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Research Article

Existence of Solutions to Anti-Periodic Boundary Value Problem for Nonlinear Fractional Differential Equations with Impulses

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This paper discusses the existence of solutions to antiperiodic boundary value problem for nonlinear impulsive fractional differential equations. By using Banach fixed point theorem, Schaefer fixed point theorem, and nonlinear alternative of Leray-Schauder type theorem, some existence results of solutions are obtained. An example is given to illustrate the main result.

1. Introduction

In this paper, we consider an antiperiodic boundary value problem for nonlinear fractional differential equations with impulses

$$\begin{gathered}
{}^{C}D^{\alpha}u(t) = f(t,u(t)), \quad t \in [0,T],\ t \neq t_k,\ k = 1,2,\dots,p, \\
\Delta u|_{t=t_k} = I_k(u(t_k)), \qquad \Delta u'|_{t=t_k} = J_k(u(t_k)), \quad k = 1,2,\dots,p, \\
u(0) + u(T) = 0, \qquad u'(0) + u'(T) = 0,
\end{gathered} \tag{1.1}$$

where T is a positive constant, $1 < \alpha \leq 2$, ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order α, $f \in C([0,T] \times R, R)$, I_k, $J_k : R \to R$ and $\{t_k\}$ satisfy that $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$.

Fractional differential equations have proved to be an excellent tool in the mathematic modeling of many systems and processes in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, electromagnetic, porous media, and so forth. In consequence, the subject of fractional differential equations is gaining much importance and attention (see [1–6] and the references therein).

The theory of impulsive differential equations has found its extensive applications in realistic mathematic modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory of impulsive differential equations, we refer the reader to [7, 8]. Recently, many authors are devoted to the study of boundary value problems for impulsive differential equations of integer order, see [9–12].

Very recently, there are only a few papers about the nonlinear impulsive differential equations and delayed differential equations of fractional order.

Agarwal et al. in [13] have established some sufficient conditions for the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the Caputo farctional derivative. Ahmad et al. in [14] have discussed some existence results for the two-point boundary value problem involving nonlinear impulsive hybrid differential equation of fractional order by means of contraction mapping principle and Krasnoselskii's fixed point theorem. By the similar way, they have also obtained the existence results for integral boundary value problem of nonlinear impulsive fractional differential equations (see [15]). Tian et al. in [16] have obtained some existence results for the three-point impulsive boundary value problem involving fractional differential equations by the means of fixed points method. Maraaba et al. in [17, 18] have established the existence and uniqueness theorem for the delay differential equations with Caputo fractional derivatives. Wang et al. in [19] have studied the existence and uniqueness of the mild solution for a class of impulsive fractional differential equations with time-varying generating operators and nonlocal conditions.

To the best of our knowledge, few papers exist in the literature devoted to the antiperiodic boundary value problem for fractional differential equations with impulses. This paper studies the existence of solutions of antiperiodic boundary value problem for fractional differential equations with impulses.

The organization of this paper is as follows. In Section 2, we recall some definitions of fractional integral and derivative and preliminary results which will be used in this paper. In Section 3, we will consider the existence results for problem (1.1). We give three results, the first one is based on Banach fixed theorem, the second one is based on Schaefer fixed point theorem, and the third one is based on the nonlinear alternative of Leray-Schauder type. In Section 4, we will give an example to illustrate the main result.

2. Preliminaries

In this section, we present some basic notations, definitions, and preliminary results which will be used throughout this paper.

Definition 2.1 (see [4]). The Caputo fractional derivative of order α of a function $f : [0, \infty) \to R$ is defined as

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad n-1 < \alpha < n, \ n = [\alpha]+1, \tag{2.1} $$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2 (see [4]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f(t)$, $t > 0$, is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{2.2}$$

provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.3 (see [4]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to R$ is given by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \tag{2.3}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α, provided that the right side is pointwise defined on $(0,\infty)$.

For the sake of convenience, we introduce the following notation.

