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Research Article

On a Chaotic Weighted Shift $z^p d^{p+1}/dz^{p+1}$ of Order p in Bargmann Space

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This paper is devoted to the study of the chaotic properties of some specific backward shift unbounded operators $H_p = A^{*^p} A^{P+1}; p = 0, 1, \ldots$ realized as differential operators in Bargmann space, where A and A^* are the standard Bose annihilation and creation operators such that $[A, A^*] = I$.

1. Introduction

It is well known that linear operators in finite-dimensional linear spaces cannot be chaotic but the nonlinear operator may be. Only in infinite-dimensional linear spaces can linear operators have chaotic properties. These last properties are based on the phenomenon of hypercyclicity or the phenomen of nonwandercity.

The study of the phenomenon of hypercyclicity originates in the papers by Birkoff [1] and Maclane [2] who show, respectively, that the operators of translation and differentiation, acting on the space of entire functions, are hypercyclic.

The theories of hypercyclic operators and chaotic operators have been intensively developed for bounded linear operators; we refer to [1, 3–5] and references therein. For a bounded operator, Ansari asserts in [6] that powers of a hypercyclic bounded operator are also hypercyclic.

For an unbounded operator, Salas exhibits in [7] an unbounded hypercyclic operator whose square is not hypercyclic. The result of Salas shows that one must be careful in the formal manipulation of operators with restricted domains. For such operators, it is often more convenient to work with vectors rather than with operators themselves.

Now, let T be an unbounded operator on a separable infinite dimensional Banach space X. A point ϕ is called wandering if there exists an open set U containing ϕ such that for some $n_0 < \infty$ and for all $n > n_0$ one has $T^n(U) \bigcap U = \emptyset$. (In other words, the neighbourhood eventually never returns). A point ϕ is called nonwandering if it is not wandering.

A closed subspace $E \subset X$ has hyperbolic structure if: $E = E^u \oplus E^s$, $TE^u = E^u$, and $TE^s = E^s$, where E^u (the unstable subspace) and E^s (the stable subspace) are closed. In addition, there exist constants $\tau(0 < \tau < 1)$ and $C > 0$, such that:

(i) for any $\Phi \in E^u$, $k \in \mathbb{N}$, $C\tau^{-k}\|\Phi\| \leq \|T^k\Phi\|$,

(ii) for any $\Psi \in E^s$, $k \in \mathbb{N}$, $\|T^k\Psi\| \leq C\tau^k\|\Psi\|$.

T is said to be a nonwandering operator relative to E which has hyperbolic structure if the set of periodic points of T is dense in E.

For the nonwandering operators, they are new linear chaotic operators. They are relative to hypercyclic operators, but different from them in the sense that some hypercyclic operators are not non-wandering operators and there also exists a non-wandering operator, which does not belong to hypercyclic operators (see [8], Remark 3.5). In fact, suppose T is a bounded linear operator and T is invertible; if T is a hypercyclic operator, then $\sigma(T) \bigcap \partial D \neq \Phi$ (see [9], Remark 4.3) but if T is a non-wandering operator, then $\sigma(T) \bigcap \partial D = \Phi$ where ∂D is the unit circle.

Now, when a linear operator is not invertible, there exist operators which are not only non-wandering but also hypercyclic. Recently, these theories began to be developed on some concrete examples of unbounded linear operators; see [1, 10–12]. On the basis of the work in [13], we study the phenomenons of chaoticity of some specific backward shift unbounded operators $H_p = A^{*^p}A^{p+1} = z^p(d^{p+1}/dz^{p+1})$; $p = 0, 1, \ldots$ realized as differential operators in Bargmann space [14] (the space of entire functions with Gaussian measure), where A and A^* are the standard Boson annihilation and creation operators satisfying the commutation relation

$$[A, A^*] = I. \tag{1.1}$$

Of special interest is a representation of these operators A and A^* as linear operators in a separable Hilbert spanned by eigenvectors $|n\rangle$; $n = 0, 1, \ldots$ of the positive semidefinite number operator $N = A^*A$.

