{\Lambda_W^\theta\}_{\theta\in\Theta}$ is the family of input-output operators associated with the system $(\mathcal{S}_A)$ and we obtain a lower bound for $r_{\text{stab}}(A; B, C)$ in terms of $\lambda_W(A; B, C)$. Thus, we point out an interesting connection between the family of input-output operators and the size of the smallest perturbation in the presence of which the perturbed system loses its exponential stability. The variational case requires a special analysis and the methods are substantially more complicated compared to those used in the nonautonomous case (see [30, 33]). We note that the study is done without any restriction or assumption on the coefficients, the obtained results being applicable for any system of variational difference equations.

2. Banach Sequence Spaces

In this section, for the sake of clarity, we will recall some basic definitions and properties of Banach sequences spaces. Let $\mathbb{Z}$ denote the set of the integers, let $\mathbb{N}$ denote the set of all non negative integers, let $\mathbb{R}$ denote the set of all real numbers, and let $\mathcal{S}(\mathbb{N}, \mathbb{R})$ be the linear space of all sequences $s : \mathbb{N} \to \mathbb{R}$. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For every set $A \subset \mathbb{N}$ we denote by χ_A the characteristic function of the set A. For every $s \in \mathcal{S}(\mathbb{N}, \mathbb{R})$ we consider the sequence $s_+ : \mathbb{N} \to \mathbb{R}$ defined by $s_+(0) = 0$ and $s_+(n) = s(n-1)$, for all $n \in \mathbb{N}^*$.

Definition 2.1. A linear space $B \subset \mathcal{S}(\mathbb{N}, \mathbb{R})$ is called a *normed sequence space* if there is a mapping $|\cdot|_B : B \to \mathbb{R}_+$ such that

(i) $|s|_B = 0$ if and only if $s = 0$;

(ii) $|\alpha s|_B = |\alpha||s|_B$, for all $(\alpha, s) \in \mathbb{R} \times B$;

(iii) $|s + \gamma|_B \leq |s|_B + |\gamma|_B$, for all $s, \gamma \in B$;

(iv) if $s, \gamma \in \mathcal{S}(\mathbb{N}, \mathbb{R})$ have the property that $|s(j)| \leq |\gamma(j)|$, for all $j \in \mathbb{N}$ and $\gamma \in B$, then $s \in B$ and $|s|_B \leq |\gamma|_B$.

If, moreover, $(B, |\cdot|_B)$ is complete, then B is called *Banach sequence space*.

Definition 2.2. A Banach sequence space $(B, |\cdot|_B)$ is called *invariant under translations* if for every $s \in B$ the sequence s_+belongs to B and $|s_+|_B = |s|_B$.

Notation. We denote by $\mathcal{Q}(\mathbb{N})$ the class of all Banach sequence spaces B which are invariant under translations and $\chi_{\{0\}} \in B$.

Example 2.3 (*Orlicz sequence spaces*). Let $\varphi : \mathbb{R}_+ \to [0,\infty]$ be a nondecreasing left continuous function which is not identically 0 or ∞ on $(0,\infty)$. The *Young function* associated with φ is $Y_\varphi(t) = \int_0^t \varphi(s)\,ds$, for all $t \geq 0$. For every $s \in \mathcal{S}(\mathbb{N},\mathbb{R})$, let $M_\varphi(s) := \sum_{k=0}^{\infty} Y_\varphi(|s(k)|)$. Then $\ell_\varphi(\mathbb{N},\mathbb{R}) := \{s \in \mathcal{S}(\mathbb{N},\mathbb{R}) : \exists c > 0$ such that $M_\varphi(cs) < \infty\}$ is a Banach space with respect to the norm $|s|_\varphi := \inf\{c > 0 : M_\varphi(s/c) \leq 1\}$. The space $\ell_\varphi(\mathbb{N},\mathbb{R})$ is called the *Orlicz sequence space* associated to φ. It is easy to see that

(i) the space $\ell_\varphi(\mathbb{N},\mathbb{R})$ belongs to the class $Q(\mathbb{N})$;

(ii) if $p \in [1,\infty)$, then $\ell^p(\mathbb{N},\mathbb{R})$ with respect to the norm $\|s\|_p = (\sum_{k=0}^{\infty} \|s(k)\|^p)^{1/p}$ is an Orlicz space, obtained for $\varphi(t) = pt^{p-1}$;

(iii) $\ell^\infty(\mathbb{N},\mathbb{R})$, with respect to the norm $\|s\|_\infty = \sup_{n\in\mathbb{N}}|s(n)|$, is also an Orlicz space corresponding to the function $\varphi(t) = 0$, for $t \in [0,1]$ and $\varphi(t) = \infty$, for $t > 1$.

Example 2.4. The space $c_0(\mathbb{N},\mathbb{R}) := \{s \in \mathcal{S}(\mathbb{N},\mathbb{R}) : \lim_{n\to\infty} s(n) = 0\}$ with respect to the norm $\|s\|_\infty = \sup_{n\in\mathbb{N}}|s(n)|$ belongs to the class $Q(\mathbb{N})$.

Remark 2.5. If $B \in Q(\mathbb{N})$, then the following properties hold:

(i) for every $A \subset \mathbb{N}$, $\chi_A \in B$;

(ii) for every $s \in B$ and every $j \in \mathbb{N}$ the sequence

$$s_j : \mathbb{N} \longrightarrow \mathbb{R}, \quad s_j(n) = \begin{cases} s(n-j), & n \geq j, \\ 0, & n < j \end{cases} \tag{2.1}$$

belongs to B and $|s_j|_B = |s|_B$,

(iii) $\ell^1(\mathbb{N},\mathbb{R}) \subset B \subset \ell^\infty(\mathbb{N},\mathbb{R})$ (see, e.g., [30, Lemma 2.1]);

(iv) if $s \in B$, then the sequence $|s| : \mathbb{N} \to \mathbb{R}_+$, $|s|(k) = |s(k)|$ also belongs to B and $||s||_B = |s|_B$.

Definition 2.6. If $(B, |\cdot|_B)$ is a Banach sequence space with $B \in Q(\mathbb{N})$ then $F_B : \mathbb{N}^* \to \mathbb{R}_+$, $F_B(n) = |\chi_{\{0,\ldots,n-1\}}|_B$ is called *the fundamental function* of B.

Lemma 2.7. *If $B \in Q(\mathbb{N})$ and $s_n \to s$ in B, then $s_n \to s$ pointwise.*

Proof. Let $j \in \mathbb{N}$. From $|s_n(j) - s(j)|\chi_{\{j\}}(k) \leq |s_n(k) - s(k)|$, for all $k \in \mathbb{N}$ and all $n \in \mathbb{N}$ we deduce that $|s_n(j) - s(j)|F_B(1) \leq |s_n - s|_B$, for all $n \in \mathbb{N}$ and the proof is complete. □

Notations. We denote by $\mathcal{V}(\mathbb{N})$ the class of all Banach sequence spaces $B \in Q(\mathbb{N})$ with the property that $\sup_{n\in\mathbb{N}} F_B(n) = \infty$ and by $\mathcal{U}(\mathbb{N})$ the class of all Banach sequence spaces $B \in Q(\mathbb{N})$ with the property that $\ell^1(\mathbb{N},\mathbb{R}) \subsetneq B$.

Lemma 2.8. *If $B \in Q(\mathbb{N})$, then $B \in Q(\mathbb{N}) \setminus \mathcal{V}(\mathbb{N})$ if and only if $c_0(\mathbb{N},\mathbb{R}) \subset B$.*

Proof. Necessity. If $B \in Q(\mathbb{N}) \setminus \mathcal{V}(\mathbb{N})$, then $\lambda_B := \sup_{n\in\mathbb{N}} F_B(n) < \infty$. Let $s \in c_0(\mathbb{N},\mathbb{R})$. Then there is a strictly increasing sequence (k_n) such that $|s(j)| \leq 1/(n+1)$, for all $j \geq k_n$ and all $n \in \mathbb{N}$. Setting $s_n = s\chi_{\{0,\ldots,k_n\}}$ we deduce that $|s_{n+p} - s_n|_B \leq \lambda_B/(n+1)$, for all $n \in \mathbb{N}$ and all $p \in \mathbb{N}^*$. It follows that (s_n) is a Cauchy sequence in B; so this is convergent. Let $u \in B$ be such that

$s_n \to u$ in B. According to Lemma 2.7 we deduce that $u = s$, and so $s \in B$. This implies that $c_0(\mathbb{N},\mathbb{R}) \subset B$.