Let $J = [0,T]$, $J_0 = [0,t_1]$, $J_i = (t_i,t_{i+1}]$, $i = 1,2,\ldots,p-1$, $J_p = (t_p,T]$. $J' = J\backslash\{t_1,t_2,\ldots,t_p\}$. We define $\mathrm{PC}(J) = \{u : [0,T] \to R \mid u \in C(J'), u(t_k^+)$ and $u(t_k^-)$ exists, and $u(t_k^-) = u(t_k),\ 1 \le k \le p\}$. Obviously, $\mathrm{PC}(J)$ is a Banach space with the norm $\|u\| = \sup_{t\in J}|u(t)|$.

Definition 2.4. A function $u \in \mathrm{PC}(J)$ is said to be a solution of (1.1) if u satisfies the equation ${}^cD^{\alpha}u(t) = f(t,u(t))$ for $t \in J'$, the equations $\Delta u|_{t=t_k} = I_k(u(t_k))$, $\Delta u'|_{t=t_k} = J_k(u(t_k))$, $k = 1,2,\ldots,p$, and the condition $u(0) + u(T) = 0$, $u'(0) + u'(T) = 0$.

Lemma 2.5 (see [20]). *Let $\alpha > 0$; then*

$$I^{\alpha C}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, \tag{2.4}$$

for some $c_i \in R$, $i = 0,1,2,\ldots,n-1$, $n = [\alpha] + 1$.

Lemma 2.6 (nonlinear alternative of Leray-Schauder type [21]). *Let E be a Banach space with $C \subseteq E$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is continuous, compact map. Then either*

(1) *A has a fixed point in $\overline{U}$, or*

(2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda A u$.*

Lemma 2.7 (Schaefer fixed point theorem [22]). *Let S be a convex subset of a normed linear space Ω and $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let*

$$\zeta(F) = \{u \in S : u = \lambda F u,\ \text{for some } 0 < \lambda < 1\}. \tag{2.5}$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Lemma 2.8. *Assume that $y \in C([0,T],R)$, $T>0$, $1<\alpha\le 2$. A function $u\in PC(J)$ is a solution of the antiperiodic boundary value problem*

$$\begin{gathered}
{}^{C}D^{\alpha}u(t)=y(t),\quad t\in[0,T],\ t\neq t_k,\ k=1,2,\dots,p,\\
\Delta u|_{t=t_k}=I_k(u(t_k)),\qquad \Delta u'|_{t=t_k}=J_k(u(t_k)),\quad k=1,2,\dots,p,\\
u(0)+u(T)=0,\qquad u'(0)+u'(T)=0,
\end{gathered}\tag{2.6}$$

if and only if u is a solution of the integral equation

$$u(t)=\begin{cases}
\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-s)^{\alpha-1}y(s)ds-\dfrac{1}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}y(s)ds\\
\quad-\dfrac{1}{2\Gamma(\alpha-1)}\displaystyle\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds\\
\quad+\dfrac{T-2t}{4\Gamma(\alpha-1)}\displaystyle\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds-\dfrac{1}{2}\sum_{i=1}^{p}(T-t_i)J_i(u(t_i))\\
\quad+\dfrac{T-2t}{4}\displaystyle\sum_{i=1}^{p}J_i(u(t_i))-\dfrac{1}{2}\sum_{i=1}^{p}I_i(u(t_i)),\quad t\in[0,t_1],\\
\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^t(t-s)^{\alpha-1}y(s)ds+\dfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}y(s)ds\\
\quad-\dfrac{1}{2\Gamma(\alpha)}\displaystyle\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}y(s)ds+\dfrac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{k}(t-t_i)\\
\quad\times\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds\\
\quad-\dfrac{1}{2\Gamma(\alpha-1)}\displaystyle\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds\\
\quad+\dfrac{T-2t}{4\Gamma(\alpha-1)}\displaystyle\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}y(s)ds+\sum_{i=1}^{k}(t-t_i)J_i(u(t_i))\\
\quad-\dfrac{1}{2}\displaystyle\sum_{i=1}^{p}(T-t_i)J_i(u(t_i))+\dfrac{T-2t}{4}\sum_{i=1}^{p}J_i(u(t_i))\\
\quad+\displaystyle\sum_{i=1}^{k}I_i(u(t_i))-\dfrac{1}{2}\sum_{i=1}^{p}I_i(u(t_i)),\quad t\in(t_k,t_{k+1}],\ 1\le k\le p.
\end{cases}\tag{2.7}$$