One has the well-known relations

$$A|n\rangle = \sqrt{n}|n-1\rangle, \qquad A^*|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{1.2}$$

We denote the Bargmann space [14] by

$$B = \left\{\phi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire}, \int_{\mathbb{C}} |\phi(z)|^2 e^{-|z|^2} dx\, dy < \infty\right\}. \tag{1.3}$$

The scalar product on B is defined by

$$\langle \phi, \psi \rangle = \int_{\mathbb{C}} \phi(z)\overline{\psi(z)} e^{-|z|^2} dx\, dy, \tag{1.4}$$

and the associated norm is denoted by $\|\cdot\|$.

An orthonormal basis of B is given by

$$e_n(z) = \frac{z^n}{\sqrt{n!}}; \quad n = 0, 1, \ldots. \tag{1.5}$$

B is closed in $L_2(\mathbb{C}, d\mu(z))$, where the measure $d\mu(z) = e^{-|z|^2} dx dy$ and is closed related to $L_2(\mathbb{R})$ by an unitary transform of $L_2(\mathbb{R})$ onto B given in [14] by the following integral transform:

$$\phi(z) = \int_{\mathbb{R}} e^{-(1/2)(z^2+u^2)+\sqrt{2}uz} f(u)\, du. \tag{1.6}$$

If $f \in L_2(\mathbb{R})$ the integral converges absolutely.

In this Bargmann representation, the annihilator and creator operators are defined by

$$\begin{gathered} A\phi(z) = \frac{d}{dz}\phi(z) \quad \text{with domain } D(A) = \left\{\phi \in B; \frac{d}{dz}\phi \in B\right\}, \\ A^*\phi(z) = z\phi(z) \quad \text{with domain } D(A^*) = \{\phi \in B; z\phi \in B\}. \end{gathered} \tag{1.7}$$

Now, we define

$$B_p = \left\{\phi \in B; \frac{d^j}{dz^j}\phi(0) = 0,\ 0 \le j \le p\right\}. \tag{1.8}$$

An orthonormal basis of B_p is given by

$$e_n(z) = \frac{z^n}{\sqrt{n!}}; \quad n = p+1,\ p+2, \ldots. \tag{1.9}$$

Hence, a family of weighted shifts H_p is defined as follows:

$$H_p = A^{*^p} A^{p+1} \quad \text{with domain } D(H_p) = \{\phi \in B; H_p\phi \in B\} \bigcap B_p. \tag{1.10}$$

Remark 1.1. (i) For $p = 0$, the operator $H_0 = A$ is the derivation in Bargmann space, and it is the celebrated quantum annihilation operator.

(ii) $H_0^* e_n = \sqrt{n+1} e_{n+1}$ is a weighted shift with weight $\omega_n = \sqrt{n+1}$ for $n = 0, 1, \ldots$.

(iii) It is known that H_0 with its domain $D(H_0)$ is a chaotic operator in Bargmann space.

(iv) $H_0\phi_\lambda(z) = \lambda\phi_\lambda(z)$ for all $\lambda \in \mathbb{C}$ where $\phi_\lambda(z) = \sum_{n=0}^{\infty}(\lambda^n/\sqrt{n!})e_n(z)$ and $\|\phi_\lambda\|^2 = e^{|\lambda|^2}$.

(v) The function $e^{-|\lambda|^2}\phi_\lambda(z)$ is called a coherent normalized quantum optics (see [15, 16]).

Remark 1.2. (i) For $p = 1$, the operator $H_1 = A^* A^2 = z(d^2/dz^2)$ has as adjoint the operator $H_1^* = z^2(d/dz)$.

(ii) $H_1^* e_n = n\sqrt{n+1}e_{n+1}$ is a weighted shift with weight $\omega_n = n\sqrt{n+1}$ for $n = 1, \ldots$ and it is known that $H_1 + H_1^*$ is a not self-adjoint operator and is chaotic in Bargmann space [13]. This operator plays an essential role in Reggeon field theory (see [17, 18]).