Sufficiency. If $c_0(\mathbb{N},\mathbb{R}) \subset B$, then there is $c > 0$ such that $|s|_B \le c\|s\|_\infty$, for all $s \in c_0(\mathbb{N},\mathbb{R})$. Then, we obtain that $F_B(n) = |\chi_{\{0,\ldots,n-1\}}|_B \le c\|\chi_{\{0,\ldots,n-1\}}\|_\infty = c$, for all $n \in \mathbb{N}^*$, and so $\lambda_B < \infty$. It follows that $B \in \mathcal{Q}(\mathbb{N}) \setminus \mathcal{U}(\mathbb{N})$. □

Remark 2.9. From Remark 2.5 and Lemma 2.8 we have that $B \in \mathcal{Q}(\mathbb{N}) \setminus \mathcal{U}(\mathbb{N})$ if and only if $c_0(\mathbb{N},\mathbb{R}) \subset B \subset \ell^\infty(\mathbb{N},\mathbb{R})$.

Another interesting property of the class $\mathcal{U}(\mathbb{N})$ is the following.

Lemma 2.10. *Let $\ell_\varphi(\mathbb{N},\mathbb{R})$ be an Orlicz space. Then either $\ell_\varphi(\mathbb{N},\mathbb{R}) \in \mathcal{U}(\mathbb{N})$ or $\ell_\varphi(\mathbb{N},\mathbb{R}) = \ell^\infty(\mathbb{N},\mathbb{R})$.*

Proof. Suppose that $\ell_\varphi(\mathbb{N},\mathbb{R}) \notin \mathcal{U}(\mathbb{N})$; so $\lambda_\varphi := \sup_{n\in\mathbb{N}} F_{\ell_\varphi}(n) < \infty$. Then $(n+1)Y_\varphi(1/\lambda_\varphi) = M_\varphi(\chi_{\{0,\ldots,n\}}/\lambda_\varphi) \le 1$, for all $n \in \mathbb{N}$. This implies that $Y_\varphi(1/\lambda_\varphi) = 0$. Let $s \in \ell^\infty(\mathbb{N},\mathbb{R})$ and let $u := s/[\lambda_\varphi(1 + \|s\|_\infty)]$. Then $|u(k)| < 1/\lambda_\varphi$, for every $k \in \mathbb{N}$. Since Y_φ is nondecreasing, this yields $Y_\varphi(|u(k)|) = 0$, for every $k \in \mathbb{N}$, so $M_\varphi(u) = 0$. It follows that $u \in \ell_\varphi(\mathbb{N},\mathbb{R})$; so $s \in \ell_\varphi(\mathbb{N},\mathbb{R})$. Hence $\ell^\infty(\mathbb{N},\mathbb{R}) \subset \ell_\varphi(\mathbb{N},\mathbb{R})$ and using Remark 2.5(iii) we obtain the conclusion. □

Lemma 2.11. *Let $B \in \mathcal{Q}(\mathbb{N})$ and let $\nu > 0$. Then, for every $s \in B$ the sequence*

$$\alpha_s : \mathbb{N} \longrightarrow \mathbb{R}, \quad \alpha_s(n) = \begin{cases} \displaystyle\sum_{k=1}^{n} e^{-\nu(n-k)} s(k-1), & n \in \mathbb{N}^*, \\ 0, & n = 0 \end{cases} \tag{2.2}$$

belongs to B. Moreover

$$|\alpha_s|_B \le \frac{1}{1-e^{-\nu}}|s|_B, \quad \forall s \in B. \tag{2.3}$$

Proof. Let $s \in B$. Then the sequence s_+ belongs to B and $|s_+|_B = |s|_B$. Using the notations from Remark 2.5(ii) we deduce that

$$\begin{aligned} |\alpha_s(n)| &= \left|\sum_{k=0}^{n} e^{-\nu(n-k)} s_+(k)\right| \le \sum_{k=0}^{n} e^{-\nu(n-k)} |s_+(k)| = \sum_{j=0}^{n} e^{-\nu j} \left|s_{j+1}(n)\right| \\ &\le \sum_{j=0}^{\infty} e^{-\nu j} \left|s_{j+1}(n)\right|, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.4}$$

From Remark 2.5(ii) we obtain that $\alpha_s \in B$ and that relation (2.3) holds. □

Notation. We denote by $\mathcal{W}(\mathbb{N})$ the class of all Banach sequence spaces $B \in \mathcal{Q}(\mathbb{N})$ with the property that if $s \in \mathcal{S}(\mathbb{N},\mathbb{R})$ and there is $M > 0$ such that $|s \cdot \chi_{\{0,\ldots,n\}}|_B \le M$, for all $n \in \mathbb{N}$, then $s \in B$ and $|s|_B \le M$.

Example 2.12. Any Orlicz sequence space O_φ belongs to the class $\mathcal{W}(\mathbb{N})$. Consequently, $\ell^p(\mathbb{N},\mathbb{R}) \in \mathcal{W}(\mathbb{N})$, for every $p \in [1,\infty]$.

Remark 2.13. The class $\mathcal{W}(\mathbb{N})$ will play a fundamental role for a new study of the robustness of uniform exponential stability presented in Section 4.

Notation. Let $(X, \|\cdot\|)$ be a real or complex Banach space. For every Banach sequence space $B \in \mathcal{Q}(\mathbb{N})$ we denote by $B(\mathbb{N}, X)$ the space of all sequences $s : \mathbb{N} \to X$ with the property that the mapping $N_s : \mathbb{N} \to \mathbb{R}_+$, $N_s(m) = \|s(m)\|$ belongs to B. $B(\mathbb{N}, X)$ is a Banach space with respect to the norm $\|s\|_{B(\mathbb{N},X)} := |N_s|_B$.

3. Stability of Variational Difference Equations

Let X be a real or complex Banach space and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X. Throughout this paper the norm on X will be denoted by $\|\cdot\|$. For every $T \in \mathcal{L}(X)$, the norm of T is defined by

$$\|T\| := \inf\{M > 0 : \|T(x)\| \le M\|x\|,\ \forall x \in X\} = \sup_{\|x\|\le 1} \|T(x)\|. \tag{3.1}$$

We denote by $\Delta(\mathbb{N}, X)$ the linear space of all sequences $s : \mathbb{N} \to X$ with the property that the set $\{j \in \mathbb{N} : s(j) \neq 0\}$ is finite and by I_d the identity operator on X.

Let (Θ, d) be a metric space and let $\mathcal{E} = X \times \Theta$. Let $\sigma : \Theta \times \mathbb{Z} \to \Theta$ be a discrete flow on Θ, that is, $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, m+n) = \sigma(\sigma(\theta, m), n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{Z}^2$.

Let $\{A(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$. We consider the variational system of variational difference equations (A). We note that in the particular case when $\Theta = \mathbb{N}$ and $\sigma(\theta, n) = \theta + n$ we obtain the case of difference equations. There are several distinct directions of generalizing the case of difference equations. One of the most interesting methods is to consider them in the general framework of dynamic equations on time scales (see [35, 36]), having a wide potential for applications in the study of population dynamics. Another method is to consider them as particular cases of variational difference equations, which often proceed from the linearization of nonlinear equations (see [2, 10] and the references therein). It is also interesting to note that the exponential stability of a variational equation is equivalent with the exponential stability of the variational difference equation associated with it. Therefore, concerning the stability of variational equations it is recommended to study the discrete-time case, because no measurability or continuity conditions are required.

