Research Article

The Hijazi Inequalities on Complete Riemannian Spin^c Manifolds

Roger Nakad

Institut Élie Cartan, Université Henri Poincaré, Nancy I, B.P 239,
54506 Vandœuvre-Lès-Nancy Cedex, France

Correspondence should be addressed to Roger Nakad, nakad@iecn.u-nancy.fr

We extend the Hijazi type inequality, involving the energy-momentum tensor, to the eigenvalues of the Dirac operator on complete Riemannian Spin^c manifolds without boundary and of finite volume. Under some additional assumptions, using the refined Kato inequality, we prove the Hijazi type inequality for elements of the essential spectrum. The limiting cases are also studied.

1. Introduction

On a compact Riemannian Spin^c manifold (M^n, g) of dimension $n \geqslant 2$, any eigenvalue λ of the Dirac operator satisfies the Friedrich type inequality [1, 2]

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \inf_M (S - c_n |\Omega|), \tag{1.1}$$

where S denotes the scalar curvature of M, $c_n = 2[n/2]^{1/2}$ and $i\Omega$ is the curvature form of the connection on the line bundle given by the Spin^c structure. Equality holds if and only if the eigenspinor ψ associated with the first eigenvalue λ_1 is a Spin^c Killing spinor; that is, for every $X \in \Gamma(TM)$, the eigenspinor ψ satisfies

$$\begin{aligned} \nabla_X \psi &= -\frac{\lambda_1}{n} X \cdot \psi, \\ \Omega \cdot \psi &= i \frac{c_n}{2} |\Omega| \psi. \end{aligned} \tag{1.2}$$

Here, $X \cdot \psi$ denotes the Clifford multiplication and ∇ the spinorial Levi-Civita connection [3, 4]. In [5], it is shown that on a compact Riemannian Spin^c manifold any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies the Hijazi type inequality [6] involving the Energy-Momentum tensor and the scalar curvature

$$\lambda^2 \geqslant \inf_M \left(\frac{1}{4} S - \frac{c_n}{4} |\Omega| + |\ell^\psi|^2 \right), \tag{1.3}$$

where ℓ^ψ is the field of symmetric endomorphisms associated with the field of quadratic forms denoted by T^ψ, called the Energy-Momentum tensor. It is defined on the complement set of zeroes of the eigenspinor ψ, for any vector field X by

$$T^\psi(X) = \mathrm{Re}\left\langle X \cdot \nabla_X \psi, \frac{\psi}{|\psi|^2} \right\rangle. \tag{1.4}$$

Equality holds in (1.3) if and only, for all $X \in \Gamma(TM)$, we have

$$\nabla_X \psi = -\ell^\psi(X) \cdot \psi,$$
$$\Omega \cdot \psi = i \frac{c_n}{2} |\Omega| \psi, \tag{1.5}$$

where ψ is an eigenspinor associated with the first eigenvalue λ_1. By definition, the trace $\mathrm{tr}(\ell^\psi)$ of ℓ^ψ, where ψ is an eigenspinor associated with an eigenvalue λ, is equal to λ. Hence, (1.3) improves (1.1) since by the Cauchy-Schwarz inequality, $|\ell^\psi|^2 \geqslant (\mathrm{tr}(\ell^\psi))^2 / n = \lambda^2 / n$. It is also shown that the sphere equipped with a special Spin^c structure satisfies the equality case in (1.3) but equality in (1.1) cannot occur.

In the same spirit as in [7], Herzlich and Moroianu (see [1]) generalized the Hijazi inequality [7], involving the first eigenvalue of the Yamabe operator L, to the case of compact Spin^c manifolds of dimension $n \geqslant 3$: any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \mu_1, \tag{1.6}$$

where μ_1 is the first eigenvalue of the perturbed Yamabe operator defined by $L^\Omega = L - c_n |\Omega|_g = 4((n-1)/(n-2))\Delta + S - c_n |\Omega|_g$. The limiting case of (1.6) is equivalent to the limiting case in (1.1). The Hijazi inequality [6], involving the energy-momentum tensor and the first eigenvalue of the Yamabe operator, is then proved by the author in [5] for compact Spin^c manifolds. In fact, any eigenvalue of the Dirac operator to which is attached an eigenspinor ψ satisfies

$$\lambda^2 \geqslant \frac{1}{4} \mu_1 + \inf_M |\ell^\psi|^2. \tag{1.7}$$

Equality in (1.7) holds if and only, for all $X \in \Gamma(TM)$, we have

$$\overline{\nabla}_X \overline{\varphi} = -\ell^{\overline{\varphi}}(X) \cdot \overline{\varphi},$$

$$\Omega \cdot \psi = i \frac{c_n}{2} |\Omega|_g \psi,$$

(1.8)

where $\overline{\varphi} = e^{-((n-1)/2)u} \overline{\psi}$, the spinor field $\overline{\psi}$ is the image of ψ under the isometry between the spinor bundles of (M^n, g) and $(M^n, \overline{g} = e^{2u}g)$, and ψ is an eigenspinor associated with the first eigenvalue λ_1 of the Dirac operator. Again, (1.7) improves (1.6). In this paper we examine these lower bounds on open manifolds, and especially on complete Riemannian Spin^c manifolds. We prove the following.