(iii) The operators H_p arising also in the Jaynes-Cummings interaction models, see for example a model introduced by Obada and Abd Al-Kader in [19], the interaction Hamiltonian for the model is

$$H_I = -\sum_{p=0}^{\infty} \left\{ \Omega_1 e^{i\phi_1} e^{-(\eta_1^2/2)} \frac{(i\eta_1)^{2p+1}}{p!(p+1)!} H_p^* + \Omega_2 e^{i\phi_2} e^{-(\eta_2^2/2)} \frac{(i\eta_2)^{2p+1}}{p!(p+1)!} H_p \right\} \sigma_- + h \cdot c, \tag{1.11}$$

where Ω_j are the Rabi frequencies and and η_j^2 are the Lamb-Dicke; $j = 1,2$. The operators σ_- and σ_+ act on the ground state $|g\rangle$ and excited state $|e\rangle$ as follows: $\sigma_\pm|g\rangle = (1 \pm 1/2)|e\rangle$ and $\sigma_\pm|e\rangle = (1 \mp 1/2)|g\rangle$.

(iv) On $B_p, p = 0, 1, \ldots$, which is the orthogonal of span $\{e_n; n \le p\}$ in Bargmann space, the adjoint of H_p is $H_p^* = z^{p+1}(d^p/dz^p)$ such that

$$H_p^* e_n = \omega_n e_{n+1} \text{ with weight } \omega_n = \sqrt{n+1}\frac{n!}{(n-p)!} \quad \text{for } n \ge p \ge 0. \tag{1.12}$$

This paper is organized as follows: in Section 2, we recall the definition of the chaoticity for an unbounded operator following Devaney and sufficient conditions on hypercyclicity of unbounded operators given by Bés-Chan-Seubert theorem [10]. As our operator H_p is an unilateral weighted backward shift with an explicit weight, we use the results of Bés et al. to proof the chaoticity of H_p in Bargmann space (we can also use the results of Bermúdez et al. [11] to proof the chaoticity of our operator H_p). Then, we construct the hyperbolic structure associated to H_p. In the appendix, we present a direct proof of the chaoticity of H_p based on the Baire Category theorem. The last theorem is essential to proof that the operator is topologically transitive and can be used for interested reader.

2. Chaoticity of the Operator $H_p = z^p(d^{p+1}/dz^{p+1})$ on B_p

Definition 2.1. Let T be an unbounded linear operator on a separable infinite dimensional Banach X with domain $D(T)$ dense in X and such that T^n is closed for all positive integers n.

(a) The operator T is hypercyclic if there exists a vector $f \in D(T)$ such that $T^n f \in D(T)$ and if the orbit $\{f, Tf, T^2 f, \ldots\}$ is dense in X. The vector f is called a hypercyclic vector of T.

(b) A vector $g \in D(T)$ is called a periodic point of T if there exists m such that $T^m g = g$. The operators having both dense sets of periodic points and hypercyclic vectors are said to be chaotic following the definition of Devaney [20, 21].

Sufficient conditions for the hypercyclicity of an unbounded operator are given in the following Bés-Chan-Seubert theorem:

Theorem 2.2 (Bés-Chan-Seubert [10]). *Let X be a separable infinite dimensional Banach, and let T be a densely defined linear operator on X. Then, T is hypercyclic if*

(i) T^m *is a closed operator for all positive integers* m,

(ii) *there exists a dense subset F of the domain $D(T)$ of T and a (possibly nonlinear and discontinuous) mapping $S : F \longrightarrow F$ so that TS is the identity on F and $T^n, S^n \longrightarrow 0$ pointwise on F as $n \longrightarrow \infty$.*

Theorem 2.3. *Let B be the Bargmann space with orthonormal basis $e_n(z) = z^n/\sqrt{n!}$. Let $H_p = z^p(d^{p+1}/dz^{p+1}) = A^{*^p}A^{p+1}$ with domain $D(H_p) = \{\phi \in B; H_p\phi \in B\} \bigcap B_p$, where $B_p = \{\phi \in B; (d^j/dz^j)\phi(0) = 0,\ 0 \le j \le p\}$. Then, H_p is a chaotic operator.*

Remark 2.4. (i) Following the ideas of Gross-Erdmann in [4, 5] or the Theorem 2.4 of Bermúdez et al. [11], we can use a test on the weight of H_p to give a proof of the chaoticity of H_p.