The discrete cocycle associated with the system (A) is

$$\Phi : \Theta \times \mathbb{N} \longrightarrow \mathcal{L}(X), \quad \Phi(\theta, n) = \begin{cases} A(\sigma(\theta, n-1)) \cdots A(\theta), & n \in \mathbb{N}^*, \\ I_d, & n = 0. \end{cases} \tag{3.2}$$

Remark 3.1. The discrete cocycle satisfies $\Phi(\theta, m+n) = \Phi(\sigma(\theta, n), m)\Phi(\theta, n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{N}^2$ (*the evolution property*).

Definition 3.2. The system (A) is said to be

(i) *uniformly stable* if there is $K > 0$ such that $\|\Phi(\theta, n)\| \leq K$, for all $(\theta, n) \in \Theta \times \mathbb{N}$;

(ii) *uniformly exponentially stable* if there are $K, \nu > 0$ such that $\|\Phi(\theta, n)\| \leq Ke^{-\nu n}$, for all $(\theta, n) \in \Theta \times \mathbb{N}$.

We associate with the system (A) the input-output system $(S_A) = \{S_\theta\}_{\theta\in\Theta}$, where for every $\theta \in \Theta$,

$$\begin{gathered} x_\theta(n+1) = A(\sigma(\theta, n))x_\theta(n) + s(n), \quad n \in \mathbb{N}, \\ x_\theta(0) = 0. \end{gathered} \tag{S_θ}$$

Remark 3.3. For every $(\theta, s) \in \Theta \times \Delta(\mathbb{N}, X)$, the solution of (2.1) has the form:

$$x_{\theta,s}(n) = \sum_{k=1}^{n} \Phi(\sigma(\theta, k), n-k)s(k-1), \quad \forall n \in \mathbb{N}^*. \tag{3.3}$$

Definition 3.4. Let U, V be two Banach sequence spaces with $U, V \in Q(\mathbb{N})$. The system (S_A) is said to be $(U(\mathbb{N}, X), V(\mathbb{N}, X))$- *stable* if the following assertions hold:

(i) for every $s \in \Delta(\mathbb{N}, X)$ and $\theta \in \Theta$ the solution $x_{\theta,s}$belongs to $V(\mathbb{N}, X)$;

(ii) there is $L > 0$ such that $\|x_{\theta,s}\|_{V(\mathbb{N},X)} \leq L\|s\|_{U(\mathbb{N},X)}$, for all $(\theta, s) \in \Theta \times \Delta(\mathbb{N}, X)$.

$U(\mathbb{N}, X)$ is called *the input space* and $V(\mathbb{N}, X)$ is called *the output space*.

Our first result provides a sufficient condition for uniform stability and is given by the following.

Theorem 3.5. *Let $U, V \in Q(\mathbb{N})$. If the system (S_A) is $(U(\mathbb{N}, X), V(\mathbb{N}, X))$-stable, then the system (A) is uniformly stable.*

Proof. Let $L > 0$ be given by Definition 3.4. Let $\theta \in \Theta$ and let $x \in X$. We consider the sequence

$$s : \mathbb{N} \longrightarrow X, \quad s(n) = \chi_{\{0\}}(n)x. \tag{3.4}$$

Then, $s \in \Delta(\mathbb{N}, X)$ and $\|s\|_{U(\mathbb{N},X)} = F_U(1)\|x\|$. We observe that $x_{\theta,s}(n) = \Phi(\sigma(\theta, 1), n-1)x$, for all $n \in \mathbb{N}^*$. Let $p \in \mathbb{N}$. Then from

$$\big\|\Phi(\sigma(\theta, 1), p)x\big\|\chi_{\{p+1\}}(k) \leq \|x_{\theta,s}(k)\|, \quad \forall k \in \mathbb{N}, \tag{3.5}$$

we obtain that

$$\big\|\Phi(\sigma(\theta, 1), p)x\big\|F_V(1) \leq \|x_{\theta,s}\|_{V(\mathbb{N},X)} \leq L\|s\|_{U(\mathbb{N},X)} = LF_U(1)\|x\|. \tag{3.6}$$

Setting $K = (LF_U(1))/F_V(1)$ it follows that

$$\|\Phi(\sigma(\theta,1),n)x\| \leq K\|x\|, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Taking into account that K does not depend on θ or x, from (3.7) we have that

$$\|\Phi(\sigma(\theta,1),n)\| \leq K, \quad \forall n \in \mathbb{N},\ \forall \theta \in \Theta. \tag{3.8}$$

Let $\theta \in \Theta$. By applying relation (3.8) for $\widetilde{\theta} = \sigma(\theta,-1)$, we deduce that $\|\Phi(\theta,n)\| \leq K$, for all $n \in \mathbb{N}$ and thus the proof is complete. □

Corollary 3.6. *The system* (A) *is uniformly stable if and only if the system* (S_A) *is* $(\ell^\infty(\mathbb{N},X), \ell^1(\mathbb{N},X))$*-stable.*

Proof. Necessity is immediate via Definition 3.4 and sufficiency follows from Theorem 3.5. □

The first main result of this section is the following.

Theorem 3.7. *Let* $U, V \in \mathcal{Q}(\mathbb{N})$ *be such that either* $U \in \mathcal{U}(\mathbb{N})$ *or* $V \in \mathcal{V}(\mathbb{N})$*. If the system* (S_A) *is* $(U(\mathbb{N},X), V(\mathbb{N},X))$*-stable, then the system* (A) *is uniformly exponentially stable.*

Proof. Let $L > 0$ be given by Definition 3.4. From Theorem 3.5 it follows that there is $M > 0$ such that

$$\|\Phi(\theta,n)x\| \leq M\|x\|, \quad \forall (\theta,n) \in \Theta \times \mathbb{N},\ \forall x \in X. \tag{3.9}$$

Case 1. If $U \in \mathcal{U}(\mathbb{N})$ then, according to Remark 2.5(iv) we deduce that there is a sequence $\delta : \mathbb{N} \to \mathbb{R}_+$ with $\delta \in U \setminus \ell^1(\mathbb{N},\mathbb{R})$. Let $h \in \mathbb{N}^*$ be such that

$$\sum_{j=0}^{h} \delta(j) \geq \frac{eLM|\delta|_U}{F_V(1)}. \tag{3.10}$$

Let $\theta \in \Theta$ and let $x \in X$. We consider the sequence

$$s : \mathbb{N} \longrightarrow X, \quad s(n) = \chi_{\{0,\ldots,h\}}(n)\delta(n)\Phi(\theta,n+1)x. \tag{3.11}$$

We have that $s \in \Delta(\mathbb{N},X)$ and using (3.9) we obtain that $\|s(n)\| \leq M\|x\|\delta(n)$, for all $n \in \mathbb{N}$, which implies that

$$\|s\|_{U(\mathbb{N},X)} \leq M\|x\||\delta|_U. \tag{3.12}$$

According to our hypothesis we deduce that

$$\|x_{\theta,s}\|_{V(\mathbb{N},X)} \leq L\|s\|_{U(\mathbb{N},X)} \leq LM\|x\||\delta|_U. \tag{3.13}$$

We observe that

$$x_{\theta,s}(h+1) = \left(\sum_{j=0}^{h}\delta(j)\right)\Phi(\theta,h+1)x. \tag{3.14}$$

Then from

$$\|x_{\theta,s}(h+1)\|\chi_{\{h+1\}}(n) \le \|x_{\theta,s}(n)\|, \quad \forall n \in \mathbb{N} \tag{3.15}$$

and using relations (3.13) and (3.14) we obtain that

$$\left(\sum_{j=0}^{h}\delta(j)\right)\|\Phi(\theta,h+1)x\|F_V(1) \le \|x_{\theta,s}\|_{V(\mathbb{N},X)} \le LM\|x\||\delta|_U. \tag{3.16}$$

From relations (3.10) and (3.16) it follows that $\|\Phi(\theta,h+1)x\| \le (1/e)\|x\|$. Setting $p = h+1$ and taking into account that p does not depend on θ or x we have that

$$\left\|\Phi(\theta,p)x\right\| \le \left(\frac{1}{e}\right)\|x\|, \quad \forall\theta \in \Theta,\ \forall x \in X. \tag{3.17}$$