Proof. Assume that y satisfies (2.6). Using Lemma 2.5, for some constants $c_0,\ c_1\in R$, we have

$$u(t)=I^{\alpha}y(t)-c_0-c_1t=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds-c_0-c_1t,\quad t\in[0,t_1].\tag{2.8}$$

Then, we obtain

$$u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)ds - c_1, \quad t \in [0,t_1]. \tag{2.9}$$

If $t \in (t_1, t_2]$, then we have

$$\begin{gathered} u(t) = \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}y(s)ds - d_0 - d_1(t-t_1), \\ u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t (t-s)^{\alpha-2}y(s)ds - d_1, \end{gathered} \tag{2.10}$$

where $d_0,\ d_1 \in R$ are arbitrary constants. Thus, we find that

$$\begin{gathered} u(t_1^-) = \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)ds - c_0 - c_1t_1, \\ u(t_1^+) = -d_0, \\ u'(t_1^-) = \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1} (t_1-s)^{\alpha-2}y(s)ds - c_1, \\ u'(t_1^+) = -d_1. \end{gathered} \tag{2.11}$$

In view of $\Delta u|_{t=t_1} = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$ and $\Delta u'|_{t=t_1} = u'(t_1^+) - u'(t_1^-) = J_1(u(t_1))$, we have

$$\begin{aligned} -d_0 &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)ds - c_0 - c_1t_1 + I_1(u(t_1)), \\ -d_1 &= \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1} (t_1-s)^{\alpha-2}y(s)ds - c_1 + J_1(u(t_1)). \end{aligned} \tag{2.12}$$

Hence, we obtain

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}y(s)ds \\ & + \frac{t-t_1}{\Gamma(\alpha-1)}\int_0^{t_1} (t_1-s)^{\alpha-2}y(s)ds + (t-t_1)J_1(u(t_1)) \\ & + I_1(u(t_1)) - c_0 - c_1t, \quad t \in (t_1, t_2]. \end{aligned} \tag{2.13}$$

Repeating the process in this way, the solution $u(t)$ for $t \in (t_k, t_{k+1}]$ can be written as

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} y(s) ds \\ & + \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{k} (t-t_i) \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds + \sum_{i=1}^{k} (t-t_i) J_i(u(t_i)) \\ & + \sum_{i=1}^{k} I_i(u(t_i)) - c_0 - c_1 t, \quad t \in (t_k, t_{k+1}],\ k = 1, 2, \ldots, p. \end{aligned} \tag{2.14}$$

On the other hand, by (2.14), we have

$$\begin{aligned} u(T) = {} & \frac{1}{\Gamma(\alpha)} \int_{t_p}^{T} (T-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} y(s) ds \\ & + \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{p} (T-t_i) \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds + \sum_{i=1}^{p} (T-t_i) J_i(u(t_i)) \\ & + \sum_{i=1}^{p} I_i(u(t_i)) - c_0 - c_1 T, \\ u'(T) = {} & \frac{1}{\Gamma(\alpha-1)} \int_{t_p}^{T} (T-s)^{\alpha-2} y(s) ds + \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds \\ & + \sum_{i=1}^{p} J_i(u(t_i)) - c_1. \end{aligned} \tag{2.15}$$

By the boundary conditions $u(0) + u(T) = 0,\ u'(0) + u'(T) = 0$, we obtain

$$\begin{aligned} c_0 = {} & \frac{1}{2\Gamma(\alpha)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} y(s) ds + \frac{1}{2\Gamma(\alpha-1)} \sum_{i=1}^{p} (T-t_i) \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds \\ & - \frac{T}{4\Gamma(\alpha-1)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds - \frac{T}{4} \sum_{i=1}^{p} J_i(u(t_i)) \\ & + \frac{1}{2} \sum_{i=1}^{p} (T-t_i) J_i(u(t_i)) + \frac{1}{2} \sum_{i=1}^{p} I_i(u(t_i)), \\ c_1 = {} & \frac{1}{2\Gamma(\alpha-1)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} y(s) ds + \frac{1}{2} \sum_{i=1}^{p} J_i(u(t_i)). \end{aligned} \tag{2.16}$$

Substituting the values of c_0 and c_1 into (2.8), (2.14), respectively, we obtain (2.7).