We choose to give a proof under lemma form based on the theorem of Bès et al. recalled above, we also indicate in the appendix the utilization of the Baire category theorem in the hepercyclicity theory and we prove that H_p possesses a certain "sensitivity to initial conditions" though this property is redundant in Devaney's definition (see Banks et al. in [20]).

(ii) Let T be an unbounded operator on separable infinite dimensional Banach X. It may happen that vector f $\in D(T)$, but Tf fails to be in the domain of T. We can exhibit a closed operator whose square is not. For example, the operator acting on $L_2(0,1) \times L_2(0,1)$ defined by $T(u,v)(x) = (v'(x), f(x)v(0))$ with domain $D(T) = L_2(0,1) \times H_1(0,1)$, where $v'(x)$ is the derivative of $v(x)$ and f is a function in $H_1(0,1)$ with $f(0) = 1$, where $H_1(0,1)$ is the classical Sobolev space. Then T, is a closed operator and $D(T^2) = D(T)$, where $D(T^2)$ is the domain of T^2 but the operator T^2 is not closed and has not closed extension.This operator can, for example, justify the asumption (a) of the Definition 2.1 for the unbounded linear operators.

Lemma 2.5. *For each positive integer m, the operator $(H_p)^m$, with domain $D((H_p)^m) = \{\phi \in B; (H_p)^m\phi \in B\} \bigcap B_p$, is a closed operator.*

Proof. As $(H_p)^m$ is closed if and only if the graph $G((H_p)^m)$ is a closed linear manifold of $B_p \times B_p$, then let $(\phi_n, (H_p)^m\phi_n)$ be a sequence in $G((H_p)^m)$ which converges to (ϕ, ψ) in $B_p \times B_p$. As ϕ_n converges to ϕ in B_p, then $z^p(d^{p+1}/dz^{p+1})\phi_n$ converges to $z^p(d^{p+1}/dz^{p+1})\phi$ pointwise on $\mathbb{C}$ and $(H_p)^m\phi_n$ converges to $(H_p)^m\phi$ pointwise on $\mathbb{C}$. As $(H_p)^m\phi_n$ converges to ψ, we deduce that $(H_p)^m)\phi = \psi$ and $\phi \in D((H_p)^m)$, hence $G((H_p)^m)$ is closed. □

Lemma 2.6. *Let $H_p = z^p(d^{p+1}/dz^{p+1})$ with domain $D(H_p) = \{\phi \in B; H_p\phi \in B\} \bigcap B_p$, where $H_pe_n = \omega_{n-1}e_{n-1}$, $e_n(z) = z^n/\sqrt{n!}$, and $\omega_n = \sqrt{n+1}(n!/(n-p)!)$ for $n \ge p \ge 0$. Then, H_p is hypercyclic.*

Proof. Let $F = \{\phi_k(z) = \sum_{n=p}^{k} a_n e_n(z)\}$. This space is dense in B_p.

Let S_p: $F \longrightarrow F$ and $S_pe_n = (1/\omega_n)e_{n+1}; n \ge p \ge 0$.

Then, $H_pS_p\phi_k(z) = \phi_k(z)$, that is, $H_pS_p = I_{|F}$.

Now, as $[H_p]^k e_n = 0$ for all $k > n \ge p$ we deduce that any element of F can be annihilated by a finite power k_n of H_p since as $[\prod_{j=n}^{k_n+n}\omega_j]^{-1} \longrightarrow 0$ when $k_\mathrm{n} \longrightarrow \infty$, we have

$$S_p^{k_n}e_n = \left[\prod_{j=n}^{k_n+n}\omega_j\right]^{-1} e_{k+n} \longrightarrow 0 \quad \text{in } B_p. \tag{2.1}$$

Now, the hypercyclicity of H_p follows from the theorem of Bés et al. recalled above. □