Let $\nu = 1/p$ and let $K = Me$. Let $\theta \in \Theta$ and let $n \in \mathbb{N}$. Then there are $k \in \mathbb{N}$ and $j \in \{0,\dots,p-1\}$ such that $n = kp + j$. Using relations (3.9) and (3.17) we have that

$$\|\Phi(\theta,n)\| \le M\left\|\Phi(\theta,kp)\right\| \le Me^{-k} \le Ke^{-\nu n}. \tag{3.18}$$

Case 2. If $V \in \mathcal{U}(\mathbb{N})$ then there is $p \in \mathbb{N}^*$ such that

$$F_V(p) \ge eLM^2F_U(1). \tag{3.19}$$

Let $\theta \in \Theta$ and let $x \in X$. We consider the sequence

$$u : \mathbb{N} \longrightarrow X, \quad u(n) = \chi_{\{0\}}(n)\Phi(\theta,1)x. \tag{3.20}$$

Then $u \in \Delta(\mathbb{N},X)$ and according to relation (3.9) we have that

$$\|u\|_{U(\mathbb{N},X)} \le MF_U(1)\|x\|. \tag{3.21}$$

It is easy to see that $x_{\theta,u}(n) = \Phi(\theta,n)x$, for all $n \in \mathbb{N}^*$. Then, using relation (3.9) we deduce that

$$\left\|\Phi(\theta,p)x\right\|\chi_{\{1,\dots,p\}}(n) \le M\|x_{\theta,u}(n)\|, \quad \forall n \in \mathbb{N} \tag{3.22}$$

which implies that

$$\|\Phi(\theta,p)x\|F_V(p) \leq M\|x_{\theta,u}\|_{V(\mathbb{N},X)} \leq LM\|u\|_{U(\mathbb{N},X)} \leq LM^2F_U(1)\|x\|. \tag{3.23}$$

From relations (3.19) and (3.23) it follows that $\|\Phi(\theta,p)x\| \leq (1/e)\|x\|$. Taking into account that p does not depend on θ or x we obtain that $\|\Phi(\theta,p)x\| \leq (1/e)\|x\|$, for all $\theta \in \Theta$ and all $x \in X$. Using similar arguments with those from Case 1, we deduce that the system (A) is uniformly exponentially stable. □

The second main result of this section is the following.

Theorem 3.8. *Let $U, V \in \mathcal{Q}(\mathbb{N})$ be such that either $U \in \mathcal{U}(\mathbb{N})$ or $V \in \mathcal{V}(\mathbb{N})$. The following assertions hold:*

(i) *if the system (S_A) is $(U(\mathbb{N},X),V(\mathbb{N},X))$-stable, then the system (A) is uniformly exponentially stable;*

(ii) *if $U \subset V$, then the system (A) is uniformly exponentially stable if and only if the system (S_A) is $(U(\mathbb{N},X),V(\mathbb{N},X))$-stable.*

Proof. (i) This follows from Theorem 3.7.

(ii) *Necessity.* Let $K,\nu > 0$ be such that $\|\Phi(\theta,n)\| \leq Ke^{-\nu n}$, for all $(\theta,n) \in \Theta \times \mathbb{N}$. Let $\theta \in \Theta$ and let $s \in \Delta(\mathbb{N},X)$. Then

$$\|x_{\theta,s}(n)\| \leq K\sum_{k=1}^{n} e^{-\nu(n-k)}\|s(k-1)\|, \quad \forall n \in \mathbb{N}^*. \tag{3.24}$$

Since $s \in \Delta(\mathbb{N},X)$, in particular $s \in V(\mathbb{N},X)$, so $\|s(\cdot)\| \in V(\mathbb{N},\mathbb{R})$. According to Lemma 2.11, the sequence

$$\alpha_s : \mathbb{R} \longrightarrow \mathbb{R}, \quad \alpha_s(n) = \begin{cases} \displaystyle\sum_{k=1}^{n} e^{-\nu(n-k)}\|s(k-1)\|, & n \in \mathbb{N}^*, \\ 0, & n = 0 \end{cases} \tag{3.25}$$

belongs to $V(\mathbb{N},\mathbb{R})$ and

$$|\alpha_s|_V \leq \frac{1}{1-e^{-\nu}}\|s\|_{V(\mathbb{N},X)}. \tag{3.26}$$

From relations (3.24) and (3.26) it follows that $x_{\theta,s} \in V(\mathbb{N},X)$ and

$$\|x_{\theta,s}\|_{V(\mathbb{N},X)} \leq \frac{K}{1-e^{-\nu}}\|s\|_{V(\mathbb{N},X)}. \tag{3.27}$$

Since $U \subset V$ there is $\lambda > 0$ such that $|u|_V \leq \lambda|u|_U$, for all $u \in U$. Setting $L = \lambda K/(1-e^{-\nu})$ from relation (3.27) we deduce that $\|x_{\theta,s}\|_{V(\mathbb{N},X)} \leq L\|s\|_{U(\mathbb{N},X)}$. Taking into account that L does not depend on θ or s, it follows that the system (S_A) is $(U(\mathbb{N},X),V(\mathbb{N},X))$-stable.

Sufficiency. This follows from (i). □

In what follows, we prove that the result obtained in Theorem 3.8 is the most general in this topic. Precisely, we will show that if $U \notin \mathcal{U}(\mathbb{N})$ and $V \notin \mathcal{V}(\mathbb{N})$, then the $(U(\mathbb{N}, X), V(\mathbb{N}, X))$-stability of the system (S_A) does not assure the uniform exponential stability of the system (A).

Example 3.9. Let $\Theta = \mathbb{R}$ and let $\sigma : \Theta \times \mathbb{Z} \to \Theta$, $\sigma(\theta, m) = \theta + m$. Then σ is a discrete flow on Θ. We consider the function

$$\varphi : \mathbb{R} \longrightarrow (0, \infty), \quad \varphi(t) = \begin{cases} t+1, & t \geq 0, \\ e^t, & t < 0. \end{cases} \tag{3.28}$$

It is easy to see that φ is nondecreasing with $\lim_{t\to\infty} \varphi(t) = \infty$.

Let X be a Banach space. For every $\theta \in \Theta$, let

$$A(\theta) : X \longrightarrow X, \quad A(\theta)x = \frac{\varphi(\theta)}{\varphi(\theta + 1)} x. \tag{3.29}$$

The discrete cocycle associated with the system(A) is

$$\Phi(\theta, n)x = \frac{\varphi(\theta)}{\varphi(\theta + n)} x, \quad \forall x \in X, \ \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{3.30}$$

We associate with the system (A) then input-output system $(S_A) = \{S_\theta\}_{\theta\in\Theta}$.

Using (3.30) it follows that

$$x_{\theta,s}(n) = \sum_{k=1}^{n} \frac{\varphi(\theta + k)}{\varphi(\theta + n)} s(k-1), \quad \forall n \in \mathbb{N}^*, \ \forall(\theta, s) \in \Theta \times \Delta(\mathbb{N}, X). \tag{3.31}$$

Let $U, V \in Q(\mathbb{N})$ with $U \notin \mathcal{U}(\mathbb{N})$ and $V \notin \mathcal{V}(\mathbb{N})$. Then, from Remark 2.5 we have that $U = \ell^1(\mathbb{N}, \mathbb{R})$ and from Remark 2.9 we obtain that $c_0(\mathbb{N}, \mathbb{R}) \subset V$. Then, there is $L > 0$ such that

$$|w|_V \leq L\|w\|_\infty, \quad \forall w \in c_0(\mathbb{N}, \mathbb{R}). \tag{3.32}$$

We prove that the system (S_A) is $(U(\mathbb{N}, X), V(\mathbb{N}, X))$-stable. Indeed, let $\theta \in \Theta$ and let $s \in \Delta(\mathbb{N}, X)$. Then, there is $h \in \mathbb{N}^*$ such that $s(j) = 0$, for $j \geq h$. From (3.31) we have that

$$x_{\theta,s}(n) = \frac{1}{\varphi(\theta + n)} \sum_{k=1}^{h+1} \varphi(\theta + k)s(k-1), \quad \forall n \geq h+1. \tag{3.33}$$