Conversely, we assume that u is a solution of the integral equation (2.7). By a direct computation, it follows that the solution given by (2.7) satisfies (2.6). The proof is completed. □

3. Main Result

In this section, our aim is to discuss the existence and uniqueness of solutions to the problem (1.1).

Theorem 3.1. *Assume that*

($H1$) *there exists a constant* $L_1 > 0$ *such that* $|f(t,u) - f(t,v)| \leq L_1|u - v|$, *for each* $t \in J$ *and all* $u,\ v \in R$;

($H2$) *there exist constants* $L_2,\ L_3 > 0$ *such that* $I_k(u) - I_k(v) \leq L_2|u - v|$, $J_k(u) - J_k(v) \leq L_3|u - v|$, *for each* $t \in J$ *and all* $u,\ v \in R,\ k = 1, 2, \ldots, p$.

If

$$L_1\left(\frac{(3p+5)T^{\alpha}}{2\Gamma(\alpha+1)} + \frac{7(p+1)T^{\alpha}}{4\Gamma(\alpha)}\right) + p\left(\frac{3}{2}L_2 + \frac{7T}{4}L_3\right) < 1, \tag{3.1}$$

then problem (1.1) *has a unique solution on* J.

Proof. We transform the problem (1.1) into a fixed point problem. Define an operator T : $\mathrm{PC}(J) \to \mathrm{PC}(J)$ by

$$\begin{aligned}
(Tu)(t) = {} & \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,u(s))ds \\
& - \frac{1}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f(s,u(s))ds \\
& + \frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}f(s,u(s))ds \\
& - \frac{1}{2\Gamma(\alpha-1)}\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f(s,u(s))ds \\
& + \frac{T-2t}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f(s,u(s))ds \\
& + \sum_{0<t_k<t}(t-t_k)J_k(u(t_k)) - \frac{1}{2}\sum_{i=1}^{p}(T-t_i)J_i(u(t_i)) + \frac{T-2t}{4}\sum_{i=1}^{p}J_i(u(t_i)) \\
& + \sum_{0<t_k<t}I_k(u(t_k)) - \frac{1}{2}\sum_{i=1}^{p}I_i(u(t_i)),
\end{aligned} \tag{3.2}$$

where $\mathrm{PC}(J)$ is with the norm $\|u\| = \sup_{t\in J}|u(t)|$. Let $u,\ v \in \mathrm{PC}(J)$; then for each $t \in J$, we have

$$\begin{aligned}
&|(Tu)(t) - (Tv)(t)| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\big|f(s,u(s)) - f(s,v(s))\big|ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\big|f(s,u(s)) - f(s,v(s))\big|ds \\
&\qquad + \frac{1}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\big|f(s,u(s)) - f(s,v(s))\big|ds \\
&\qquad + \frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\big|f(s,u(s)) - f(s,v(s))\big|ds \\
&\qquad + \frac{1}{2\Gamma(\alpha-1)}\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\big|f(s,u(s)) - f(s,v(s))\big|ds \\
&\qquad + \frac{|T-2t|}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\big|f(s,u(s)) - f(s,v(s))\big|ds \\
&\qquad + \sum_{0<t_k<t}(t-t_k)|J_k(u(t_k)) - J_k(v(t_k))| + \frac{1}{2}\sum_{i=1}^{p}(T-t_i)|J_i(u(t_i)) - J_i(v(t_i))| \\
&\qquad + \frac{|T-2t|}{4}\sum_{i=1}^{p}|J_i(u(t_i)) - J_i(v(t_i))| + \sum_{0<t_k<t}|I_k(u(t_k)) - I_k(v(t_k))| \\
&\qquad + \frac{1}{2}\sum_{i=1}^{p}|I_i(u(t_i)) - I_i(v(t_i))| \\
&\quad\le \frac{L_1\|u-v\|}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}ds + \frac{3L_1\|u-v\|}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds \\
&\qquad + \frac{7TL_1\|u-v\|}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds + \frac{3p}{2}L_2\|u-v\| + \frac{7Tp}{4}L_3\|u-v\| \\
&\quad\le \frac{T^\alpha L_1}{\Gamma(\alpha+1)}\|u-v\| + \frac{3(p+1)T^\alpha L_1}{2\Gamma(\alpha+1)}\|u-v\| + \frac{7(p+1)T^\alpha L_1}{4\Gamma(\alpha)}\|u-v\| \\
&\qquad + \frac{3p}{2}L_2\|u-v\| + \frac{7Tp}{4}L_3\|u-v\| \\
&\quad= \left[L_1\left(\frac{(3p+5)T^\alpha}{2\Gamma(\alpha+1)} + \frac{7(p+1)T^\alpha}{4\Gamma(\alpha)}\right) + p\left(\frac{3}{2}L_2 + \frac{7T}{4}L_3\right)\right]\|u-v\|.
\end{aligned} \tag{3.3}$$