Lemma 2.7. *Let $H_p = z^p(d^{p+1}/dz^{p+1})$ with domain $D(H_p) = \{\phi \in B; H_p\phi \in B\} \bigcap B_p$, where $H_p e_n = \omega_{n-1}e_{n-1}$, $e_n(z) = (z^n/\sqrt{n!})$, and $\omega_n = \sqrt{n+1}(n!/(n-p)!)$ for $n \geq p \geq 0$. Then, there exist $k > 0$ and $g \in D(H_p^k)$ such that $H_p^k g(z) = g(z)$.*

Proof. Let $\lambda \in \mathbb{C}$ and

$$g_\lambda(z) = e_p(z) + \sum_{n=p+1}^{\infty} \frac{\lambda^{n-p}}{\omega_p\omega_{p+1}\cdots\omega_{n-1}} e_n(z), \tag{2.2}$$

then g_λ is in the domain of H_p and it is an eigenvector for H_p corresponding to eigenvalue λ, therefore it is a periodic point of H_p. λ is a root of unity.

In fact, let $r > 0$ and $|\lambda| < r$, then as

$$\lim \prod_{j=p}^{n-1} \omega_j = \infty \quad \text{when } n \longrightarrow \infty, \tag{2.3}$$

there exist $n_0 > 0$ and $q < 1$ such that

$$\frac{r}{(\omega_p\omega_{p+1}\cdots\omega_{n-1})^{(1/n)}} \leq q \quad \text{for } n \geq n_0 , \tag{2.4}$$

since for $|\lambda| < r$, we have

$$\frac{|\lambda|^{(n-p)}}{(\omega_p\omega_{p+1}\cdots\omega_{n-1})^2} \leq q^{2n}; \quad n \geq n_0 \tag{2.5}$$

and g_λ is in Bargmann space. Now as,

$$\begin{gathered} \langle g_\lambda, e_p \rangle = 1, \\ \langle g_\lambda, e_{n+1} \rangle = \frac{\lambda^{n-p+1}}{\omega_p\omega_{p+1}\cdots\omega_n} , \end{gathered} \tag{2.6}$$

we get

$$\begin{aligned} |\langle g_\lambda, e_{n+1} \rangle|^2 &= \frac{\lambda^{2(n-p+1)}}{(\omega_p\omega_{p+1}\cdots\omega_n)^2}, \\ |\langle g_\lambda, e_{n+1} \rangle|^2 (\omega_n)^2 &= \frac{\lambda^{2(n-p+1)}}{(\omega_p\omega_{p+1}\cdots\omega_{n-1})^2} \leq q^{2n}|\lambda|^2 \quad \text{for } n \geq n_0. \end{aligned} \tag{2.7}$$

We deduce that

$$\sum_{n=p}^{\infty} \left| \langle g_\lambda, e_{n+1} \rangle \right|^2 (\omega_n)^2 < \infty, \tag{2.8}$$

that is, $g_\lambda \in D(H_p)$.

Thus, we get

$$\begin{aligned} H_p g_\lambda &= H_p e_p + \sum_{n=p+1}^{\infty} \frac{\lambda^{n-p}}{\omega_p \omega_{p+1} \cdots \omega_{n-1}} \cdot \omega_{n-1} e_{n-1} \\ &= \lambda \sum_{m=p}^{\infty} \frac{\lambda^{m-p}}{\omega_p \omega_{p+1} \cdots \omega_{m-1}} e_m \\ &= \lambda \left\{ e_p + \sum_{m=p+1}^{\infty} \frac{\lambda^{m-p}}{\omega_p \omega_{p+1} \cdots \omega_{m-1}} e_m \right\} \\ &= \lambda g_\lambda. \end{aligned} \tag{2.9}$$

Therefore, g_λ is the eigenvector corresponding to the eigenvalue λ and g_λ is a periodic point of H_p, where λ is a root of unity. □

Lemma 2.8. *The set of periodic points of H_p is dense in B_p.*

Proof. Let

$$G = \operatorname{span} \{ g_\lambda; \lambda \text{ is a root of unity} \}. \tag{2.10}$$