Since $\varphi(t) \to \infty$ as $t \to \infty$, we obtain that $x_{\theta,s}(n) \to 0$ as $n \to \infty$. This shows that $x_{\theta,s} \in c_0(\mathbb{N}, X)$, so $x_{\theta,s} \in V(\mathbb{N}, X)$. Moreover, from

$$\|x_{\theta,s}(n)\| \le \sum_{k=1}^{n} \|s(k-1)\| \le \|s\|_{\ell^1(\mathbb{N},X)}, \quad \forall n \in \mathbb{N}, \tag{3.34}$$

we have that

$$\|x_{\theta,s}\|_{c_0(\mathbb{N},X)} \le \|s\|_{\ell^1(\mathbb{N},X)}. \tag{3.35}$$

From relations (3.32) and (3.35) it follows that

$$\|x_{\theta,s}\|_{V(\mathbb{N},X)} \le L\|s\|_{\ell^1(\mathbb{N},X)}. \tag{3.36}$$

Taking into account that L does not depend on θ or s it follows that the system (S_A) is $(\ell^1(\mathbb{N}, X), V(\mathbb{N}, X))$-stable. But, for all that, it is easy to verify that there are not $K, \nu > 0$ such that $\|\Phi(\theta, n)\| \le Ke^{-\nu n}$, for all $(\theta, n) \in \Theta \times \mathbb{N}$, so the system (A) is not uniformly exponentially stable.

Corollary 3.10. *Let $p, q \in [1, \infty]$ with $(p, q) \ne (1, \infty)$. The following assertions hold:*

(i) *if the system (S_A) is $(\ell^p(\mathbb{N}, X), \ell^q(\mathbb{N}, X))$-stable, then the system (A) is uniformly exponentially stable;*

(ii) *if $p \le q$, then the system (A) is uniformly exponentially stable if and only if the system (S_A) is $(\ell^p(\mathbb{N}, X), \ell^q(\mathbb{N}, X))$-stable.*

Corollary 3.11. *Let $W \in Q(\mathbb{N})$. The system (A) is uniformly exponentially stable if and only if the system (S_A) is $(W(\mathbb{N}, X), W(\mathbb{N}, X))$-stable.*

Proof. This follows from Theorem 3.8 observing that if $W \notin \mathcal{U}(\mathbb{N})$, then $W = \ell^1(\mathbb{N}, \mathbb{R})$, and so $W \in \mathcal{V}(\mathbb{N})$. □

Definition 3.12. Let $U, V \in Q(\mathbb{N})$. The system (S_A) is said to be *completely* $(U(\mathbb{N}, X), V(\mathbb{N}, X))$-*stable* if the following properties hold:

(i) for every $s \in U(\mathbb{N}, X)$ and every $\theta \in \Theta$ the solution $x_{\theta,s}$ belongs to $V(\mathbb{N}, X)$;

(ii) there is $L > 0$ such that $\|x_{\theta,s}\|_{V(\mathbb{N},X)} \le L\|s\|_{U(\mathbb{N},X)}$, for all $(\theta, s) \in \Theta \times U(\mathbb{N}, X)$.

As consequences of Theorem 3.8 we obtain the following.

Theorem 3.13. *Let $U, V \in \mathcal{Q}(\mathbb{N})$ with $U \in \mathcal{U}(\mathbb{N})$ or $V \in \mathcal{V}(\mathbb{N})$. The following assertions hold:*

(i) *if the system (S_A) is completely $(U(\mathbb{N}, X), V(\mathbb{N}, X))$-stable, then the system (A) is uniformly exponentially stable;*

(ii) *if $U \subset V$, then the system (A) is uniformly exponentially stable if and only if the system (S_A) is completely $(U(\mathbb{N}, X), V(\mathbb{N}, X))$-stable.*

Corollary 3.14. *Let $p, q \in [1, \infty]$ with $(p, q) \neq (1, \infty)$. The following assertions hold:*

(i) *if the system (S_A) is completely $(\ell^p(\mathbb{N}, X), \ell^q(\mathbb{N}, X))$-stable, then the system (A) is uniformly exponentially stable;*

(ii) *if $p \leq q$, then the system (A) is uniformly exponentially stable if and only if the system (S_A) is completely $(\ell^p(\mathbb{N}, X), \ell^q(\mathbb{N}, X))$-stable.*

Corollary 3.15. *Let $W \in \mathcal{Q}(\mathbb{N})$. The system (A) is uniformly exponentially stable if and only if the system (S_A) is completely $(W(\mathbb{N}, X), W(\mathbb{N}, X))$-stable.*

Remark 3.16. Let $W \in \mathcal{Q}(\mathbb{N})$. If the system (A) is uniformly exponentially stable, then according to Corollary 3.15, for every $\theta \in \Theta$, the operator

$$\Gamma_\theta : W(\mathbb{N}, X) \longrightarrow W(\mathbb{N}, X), \quad \Gamma_\theta(s) = x_{\theta,s} \tag{3.37}$$

is correctly defined and this is a bounded linear operator. Moreover, if $L > 0$ is given by Definition 3.12, then we have that $\sup_{\theta \in \Theta} \|\Gamma_\theta\| \leq L$.

Remark 3.17. From the above results it follows that in the study of exponential stability of variational difference equations using input-output techniques one may work with Banach sequence spaces which are invariant under translations, such that *either* the input space contains at least a sequence whose series is divergent *or* the output space has unbounded fundamental function. Moreover, according to Corollary 3.11 we deduce that when the input space and the output space coincide, then there is no other requirement on the underlying sequence spaces and it is sufficient to consider any Banach sequence space which is invariant under translations and contains at least a characteristic function of a singleton.

4. Applications to Robustness of Exponential Stability

In this section, by applying our main results we will study the persistence of the exponential stability in the presence of variational structured perturbations.

Notations. If Y, Z are two Banach spaces, we denote by $\mathcal{L}(Y, Z)$ the Banach space of all bounded linear operators $T : Y \to Z$. If $Y = Z$, we denote $\mathcal{L}(Y, Y) = \mathcal{L}(Y)$. If (Θ, d) is a metric space, we consider $\ell^\infty(\Theta, \mathcal{L}(Y, Z)) := \{R : \Theta \to \mathcal{L}(Y, Z) \mid \sup_{\theta \in \Theta} \|R(\theta)\| < \infty\}$, which is a Banach space with respect to the norm

$$|||R||| := \sup_{\theta \in \Theta} \|R(\theta)\|. \tag{4.1}$$

Let (Θ, d) be a metric space and let $\sigma : \Theta \times \mathbb{Z} \to \Theta$ be a discrete flow on Θ. Let X be a Banach space and let $\{A(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$. We consider the linear system of variational difference equations (A). For every $D \in \ell^\infty(\Theta, \mathcal{L}(X))$ we consider the perturbed system

$$x(\theta)(n+1) = [A(\sigma(\theta,n)) + D(\sigma(\theta,n))]x(\theta)(n), \quad (\theta,n) \in \Theta \times \mathbb{N}. \tag{$A+D$}$$

Remark 4.1. If Φ_A is the discrete cocycle associated to the system (A) and Φ_{A+D} is the discrete cocycle associated to the system (4.2), then the following perturbation formulas are satisfied:

$$\begin{aligned}\Phi_{A+D}(\theta,n) &= \Phi_A(\theta,n) + \sum_{k=1}^{n} \Phi_A(\sigma(\theta,k), n-k) D(\sigma(\theta,k-1)) \Phi_{A+D}(\theta,k-1),\\ \Phi_{A+D}(\theta,n) &= \Phi_A(\theta,n) + \sum_{k=1}^{n} \Phi_{A+D}(\sigma(\theta,k), n-k) D(\sigma(\theta,k-1)) \Phi_A(\theta,k-1)\end{aligned} \tag{4.2}$$

for all $(\theta,n) \in \Theta \times \mathbb{N}^*$.

In what follows we suppose that the system (A) is uniformly exponentially stable. The main question is how large may be the norm of the perturbation $D \in \ell^\infty(\Theta, \mathcal{L}(X))$ such that the perturbed system (4.2) remains uniformly exponentially stable. With this purpose we introduce in the following.