Therefore,

$$\|Tu - Tv\| \leq \left[L_1\left(\frac{(3p+5)T^\alpha}{2\Gamma(\alpha+1)} + \frac{7(p+1)T^\alpha}{4\Gamma(\alpha)}\right) + p\left(\frac{3}{2}L_2 + \frac{7T}{4}L_3\right)\right]\|u - v\|. \tag{3.4}$$

Since

$$L_1\left(\frac{(3p+5)T^\alpha}{2\Gamma(\alpha+1)} + \frac{7(p+1)T^\alpha}{4\Gamma(\alpha)}\right) + p\left(\frac{3}{2}L_2 + \frac{7T}{4}L_3\right) < 1, \tag{3.5}$$

consequently T is a contraction; as a consequence of Banach fixed point theorem, we deduce that T has a fixed point which is a solution of the problem (1.1). □

Theorem 3.2. *Assume that*

($H3$) *the function* $f : J \times R \to R$ *is continuous and there exists a constant* $N_1 > 0$ *such that* $|f(t,u)| \leq N_1$ *for each* $t \in J$ *and all* $u \in R$;

($H4$) *the functions* I_k, $J_k : R \to R$ *are continuous and there exist constants* N_2, $N_3 > 0$ *such that* $|I_k(u)| \leq N_2$, $|J_k(u)| \leq N_3$, *for all* $u \in R$, $k = 1, 2, \ldots, p$.

Then the problem (1.1) *has at least one solution on* J.

Proof. We will use Schaefer fixed-point theorem to prove T has a fixed point. The proof will be given in several steps.

Step 1. T is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathrm{PC}(J)$; we have

$$\begin{aligned}
&|(Tu_n)(t) - (Tu)(t)| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f(s,u_n(s)) - f(s,u(s))\right|ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\left|f(s,u_n(s)) - f(s,u(s))\right|ds \\
&\qquad + \frac{1}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u_n(s)) - f(s,u(s))\right|ds \\
&\qquad + \frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\left|f(s,u_n(s)) - f(s,u(s))\right|ds \\
&\qquad + \frac{1}{2\Gamma(\alpha-1)}\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u_n(s)) - f(s,u(s))\right|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|T-2t|}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u_n(s))-f(s,u(s))\right|ds\\
&+\sum_{0<t_k<t}(t-t_k)|J_k(u_n(t_k))-J_k(u(t_k))|+\frac{1}{2}\sum_{i=1}^{p}(T-t_i)|J_i(u_n(t_i))-J_i(u(t_i))|\\
&+\frac{|T-2t|}{4}\sum_{i=1}^{p}|J_i(u_n(t_i))-J_i(u(t_i))|+\sum_{0<t_k<t}|I_k(u_n(t_k))-I_k(u(t_k))|\\
&+\frac{1}{2}\sum_{i=1}^{p}|I_i(u_n(t_i))-I_i(u(t_i))|\\
\le\;&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f(s,u_n(s))-f(s,u(s))\right|ds\\
&+\frac{3}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u_n(s))-f(s,u(s))\right|ds\\
&+\frac{7T}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u_n(s))-f(s,u(s))\right|ds\\
&+\frac{3}{2}\sum_{i=1}^{p}|I_i(u_n(t_i))-I_i(u(t_i))|+\frac{7T}{4}\sum_{i=1}^{p}|J_i(u_n(t_i))-J_i(u(t_i))|.
\end{aligned}
\tag{3.6}
$$

Since $f,\ I,\ J$ are continuous functions, then we have

$$
\|Tu_n-Tu\|\longrightarrow 0,\quad n\longrightarrow\infty. \tag{3.7}
$$

Step 2. T maps bounded sets into bounded sets in $\mathrm{PC}(J)$.