G is dense in B_p, otherwise there exists nonzero vector $g \in B_p$ which is orthogonal to G.
Let

$$\begin{gathered} g(z) = \sum_{n=p}^{\infty} b_n e_n(z) \quad \text{such that } \langle g, g_\lambda \rangle = 0, \quad \text{for each } g_\lambda \in G, \\ f(\lambda) = \langle g, g_\lambda \rangle \quad \text{for } |\lambda| < 1, \qquad f(\lambda) = 0 \quad \text{for } |\lambda| = 1 . \end{gathered} \tag{2.11}$$

$f(\lambda)$ is a continuous function on the closed unit disc which is holomorphic on the interior and vanishes at each root of unity, hence on the entire unit circle, hence $f(\lambda)$ vanishes for all $|\lambda| \leq 1$. We deduce that $b_n = 0$ for $n \geq p$, then G is dense in B_p. □

Remark 2.9. (i) The Lemmas 2.5, 2.6, and 2.8 show the chaoticity of H_p.

(ii) The Theorem 2.3 generalizes the result of [12] on the annihilation operator in Bargmann space.

Definition 2.10. Let T be an unbounded linear operator on a separable infinite dimensional Banach X whose domain $D(T)$ is dense in X, and let T^n be closed for all positive integers n.

(a) A closed subspace $E \subset X$ has hyperbolic structure if $E = E^u \oplus E^s$, $TE^u = E^u$, and $TE^s = E^s$, where E^u (the unstable subspace) and E^s (the stable subspace) are closed. In addition, there exist constants $\tau(0 < \tau < 1)$ and $C > 0$, such that:

(i) For any $\Phi \in E^u$, $k \in \mathbb{N}$, $C\tau^{-k}\|\Phi\| \leq \|T^k\Phi\|$ (the vectors of E^u are exponentially expanded, we say they belong to the unstable subspace E^u).

(ii) For any $\Psi \in E^s$, $k \in \mathbb{N}$, $\|T^k\Psi\| \leq C\tau^k\|\Psi\|$ (some vectors are contracted exponentially fast by the iterates of the operator T, we say they belong to the stable subspace E^s).

(b) If there exists a closed subspace $E \subset X$ which has hyperbolic structure relative to T and the set of periodic points of T is dense in E, then T is said to be a nonwandering operator relative to E following the definition of Tian et al. [8].

Since H_p is chaotic operator on B_p, so H_p has dense set of periodic points on B_p, we only need to construct an hyperbolic structure associated to it in B_p to obtain:

Theorem 2.11. *Let B be the Bargmann space with orthonormal basis $e_n(z) = z^n/\sqrt{n!}$.*

Let $H_p = z^p(d^{p+1}/dz^{p+1}) = A^{*^p}A^{p+1}$ with domain $D(H_p) = \{\phi \in B; H_p\phi \in B\} \bigcap B_p$, where $B_p = \{\phi \in B; (d^j/dz^j)\phi(0) = 0,\ 0 \leq j \leq p\}$.

Then, H_p is a nonwandering operator.

Proof. We construct a closed invariant subspace $E \subset B_p$ such that E has hyperbolic structure.

For $\lambda \in \mathbb{C}$, the function defined by (2.2).

$g_\lambda(z) = e_p(z) + \sum_{n=p+1}^{\infty}(\lambda^{n-p}/\omega_p\omega_{p+1}\cdots\omega_{n-1})e_n(z)$ is in the domain of H_p and is an eigenvector for H_p corresponding to the eigenvalue λ.

Let $E^u = \overline{\operatorname{span}\{g_\lambda; |\lambda| > 1\}}$, $E^s = \overline{\operatorname{span}\{g_\lambda; 0 < |\lambda| < 1\}}$, and $E = E^u \oplus E^s$, where $\oplus$ represents direct sum.

We will verify that E has an hyperbolic structure.