Definition 4.2. The number

$$\begin{aligned}r_{\text{stab}}(A) := \sup\{r > 0 :\ &\forall D \in \ell^\infty(\Theta, \mathcal{L}(X))\\ &\text{with } |||D||| < r \Longrightarrow (A+D) \text{ is uniformly exponentially stable}\}\end{aligned} \tag{4.3}$$

is called *the stability radius* of the system (A).

We note that in the existent literature there is not an explicit computational formula for the stability radius of systems, only in some sporadic special cases. For linear retarded systems of differential equations on $\mathbb{R}^n$, some interesting formulas were obtained by Ngoc and Son in [26, Theorems 3.5 and 3.10], but the estimates are still complicated. For the case of positive linear retarded systems which are Hurwitz stable, the authors succeeded to deduce a formula for the stability radius corresponding to multiaffine perturbations (see Theorem 4.7 and the following example). In [25] Murakami and Nagabuchi obtained an explicit formula for the stability radius of uniformly asymptotically stable positive Volterra difference equations on Banach lattices (see [25], Theorem 4.3). Generally, in order to analyze the persistence of the exponential stability in the presence of perturbations, it is interesting to find a lower bound for the stability radius of systems (see [3, 25, 26, 30, 33]) because in this manner we estimate the possible size of the disturbance operator under which the (additively) perturbed system remains exponentially stable.

In what follows, using the results obtained in the previous section we will estimate a lower bound for the stability radius of the system (A). We will provide a detailed study, when the system (A) is subject to a very general perturbation structure.

Let U, Y be two Banach spaces. Let $B \in \ell^\infty(\Theta, \mathcal{L}(U,X))$, $C \in \ell^\infty(\Theta, \mathcal{L}(X,Y))$.

Definition 4.3. The number

$$r_{\text{stab}}(A;B,C) := \sup\{r > 0 : \forall P \in \ell^{\infty}(\Theta, \mathcal{L}(Y,U)) \text{ with } |||P||| < r \Longrightarrow (A+BPC) \text{ is uniformly exponentially stable}\} \tag{4.4}$$

is called *the stability radius* of the system (A) subject to the perturbation structure (B,C).

Remark 4.4. In particular, if $U = Y = X$ and $B_I(\theta) = C_I(\theta) = I_d$, for all $\theta \in \Theta$, then

$$r_{\text{stab}}(A;B_I,C_I) = r_{\text{stab}}(A). \tag{4.5}$$

Up to now, there is not an explicit formula for the computation of the stability radius of variational systems. Based on our stability results, we will deduce a lower bound for $r_{\text{stab}}(A;B,C)$. With this purpose we associate with the system (A) the input-output control system $(\mathcal{S}_A) = \{\mathcal{S}_\theta\}_{\theta\in\Theta}$, where for every $\theta \in \Theta$,

$$\begin{aligned} x_\theta(n+1) &= A(\sigma(\theta,n))x_\theta(n) + B(\sigma(\theta,n))u(n), \quad n \in \mathbb{N} \\ x_\theta(0) &= 0 \\ y_\theta(n) &= C(\sigma(\theta,n))x_\theta(n), \quad n \in \mathbb{N} \end{aligned} \tag{$\mathcal{S}_\theta$}$$

Let $W \in \mathcal{W}(\mathbb{N})$. Since the system (A) is uniformly exponentially stable, according to Corollary 3.15 there is $L > 0$ such that for every $(\theta,s) \in \Theta \times W(\mathbb{N},X)$ the corresponding solution

$$x_{\theta,s}(n) = \sum_{k=1}^{n} \Phi_A(\sigma(\theta,k), n-k)s(k-1), \quad n \in \mathbb{N}^* \tag{4.6}$$

has the property

$$\|x_{\theta,s}\|_{W(\mathbb{N},X)} \le L\|s\|_{W(\mathbb{N},X)}. \tag{4.7}$$

Let $\theta \in \Theta$ and let $u \in W(\mathbb{N},U)$. Then, the sequence

$$s_u : \mathbb{N} \longrightarrow X, \quad s_u(n) = B(\sigma(\theta,n))u(n) \tag{4.8}$$

has the property

$$\|s_u(n)\| \le |||B|||\,\|u(n)\|, \quad \forall n \in \mathbb{N}. \tag{4.9}$$

This implies that $s_u \in W(\mathbb{N}, X)$. According to (4.7) we have that $x_{\theta,s_u} \in W(\mathbb{N}, X)$ and from (4.7) and (4.9) it follows that

$$\|x_{\theta,s_u}\|_{W(\mathbb{N},X)} \leq L|||B|||\,\|u\|_{W(\mathbb{N},U)}. \tag{4.10}$$

Observing that the solution $y_{\theta,u}$ of the system (2.2) has the property

$$y_{\theta,u}(n) = C(\sigma(\theta,n))x_{\theta,s_u}(n), \quad \forall n \in \mathbb{N} \tag{4.11}$$

and since

$$\left\|y_{\theta,u}(n)\right\| \leq |||C|||\,\|x_{\theta,s_u}(n)\|, \quad \forall n \in \mathbb{N}, \tag{4.12}$$

we deduce that $y_{\theta,u} \in W(\mathbb{N}, Y)$ and from (4.10) and (4.12) we have that

$$\left\|y_{\theta,u}\right\|_{W(\mathbb{N},Y)} \leq \lambda\|u\|_{W(\mathbb{N},U)}, \tag{4.13}$$

where $\lambda = |||C|||L|||B|||$. Taking into account that λ does not depend on θ or u it follows that

$$\left\|y_{\theta,u}\right\|_{W(\mathbb{N},Y)} \leq \lambda\|u\|_{W(\mathbb{N},U)}, \quad \forall(\theta,u) \in \Theta \times W(\mathbb{N},U). \tag{4.14}$$

In this context, for every $\theta \in \Theta$, it makes sense to consider the operator

$$\Lambda_W^\theta : W(\mathbb{N},U) \longrightarrow W(\mathbb{N},Y), \quad \Lambda_W^\theta(u) = y_{\theta,u}. \tag{4.15}$$

We have that Λ_W^θ is a correctly defined and bounded linear operator. Moreover, from (4.14) we have that

$$\lambda_W(A;B,C) := \sup_{\theta\in\Theta}\left\|\Lambda_W^\theta\right\| < \infty. \tag{4.16}$$

Remark 4.5. The family $\{\Lambda_W^\theta\}_{\theta\in\Theta}$ is called *the family of input-output operators* associated to the system $(\mathcal{S}_A)$.

Remark 4.6. For every $(\theta,u) \in \Theta \times W(\mathbb{N},U)$ we have that

$$\left\|y_{\theta,u}\right\|_{W(\mathbb{N},Y)} \leq \lambda_W(A;B,C)\|u\|_{W(\mathbb{N},U)}. \tag{4.17}$$

In what follows we suppose that there is $\delta > 0$ such that $\|C(\theta)x\| \geq \delta\|x\|$, for all $(\theta,x) \in \Theta \times X$.