Indeed, it is enough to show that for any $r>0$, there exists a positive constant L such that, for each $u\in\Omega_r=\{u\in\mathrm{PC}(J):\|u\|\le r\}$, we have $\|Tu\|\le L$. By $(H3)$ and $(H4)$, for each $t\in J$, we can obtain

$$
\begin{aligned}
|(Tu)(t)|\le\;&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f(s,u(s))\right|ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\left|f(s,u(s))\right|ds\\
&+\frac{1}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\frac{1}{2\Gamma(\alpha-1)}\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|T-2t|}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\sum_{0<t_k<t}(t-t_k)|J_k(u(t_k))|+\frac{1}{2}\sum_{i=1}^{p}(T-t_i)|J_i(u(t_i))|\\
&+\frac{|T-2t|}{4}\sum_{i=1}^{p}|J_i(u(t_i))|+\sum_{0<t_k<t}|I_k(u(t_k))|+\frac{1}{2}\sum_{i=1}^{p}|I_i(u(t_i))|\\
&\le\frac{N_1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}ds+\frac{3N_1}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds\\
&+\frac{7TN_1}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds+\frac{3p}{2}N_2+\frac{7Tp}{4}N_3\\
&\le\left[N_1\left(\frac{(3p+5)T^{\alpha}}{2\Gamma(\alpha+1)}+\frac{7(p+1)T^{\alpha}}{4\Gamma(\alpha)}\right)+p\left(\frac{3}{2}N_2+\frac{7T}{4}N_3\right)\right].
\end{aligned}
\tag{3.8}
$$

Therefore,

$$
\|Tu\|\le\left[N_1\left(\frac{(3p+5)T^{\alpha}}{2\Gamma(\alpha+1)}+\frac{7(p+1)T^{\alpha}}{4\Gamma(\alpha)}\right)+p\left(\frac{3}{2}N_2+\frac{7T}{4}N_3\right)\right]:=L.
\tag{3.9}
$$

Step 3. T maps bounded sets into equicontinuous sets in $PC(J)$.

Let Ω_r be a bounded set of $PC(J)$ as in Step 2, and let $u\in\Omega_r$. For each $t\in J$, we can estimate the derivative $(Tu)'(t)$:

$$
\begin{aligned}
\left|(Tu)'(t)\right|\le&\frac{1}{\Gamma(\alpha-1)}\int_{t_k}^{t}(t-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\frac{1}{2\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\sum_{0<t_k<t}|J_k(u(t_k))|+\frac{1}{2}\sum_{i=1}^{p}|J_i(u(t_i))|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{N_1}{\Gamma(\alpha-1)}\int_{t_k}^{t}(t-s)^{\alpha-2}ds+\frac{3N_1}{2\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}ds+\frac{3p}{2}N_3\\
&\leq \frac{N_1T^{\alpha-1}}{\Gamma(\alpha)}+\frac{3(p+1)N_1T^{\alpha-1}}{2\Gamma(\alpha)}+\frac{3p}{2}N_3\\
&=\left(\frac{(3p+5)T^{\alpha-1}}{2\Gamma(\alpha)}\right)N_1+\frac{3p}{2}N_3:=M.
\end{aligned}\tag{3.10}$$

Hence, let $t',\ t''\in J,\ t'<t''$; we have

$$\left|(Tu)(t'')-(Tu)(t')\right|=\int_{t'}^{t''}\left|(Tu)'(s)\right|ds\leq M(t''-t').\tag{3.11}$$