For $\phi \in E^u$, there exists a sequence (a_i), $i = 1, 2, \ldots$ such that

$$\phi(z) = \sum_{i=1}^{\infty} a_i g_{\lambda_i}(z) = \sum_{i=1}^{\infty} a_i \sum_{n=p}^{\infty} \frac{\lambda_i^{n-p}}{\omega_p\omega_{p+1}\cdots\omega_{n-1}} e_n(z). \tag{2.12}$$

And for each positive integer m, we have

$$\|(H_p)^m\phi\| = \left\|(H_p)^m \sum_{i=1}^{\infty} a_i g_{\lambda_i}\right\| = \left\|(H_p)^m \sum_{i=1}^{\infty} a_i \lambda_i^m \sum_{n=p}^{\infty} \frac{\lambda_i^{n-p}}{\omega_p\omega_{p+1}\cdots\omega_{n-1}} e_n\right\| \geq \mu^m\|\phi\|, \tag{2.13}$$

where $\mu = \min\{|\lambda_i|; |\lambda_i| > 1\}$.

Next, we will prove E^u is the invariant subspace of H_p.

Let $\phi \in E^u$, then

$$\phi(z) = \sum_{i=1}^{\infty} a_i g_{\lambda_i}(z) = \sum_{i=1}^{\infty} a_i \sum_{n=p}^{\infty} \frac{\lambda_i^{n-p}}{\omega_p\omega_{p+1}\cdots\omega_{n-1}} e_n(z) = H_{\mathrm{p}} \sum_{i=1}^{\infty} b_i g_{\lambda_i}(z), \tag{2.14}$$

where $b_i = a_i/\lambda_i$ then $E^u \subset H_pE^u$.

Now for $\psi \in H_p E^u$, then there exists $\phi \in E^u$, such that $\psi = H_p \Phi = \sum_{i=1}^{\infty} c_i g_{\lambda_i}$, where $c_i = \lambda_i a_i$. Therefore, $H_p E^u \subset E^u$.

Similarly, let $E^s = \overline{\operatorname{span}\{g_\lambda; 0 < |\lambda| < 1\}}$, we deduce that $H_p E^s = E^s$ and if we chose $\tau = 1/\mu$, then we have

$$\|(H_p)^m \phi\| \geq \frac{\|\phi\|}{\tau^m}. \tag{2.15}$$

Then, E has hyperbolic structure and H_p is nonwandering operator relative to E.
Here, the linear space E^s is formed by the (spectral) subspace corresponding to the eigenvalues of H_p of modulus less than 1, while the unstable subspace E^u corresponds to those of modulus greater than 1. □

As in [22], we can use the Gazeau-Klauder formalism to construct the coherent states of this operator H_p and investigate some properties of these coherent states (see [23]).

We conclude that main results of this work can be considered in [24] as an introduction to study of the operators $H_{p,m} = z^p(\partial^{p+m}/dz^{p+m})$ with $p = 0, 1, \ldots$ and $m = 1, 2, \ldots$ particularly, to study the chaoticity of $H_{p,m} + H_{p,m}^*$.

Appendix

Let us recall below the essential spaces and operators used in above sections

(i) $B = \{\phi : \mathbb{C} \to \mathbb{C} \text{ entire}; \int_{\mathbb{C}} |\phi(z)|^2 e^{-|z|^2} dx\, dy < \infty\}$,

(ii) $B_p = \{\phi \in B; (d^j/dz^j)\phi(0) = 0,\ 0 \leq j \leq p\}$,

(iii) $H_p e_n = \omega_{n-1} e_{n-1}$ with $e_n(z) = z^n/\sqrt{n!}$ and $\omega_n = \sqrt{n+1}(n!/(n-p)!)$ for $n \geq p \geq 0$,

(iv) $F = \{\phi_k(z) = \sum_{n=p}^{k} a_n e_n(z),\ k = p,\ p+1, \ldots\}$,

(v) $S_p e_n = (1/\omega_n) e_{n+1}; n \geq p \geq 0$.

Then, we have the following.

Lemma A.1. *For arbitrary $\phi, \psi \in B_p$, there exists $\phi_k \in F$ such that $\phi_k \to \phi$ and $H_p^k \phi_k \to \psi$.*

Proof. As F is dense in B_p, then for arbitrary $\psi \in B_p$, there exists $\psi_k \in F$ such that $\psi_k \to \psi$.