Theorem 4.7. *If $P \in \ell^\infty(\Theta, \mathcal{B}(Y,U))$ has the property that*

$$|||P||| < \frac{1}{\lambda_W(A;B,C)}, \tag{4.18}$$

then for every $(\theta, x) \in \Theta \times X$, *the sequence*

$$\gamma_{\theta,x} : \mathbb{N} \longrightarrow X, \quad \gamma_{\theta,x}(n) = \Phi_{A+BPC}(\theta, n)x \tag{4.19}$$

belongs to $W(\mathbb{N}, X)$. *Moreover, there is* $M > 0$ *such that*

$$\left\|\gamma_{\theta,x}\right\|_{W(\mathbb{N},X)} \leq M\|x\|, \quad \forall(\theta, x) \in \Theta \times X. \tag{4.20}$$

Proof. Let $K, \nu > 0$ be such that

$$\|\Phi_A(\theta, n)\| \leq Ke^{-\nu n}, \quad \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{4.21}$$

Let $P \in \ell^\infty(\Theta, \mathcal{B}(Y, U))$ be such that $|||P|||\lambda_W(A; B, C) < 1$. We set $\alpha := |||P|||\lambda_W(A; B, C)$. Let $(\theta, x) \in \Theta \times X$. For every $n \in \mathbb{N}^*$ we denote by

$$\begin{aligned} \gamma_n &: \mathbb{N} \longrightarrow X, \quad \gamma_n(k) = \chi_{\{0,\ldots,n\}}(k)\gamma_{\theta,x}(k), \\ q_n &: \mathbb{N} \longrightarrow Y, \quad q_n(k) = C(\sigma(\theta, k))\gamma_n(k), \\ u_n &: \mathbb{N} \longrightarrow U, \quad u_n(k) = P(\sigma(\theta, k))q_n(k). \end{aligned} \tag{4.22}$$

Let $n \in \mathbb{N}^*$. We have that $u_n \in \Delta(\mathbb{N}, U)$, so $u_n \in W(\mathbb{N}, U)$. Since

$$\|u_n(k)\| \leq \|P(\sigma(\theta, k))\|\left\|q_n(k)\right\| \leq |||P|||\left\|q_n(k)\right\|, \quad \forall k \in \mathbb{N}, \tag{4.23}$$

we deduce that $\|u_n\|_{W(\mathbb{N},U)} \leq |||P|||\, \|q_n\|_{W(\mathbb{N},Y)}$. From $u_n \in W(\mathbb{N}, U)$, using Remark 4.6 we have that

$$\left\|y_{\theta,u_n}\right\|_{W(\mathbb{N},Y)} \leq \lambda_W(A; B, C)\|u_n\|_{W(\mathbb{N},U)} \leq \lambda_W(A; B, C)|||P|||\left\|q_n\right\|_{W(\mathbb{N},Y)} = \alpha\left\|q_n\right\|_{W(\mathbb{N},Y)}. \tag{4.24}$$

From Remark 4.1 we obtain that

$$q_n(k) = C(\sigma(\theta, k))\Phi_A(\theta, k)x + y_{\theta,u_n}(k), \quad \forall k \in \{1, \ldots, n\}, \tag{4.25}$$

which via (4.21) implies that

$$\left\|q_n(k)\right\| \leq |||C|||Ke^{-\nu k}\|x\| + \left\|y_{\theta,u_n}(k)\right\|, \quad \forall k \in \mathbb{N}^*. \tag{4.26}$$

Since $\|q_n(0)\| = \|C(\theta)x\| \leq |||C|||\|x\|$, using (4.26) we deduce that

$$\left\|q_n(k)\right\| \leq |||C|||K\|x\|e_\nu(k) + \left\|y_{\theta,u_n}(k)\right\|, \quad \forall k \in \mathbb{N}, \tag{4.27}$$

where $e_\nu : \mathbb{N} \to \mathbb{R}$, $e_\nu(k) = e^{-\nu k}$. Since $e_\nu \in \ell^1(\mathbb{N}, \mathbb{R})$, we have that $e_\nu \in W$. Then, setting $m = \||C\||K|e_\nu|_W$ from (4.27) and (4.24) it follows that

$$\|q_n\|_{W(\mathbb{N},Y)} \leq m\|x\| + \|y_{\theta,u_n}\|_{W(\mathbb{N},Y)} \leq m\|x\| + \alpha\|q_n\|_{W(\mathbb{N},Y)}, \tag{4.28}$$

which implies that

$$\|q_n\|_{W(\mathbb{N},Y)} \leq \frac{m}{1-\alpha}\|x\|. \tag{4.29}$$

According to our hypothesis we have that $\delta\|\gamma_n(k)\| \leq \|q_n(k)\|$, for all $k \in \mathbb{N}$, so $\delta\|\gamma_n\|_{W(\mathbb{N},X)} \leq \|q_n\|_{W(\mathbb{N},Y)}$. Setting $M = m/(\delta - \alpha\delta)$, from (4.29) we deduce that $\|\gamma_n\|_{W(\mathbb{N},X)} \leq M\|x\|$. Since M does not depend on n, θ, or x, we have that

$$\|\gamma_{\theta,x}\chi_{\{0,\ldots,n\}}\|_{W(\mathbb{N},X)} \leq M\|x\|, \quad \forall n \in \mathbb{N},\ \forall(\theta, x) \in \Theta \times X. \tag{4.30}$$

Taking into account that $W \in \mathcal{W}(\mathbb{N})$ from (4.30) it follows that $\gamma_{\theta,x} \in W(\mathbb{N}, X)$, for all $(\theta, x) \in \Theta \times X$ and $\|\gamma_{\theta,x}\|_{W(\mathbb{N},X)} \leq M\|x\|$, for all $(\theta, x) \in \Theta \times X$. □

The main result of this section is the following.

Theorem 4.8. *The following estimate holds:*

$$r_{stab}(A; B, C) \geq \frac{1}{\lambda_W(A; B, C)}. \tag{4.31}$$

Proof. Since (A) is uniformly exponentially stable, according to Corollary 3.11 we have that the associated system $(S_A) = \{S_\theta\}_{\theta\in\Theta}$ (see p. 7) is $(W(\mathbb{N}, X), W(\mathbb{N}, X))$-stable. So there is $L_A > 0$ such that

$$\|x_{\theta,s}\|_{W(\mathbb{N},X)} \leq L_A\|s\|_{W(\mathbb{N},X)}, \quad \forall(\theta, s) \in \Theta \times \Delta(\mathbb{N}, X). \tag{4.32}$$

Let $P \in \ell^\infty(\Theta, \mathcal{B}(Y, U))$ with $\||P\||\lambda_W(A; B, C) < 1$. Denote by $\alpha := \||P\||\lambda_W(A; B, C)$. Let $M > 0$ be given by Theorem 4.7.

We consider the system $(S_{A+BPC}) = \{S_\theta^P\}_{\theta\in\Theta}$, where

$$\begin{aligned} z_\theta(n+1) &= [A(\sigma(\theta, n)) + (BPC)(\sigma(\theta, n))]z_\theta(n) + s(n), \quad n \in \mathbb{N}, \\ z_\theta(0) &= 0 \end{aligned} \tag{S_θ^P}$$

associated with the perturbed system $(A + BPC)$. In what follows we shall prove that the system (S_{A+BPC}) is $(W(\mathbb{N}, X), W(\mathbb{N}, X))$-stable.

Let $s \in \Delta(\mathbb{N}, X)$ and let $\theta \in \Theta$. Since $s \in \Delta(\mathbb{N}, X)$, there is $h \geq 2$ such that $s(j) = 0$, for all $j \geq h$. Then we observe that

$$\begin{aligned} z_{\theta,s}(n) &= \sum_{k=1}^{h} \Phi_{A+BPC}(\sigma(\theta,k), n-k)s(k-1) \\ &= \Phi_{A+BPC}(\sigma(\theta,h), n-h) \sum_{k=1}^{h} \Phi_{A+BPC}(\sigma(\theta,k), h-k)s(k-1) \\ &= \Phi_{A+BPC}(\sigma(\theta,h), n-h) z_{\theta,s}(h), \quad \forall n \geq h. \end{aligned} \tag{4.33}$$

According to Theorem 4.7 the sequence

$$\gamma : \mathbb{N} \longrightarrow X, \quad \gamma(n) = \Phi_{A+BPC}(\sigma(\theta,h), n) z_{\theta,s}(h) \tag{4.34}$$

belongs to $W(\mathbb{N}, X)$ and

$$\|\gamma\|_{W(\mathbb{N},X)} \leq M \|z_{\theta,s}(h)\|. \tag{4.35}$$

Since W is invariant under translations, we have that

$$\gamma_h : \mathbb{N} \longrightarrow X, \quad \gamma_h(n) = \begin{cases} \gamma(n-h), & n \geq h, \\ 0, & n \in \{0, \dots, h-1\} \end{cases} \tag{4.36}$$

belongs to $W(\mathbb{N}, X)$ and

$$\|\gamma_h\|_{W(\mathbb{N},X)} = \|\gamma\|_{W(\mathbb{N},X)}. \tag{4.37}$$

Then, from

$$\|z_{\theta,s}(n)\| \leq \chi_{\{0,\dots,h-1\}}(n) \|z_{\theta,s}(n)\| + \|\gamma_h(n)\|, \quad \forall n \in \mathbb{N}, \tag{4.38}$$

we deduce that $z_{\theta,s} \in W(\mathbb{N}, X)$. In what follows, we prove that the second condition from Definition 3.4 is fulfilled.