So $T(\Omega_r)$ is equicontinuous in $\mathrm{PC}(J)$. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $T:\mathrm{PC}(J)\to\mathrm{PC}(J)$ is completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$\zeta(T)=\{u\in \mathrm{PC}(J): u=\lambda Tu \text{ for some } 0<\lambda<1\}\tag{3.12}$$

is bounded. Let $u=\lambda Tu$ for some $0<\lambda<1$. Thus, for each $t\in J$, we have

$$\begin{aligned}
u(t)={}&\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,u(s))ds+\frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,u(s))ds\\
&-\frac{\lambda}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f(s,u(s))ds\\
&+\frac{\lambda}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}f(s,u(s))ds\\
&-\frac{\lambda}{2\Gamma(\alpha-1)}\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f(s,u(s))ds\\
&+\frac{\lambda(T-2t)}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f(s,u(s))ds\\
&+\lambda\sum_{0<t_k<t}(t-t_k)J_k(u(t_k))-\frac{\lambda}{2}\sum_{i=1}^{p}(T-t_i)J_i(u(t_i))+\frac{\lambda(T-2t)}{4}\sum_{i=1}^{p}J_i(u(t_i))\\
&+\lambda\sum_{0<t_k<t}I_k(u(t_k))-\frac{\lambda}{2}\sum_{i=1}^{p}I_i(u(t_i)).
\end{aligned}\tag{3.13}$$

For each $t \in J$, by $(H3)$ and $(H4)$, we have

$$\|u\| \leq N_1\left(\frac{(3p+5)T^{\alpha}}{2\Gamma(\alpha+1)} + \frac{7(p+1)T^{\alpha}}{4\Gamma(\alpha)}\right) + p\left(\frac{3}{2}N_2 + \frac{7T}{4}N_3\right). \tag{3.14}$$

This shows that the set $\zeta(T)$ is bounded. As a consequence of Schaefer fixed-point theorem, we deduce that T has a fixed point which is a solution of the problem (1.1). □

In the following theorem we give an existence result for the problem (1.1) by applying the nonlinear alternative of Leray-Schauder type and by which the conditions $(H3)$ and $(H4)$ are weakened.

Theorem 3.3. *Assume that* $(H2)$ *and the following conditions hold.*

$(H5)$ *There exists* $\phi \in C(J)$ *and* $\psi : [0,\infty) \to (0,\infty)$ *continuous and nondecreasing such that*

$$\left|f(t,u)\right| \leq \phi(t)\psi(|u|), \quad t \in J,\ u \in R. \tag{3.15}$$

$(H6)$ *There exist* $\psi^*, \overline{\psi^*} : [0,\infty) \to (0,\infty)$ *continuous and nondecreasing such that*

$$|I_k(u)| \leq \psi^*(|u|), \quad |J_k(u)| \leq \overline{\psi^*}(|u|),\ u \in R. \tag{3.16}$$

$(H7)$ *There exists a number* $M^* > 0$ *such that*

$$\frac{M^*}{\phi^*\psi(M^*)\left((3p+5)T^{\alpha}/2\Gamma(\alpha+1)+7(p+1)T^{\alpha}/4\Gamma(\alpha)\right)+p\left((3/2)\psi^*(M^*)+(7T/4)\overline{\psi^*}(M^*)\right)} > 1, \tag{3.17}$$

where $\phi^* = \sup\{\phi(t) : t \in J\}$.

Then (1.1) has at least one solution on J.

Proof. Consider the operator T defined in Theorem 3.1. It can be easily shown that T is continuous and completely continuous. For $\lambda \in (0,1)$ and each $t \in J$, let $u = \lambda T u$. Then from $(H5)$ and $(H6)$, and we have