Let m a natural number, as $\omega_n = \sqrt{n+1}(n!/(n-p)!)$ for $n \geq p \geq 0$, then we get

$$\begin{gathered}
\omega_n \omega_{n+1} \cdots \omega_{n+m} = \sqrt{(n+m)!} \prod_{j=n}^{n+m} \frac{j!}{(j-p)!} \quad \text{for } n \geq p \geq 0, \\
\frac{1}{\omega_n \omega_{n+1} \cdots \omega_{n+m}} \leq \frac{1}{\sqrt{(n+m)!}} \quad \text{for } n \geq p \geq 0\ .
\end{gathered} \tag{A.1}$$

Now, for arbitrary $\phi \in F$ and $\|\phi\| = 1$, $\phi = \sum_{n=p}^{\infty} a_n e_n$, we have

$$S_p^m \phi = \sum_{n=p}^{\infty} a_n \frac{1}{\omega_n \omega_{n+1} \cdots \omega_{n+m}} e_{n+m},$$
$$\left\| S_p^m \phi \right\| \le \sum_{n=p}^{\infty} |a_n| \frac{1}{\sqrt{(n+m)!}}. \tag{A.2}$$

As for $m \ge 3$, we have $(1/m!) \le (1/2^m)$, then

$$\left\| S_p^m \phi \right\| \le \frac{1}{\sqrt{2}^m} + \frac{1}{\sqrt{2}^{m+1}} + \cdots \equiv \frac{1}{2-\sqrt{2}} \frac{1}{\sqrt{2}^n} \longrightarrow 0 \quad (m \longrightarrow \infty), \tag{A.3}$$

hence S_p^m tends pointwise to zero on F.

By choosing diagonal element of $\{S_p^m \psi_k\}$, we get

$$S_p^k \psi_k \longrightarrow 0 \quad \text{when} \quad k \longrightarrow \infty. \tag{A.4}$$

As $H_p S_p = I$ on F, then we can write $\psi_k = H_p S_p \psi_k$, that is,

$$H_p S_p^k \psi_k \longrightarrow \psi \quad \text{when} \quad k \longrightarrow \infty. \tag{A.5}$$

Now as $\mathrm{Ker} H_p^m = \mathrm{span}\{e_j\ ,\ p \le j \le p+m-1\}$, where $\mathrm{Ker} H_p^m$, is the kernel space of H_p^m then $\prod_{m=p}^{\infty} \mathrm{Ker} H_p^m$, is dense in B_p and for arbitrary $\Phi \in \bigcup \prod_{m=p}^{\infty} \mathrm{Ker} H_p^m$, there exists m such that $\phi \in \mathrm{Ker} H_p^m$. Therefore, H_p^m tends pointwise to zero on a dense subset of B_p.

For arbitrary $\phi \in B_p$, there exists $f_k \in \prod_{m=p}^{\infty} \mathrm{Ker} H_p^m$ such that $f_k \to \phi$, therefore

$$H_p^m f_k \longrightarrow 0 \quad (m \longrightarrow \infty). \tag{A.6}$$

Particularly,

$$H_p^k f_k \longrightarrow 0 \quad (k \longrightarrow \infty). \tag{A.7}$$

Let

$$\phi_k = f_k + S_p^k \psi_k. \tag{A.8}$$

Then, $H_p^k \phi_k \to \psi$ $(k \to \infty)$. □

Lemma A.2. *Let $\phi \in G = \bigcap_{j=0}^{\infty} \bigcup_{m=0}^{\infty} H_p^m D_j$, where D_j, is an enumeration of open ball in B_p with centers in a countable dense subset of B_p, then $\{\phi, H_p\phi, \ldots, H_p^m \Phi, \ldots\}$ is dense in B_p.*

Proof. The above lemma imply for arbitrary $\phi \in D_j$ and $\psi \in B_p$ that there exists $\phi_k \in D_j$ such that $\phi_k \to \phi$, and $\overline{H}_p^k \phi_k \to \psi$, hence $\bigcup_{m=0}^{\infty} H_p^m D_j$ is dense in B_p and Baire category theorem implies G is dense in B_p. Hence, H_p is topologically transitive. □

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