For $n \in \{2, \dots, h\}$ using Remark 4.1 we successively obtain that

$$\begin{aligned}
z_{\theta,s}(n) &= \sum_{k=1}^{n} \Phi_{A+BPC}(\sigma(\theta,k), n-k) s(k-1) \\
&= x_{\theta,s}(n) + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \Phi_A\big(\sigma(\theta, k+j), n-k-j\big)(BPC)\big(\sigma(\theta, k+j-1)\big) \\
&\quad \cdot \Phi_{A+BPC}\big(\sigma(\theta,k), j-1\big) s(k-1) = x_{\theta,s}(n) \\
&\quad + \sum_{k=1}^{n-1} \sum_{i=k}^{n-1} \Phi_A(\sigma(\theta, i+1), n-i-1)(BPC)(\sigma(\theta,i)) \Phi_{A+BPC}(\sigma(\theta,k), i-k) s(k-1) \\
&= x_{\theta,s}(n) + \sum_{i=1}^{n-1} \sum_{k=1}^{i} \Phi_A(\sigma(\theta, i+1), n-i-1)(BPC)(\sigma(\theta,i)) \\
&\quad \cdot \Phi_{A+BPC}(\sigma(\theta,k), i-k) s(k-1) = x_{\theta,s}(n) \\
&\quad + \sum_{i=1}^{n-1} \Phi_A(\sigma(\theta, i+1), n-i-1)(BPC)(\sigma(\theta,i)) z_{\theta,s}(i).
\end{aligned} \tag{4.39}$$

Since $z_{\theta,s}(0) = 0$, from the above estimate we have that

$$\begin{aligned}
z_{\theta,s}(n) &= x_{\theta,s}(n) + \sum_{i=0}^{n-1} \Phi_A(\sigma(\theta, i+1), n-i-1)(BPC)(\sigma(\theta,i)) z_{\theta,s}(i) \\
&= x_{\theta,s}(n) + \sum_{\xi=1}^{n} \Phi_A(\sigma(\theta, \xi), n-\xi)(BPC)(\sigma(\theta,\xi-1)) z_{\theta,s}(\xi-1).
\end{aligned} \tag{4.40}$$

Denoting by

$$u : \mathbb{N} \longrightarrow U, \quad u(n) = P(\sigma(\theta,n)) C(\sigma(\theta,n)) z_{\theta,s}(n) \tag{4.41}$$

and taking into account that $\|u(n)\| \le |||P|||\,|||C|||\,\|z_{\theta,s}(n)\|$, for all $n \in \mathbb{N}$, we have that $u \in W(\mathbb{N}, U)$. Then, using (4.40) we deduce that

$$C(\sigma(\theta,n)) z_{\theta,s}(n) = C(\sigma(\theta,n)) x_{\theta,s}(n) + \left(\Lambda_W^{\theta} u\right)(n), \quad \forall n \in \{2, \dots, h\}, \tag{4.42}$$

which implies that

$$\begin{aligned}
\|C(\sigma(\theta,n)) z_{\theta,s}(n)\| &\le |||C|||\,\|x_{\theta,s}(n)\| + \left\|\Lambda_W^{\theta}\right\| |||P|||\|C(\sigma(\theta,n)) z_{\theta,s}(n)\| \\
&\le |||C|||\,\|x_{\theta,s}(n)\| + \lambda_W(A;B,C) |||P|||\,\|C(\sigma(\theta,n)) z_{\theta,s}(n)\|.
\end{aligned} \tag{4.43}$$

From (4.43) we obtain that

$$\|C(\sigma(\theta,n))z_{\theta,s}(n)\| \leq \frac{|||C|||}{1-\alpha}\|x_{\theta,s}(n)\|. \tag{4.44}$$

Since $\|C(\sigma(\theta,n))z_{\theta,s}(n)\| \geq \delta\|z_{\theta,s}(n)\|$, using (4.44) it follows that

$$\|z_{\theta,s}(n)\| \leq \frac{|||C|||}{\delta(1-\alpha)}\|x_{\theta,s}(n)\|, \quad \forall n \in \{2,\dots,h\}. \tag{4.45}$$

We note that $z_{\theta,s}(0) = x_{\theta,s}(0) = 0$ and $z_{\theta,s}(1) = x_{\theta,s}(1) = s(0)$. Then, setting $q = \max\{|||C|||/(\delta - \alpha\delta), 1\}$ from relation (4.45) we have that

$$\|z_{\theta,s}(n)\| \leq q\|x_{\theta,s}(n)\|, \quad \forall n \in \{0,\dots,h\}. \tag{4.46}$$

Then, from relations (4.38) and (4.46) we deduce that

$$\|z_{\theta,s}(n)\| \leq q\|x_{\theta,s}(n)\| + \left\|\gamma_h(n)\right\|, \quad \forall n \in \mathbb{N}. \tag{4.47}$$

This implies that

$$\|z_{\theta,s}\|_{W(\mathbb{N},X)} \leq q\|x_{\theta,s}\|_{W(\mathbb{N},X)} + \left\|\gamma_h\right\|_{W(\mathbb{N},X)}. \tag{4.48}$$

From relations (4.35), (4.37), and (4.46) we obtain that

$$\left\|\gamma_h\right\|_{W(\mathbb{N},X)} \leq M\|z_{\theta,s}(h)\| \leq Mq\|x_{\theta,s}(h)\|. \tag{4.49}$$

From

$$\|x_{\theta,s}(h)\|\chi_{\{h\}}(n) \leq \|x_{\theta,s}(n)\|, \quad \forall n \in \mathbb{N} \tag{4.50}$$

using the translation invariance of W we have that

$$\|x_{\theta,s}(h)\|F_W(1) \leq \|x_{\theta,s}\|_{W(\mathbb{N},X)}. \tag{4.51}$$

Then from relations (4.48)–(4.51) it follows that

$$\|z_{\theta,s}\|_{W(\mathbb{N},X)} \leq q\left(1 + \frac{M}{F_W(1)}\right)\|x_{\theta,s}\|_{W(\mathbb{N},X)}. \tag{4.52}$$

Finally, setting $L = qL_A[1 + (M/F_W(1))]$, from relations (4.32) and (4.52) we deduce that

$$\|z_{\theta,s}\|_{W(\mathbb{N},X)} \leq L\|s\|_{W(\mathbb{N},X)}. \tag{4.53}$$

Taking into account that L does not depend on θ or s it follows that the system (S_{A+BPC}) is $(W(\mathbb{N}, X), W(\mathbb{N}, X))$-stable. By applying Theorem 3.8 we deduce that the system $(A + BPC)$ is uniformly exponentially stable and the proof is complete. □

Remark 4.9. From Theorem 4.8 we deduce that the property of uniform exponential stability of a system of variational difference equations is preserved in the presence of structured perturbations, provided that the norm of the perturbation factor is less than $1/\lambda_W(A; B, C)$ and this estimate holds for any sequence space W in the general class $\mathcal{W}(\mathbb{N})$. The study points out an interesting connection between the family of the input-output operators associated with the control system $(\mathcal{S}_A)$ and the size of the "largest" perturbation in the presence of which the perturbed system still remains uniformly exponentially stable.

The central result of this section is the following.

Theorem 4.10. *The following estimate holds:*

$$r_{stab}(A; B, C) \geq \sup_{W\in\mathcal{W}(\mathbb{N})} \frac{1}{\lambda_W(A; B, C)}. \tag{4.54}$$

As a consequence, we deduce the following.

Corollary 4.11. *Setting* $\lambda_p(A; B, C) = \lambda_{\ell^p(\mathbb{N},\mathbb{R})}(A; B, C)$ *one has that*

$$r_{stab}(A; B, C) \geq \sup_{p\in[1,\infty]} \frac{1}{\lambda_p(A; B, C)}. \tag{4.55}$$

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