$$\begin{aligned}|u(t)| \leq{}& \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left|f(s,u(s))\right|ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\left|f(s,u(s))\right|ds \\ &+ \frac{1}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|f(s,u(s))\right|ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\frac{1}{2\Gamma(\alpha-1)}\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\frac{|T-2t|}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|f(s,u(s))\right|ds\\
&+\sum_{0<t_k<t}(t-t_k)|J_k(u(t_k))|+\frac{1}{2}\sum_{i=1}^{p}(T-t_i)|J_i(u(t_i))|\\
&+\frac{|T-2t|}{4}\sum_{i=1}^{p}|J_i(u(t_i))|+\sum_{0<t_k<t}|I_k(u(t_k))|+\frac{1}{2}\sum_{i=1}^{p}|I_i(u(t_i))|\\
\le{}&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\phi(s)\psi(|u(s)|)ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\phi(s)\psi(|u(s)|)ds\\
&+\frac{1}{2\Gamma(\alpha)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\phi(s)\psi(|u(s)|)ds\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-2}\phi(s)\psi(|u(s)|)ds\\
&+\frac{1}{2\Gamma(\alpha-1)}\sum_{i=1}^{p}(T-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi(s)\psi(|u(s)|)ds\\
&+\frac{|T-2t|}{4\Gamma(\alpha-1)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi(s)\psi(|u(s)|)ds\\
&+\sum_{0<t_k<t}(t-t_k)\overline{\psi^*}(|u(t_k)|)+\frac{1}{2}\sum_{i=1}^{p}(T-t_i)\overline{\psi^*}(|u(t_i)|)\\
&+\frac{|T-2t|}{4}\sum_{i=1}^{p}\overline{\psi^*}(|u(t_i)|)+\sum_{0<t_k<t}\psi^*(|u(t_k)|)+\frac{1}{2}\sum_{i=1}^{p}\psi^*(|u(t_i)|)\\
\le{}&\phi^*\psi(\|u\|)\frac{T^\alpha}{\Gamma(\alpha+1)}+\phi^*\psi(\|u\|)\frac{3(p+1)T^\alpha}{2\Gamma(\alpha+1)}\\
&+\phi^*\psi(\|u\|)\frac{7(p+1)T^\alpha}{4\Gamma(\alpha)}+\frac{7pT}{4}\overline{\psi^*}(\|u\|)+\frac{3p}{2}\psi^*(\|u\|)\\
={}&\phi^*\psi(\|u\|)\left(\frac{(3p+5)T^\alpha}{2\Gamma(\alpha+1)}+\frac{7(p+1)T^\alpha}{4\Gamma(\alpha)}\right)+p\left(\frac{3}{2}\psi^*(\|u\|)+\frac{7T}{4}\overline{\psi^*}(\|u\|)\right).
\end{aligned}
\tag{3.18}
$$

Thus,

$$\frac{\|u\|}{\phi^*\psi(\|u\|)\left((3p+5)T^\alpha/2\Gamma(\alpha+1)+7(p+1)T^\alpha/4\Gamma(\alpha)\right)+p\left((3/2)\varphi^*(\|u\|)+(7T/4)\overline{\varphi^*}(\|u\|)\right)} \leq 1. \tag{3.19}$$

Then by $(H7)$, there exists M^* such that $\|u\| \neq M^*$. Let

$$U = \{u \in \mathrm{PC}(J) : \|u\| < M^*\}. \tag{3.20}$$

The operator $T : \overline{U} \to \mathrm{PC}(J)$ is a continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda T u$ for some $0 < \lambda < 1$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that T has a fixed point u in $\overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

4. Example

Let $\alpha = 3/2$, $T = 2\pi$, $p = 1$. We consider the following boundary value problem:

$$\begin{gathered} {}^C D^{3/2} u(t) = f(t, u(t)), \quad 0 \leq t \leq 2\pi,\ t \neq \frac{1}{2}, \\ \Delta u|_{t=1/2} = I\left(u\left(\frac{1}{2}\right)\right), \qquad \Delta u'|_{t=1/2} = J\left(u\left(\frac{1}{2}\right)\right), \\ u(0) + u(2\pi) = 0, \qquad u'(0) + u'(2\pi) = 0, \end{gathered} \tag{4.1}$$

where

$$\begin{gathered} f(t,u) = \frac{\cos tu}{(t+20)^2(1+u)}, \quad (t,u) \in J \times [0,\infty), \\ I(u) = \frac{u}{10+u}, \qquad J(u) = \frac{u}{25+u}. \end{gathered} \tag{4.2}$$

Obviously $L_1 = 1/400$, $L_2 = 1/10$, $L_3 = 1/25$. Further,

$$\begin{aligned} &L_1\left(\frac{(3p+5)T^\alpha}{2\Gamma(\alpha+1)} + \frac{7(p+1)T^\alpha}{4\Gamma(\alpha)}\right) + p\left(\frac{3}{2}L_2 + \frac{7T}{4}L_3\right) \\ &\quad = \frac{1}{400}\left(\frac{32\sqrt{2}}{3}\pi + 14\sqrt{2}\pi\right) + \frac{3}{20} + \frac{7\pi}{50} < 1. \end{aligned} \tag{4.3}$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, the impulsive fractional antiperiodic boundary value problem has a unique solution on $[0, 2\pi]$.

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