Theorem 1.1. *Let (M^n, g) be a complete Riemannian Spin^c manifold of finite volume. Then, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies the Hijazi type (1.3). Equality holds if and only if the eigenspinor associated with the first eigenvalue λ_1 satisfies (1.5).*

The Friedrich type (1.1) is derived for complete Riemannian Spin^c manifolds of finite volume and equality also holds if and only if the eigenspinor associated with the first eigenvalue λ_1 is a Killing Spin^c spinor. This was proved by Grosse in [8, 9] for complete spin manifolds of finite volume. Using the conformal covariance of the Dirac operator, we prove the following.

Theorem 1.2. *Let (M^n, g) be a complete Riemannian Spin^c manifold of finite volume and dimension $n > 2$. Any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies the Hijazi type (1.7). Equality holds if and only if (1.8) holds.*

Now, the Hijazi type (1.6) can be derived for complete Riemannian Spin^c manifolds of finite volume and dimension $n > 2$ and equality holds if and only if the eigenspinor associated with the first eigenvalue λ_1 is a Killing Spin^c spinor. This was also proved by Grosse in [8, 9] for complete spin manifolds of finite volume and dimension $n > 2$. On complete manifolds, the Dirac operator is essentially self-adjoint and, in general, its spectrum consists of eigenvalues and the essential spectrum. For elements of the essential spectrum, we also extend to Spin^c manifolds the Hijazi type (1.6) obtained by Grosse in [9] on spin manifolds.

Theorem 1.3. *Let (M^n, g) be a complete Riemannian Spin^c manifold of dimension $n \geq 5$ with finite volume. Furthermore, assume that $S - c_n |\Omega|$ is bounded from below. If λ is in the essential spectrum of the Dirac operator $\sigma_{ess}(D)$, then λ satisfies the Hijazi type (1.6).*

For the 2-dimensional case, Grosse proved in [8] that for any Riemannian spin surface of finite area, homeomorphic to $\mathbb{R}^2$, we have

$$\lambda^+ \geq \frac{4\pi}{\text{Area}(M^2, g)},$$

(1.9)

where $\lambda^+ = \inf_{\varphi \in C_c^\infty(M)}((D^2\varphi, \varphi)/(\varphi, \varphi))$ (in the compact case, λ^+ coincides with the first eigenvalue of the square of the Dirac operator). Recently, in [10], Bär showed the same inequality for any connected 2-dimensional Riemannian manifold of genus 0, with finite area and equipped with a spin structure which is *bounding at infinity*. A spin structure on M is said

to be *bounding at infinity* if M can be embedded into $\mathbb{S}^2$ in such a way that the spin structure extends to the unique spin structure of $\mathbb{S}^2$.

Studying the energy-momentum tensor on a compact Riemannian spin or Spin^c manifolds has been done by many authors, since it is related to several geometric situations. Indeed, on compact spin manifolds, Bourguignon and Gauduchon [11] proved that the energy-momentum tensor appears naturally in the study of the variations of the spectrum of the Dirac operator. Hence, when deforming the Riemannian metric in the direction of this tensor, the eigenvalues of the Dirac operator are then critical. Using this, Kim and Friedrich [12] obtained the Einstein-Dirac equation as the Euler-Lagrange equation of a certain functional. The author extends these last results to compact Spin^c manifolds [13]. Even it is not a computable geometric invariant, the energy-momentum tensor is, up to a constant, the second fundamental form of an isometric immersion into a Spin^c manifold carrying a parallel spinor [13, 14]. Moreover, in low dimensions, the existence, on a spin or Spin^c manifold M, of a spinor ψ satisfying (1.5) is, under some additional assumptions, equivalent to the existence of a local immersion of M into $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{C}P^2$, $\mathbb{S}^2 \times \mathbb{R}$ or some others manifolds [14–16].

2. Preliminaries

In this section, we briefly introduce basic notions concerning Spin^c manifolds, the Dirac operator and its conformal covariance. Then, we recall the refined Kato inequality which is crucial for the proof.

The Dirac Operator on Spin^c Manifolds

Let (M^n, g) be a connected oriented Riemannian manifold of dimension $n \geqslant 2$ without boundary. Furthermore, let SOM be the SO_n-principal bundle over M of positively oriented orthonormal frames. A Spin^c structure of M is a Spin$_n^c$-principal bundle $(\text{Spin}^c M, \pi, M)$ and an $\mathbb{S}^1$-principal bundle $(\mathbb{S}^1 M, \pi, M)$ together with a double covering given by $\theta : \text{Spin}^c M \to SOM \times_M \mathbb{S}^1 M$ such that $\theta(ua) = \theta(u)\xi(a)$, for every $u \in \text{Spin}^c M$ and $a \in \text{Spin}_n^c$, where ξ is the 2-fold covering of Spin_n^c over $SO_n \times \mathbb{S}^1$. Let $\Sigma M := \text{Spin}^c M \times_{\rho_n} \Sigma_n$ be the associated spinor bundle, where $\Sigma_n = \mathbb{C}^{2^{[n/2]}}$ and $\rho_n : \text{Spin}_n^c \to \text{End}(\Sigma_n)$ the complex spinor representation. A section of ΣM will be called a spinor and the set of all spinors will be denoted by $\Gamma(\Sigma M)$ and those of compactly supported smooth spinors by $\Gamma_c(\Sigma M)$. The spinor bundle ΣM is equipped with a natural Hermitian scalar product, denoted by $\langle \cdot, \cdot \rangle$, satisfying

$$\langle X \cdot \psi, \varphi \rangle = -\langle \psi, X \cdot \varphi \rangle \quad \text{for every } X \in \Gamma(TM),\ \psi, \varphi \in \Gamma(\Sigma M), \tag{2.1}$$

where $X \cdot \psi$ denotes the Clifford multiplication of X and ψ. With this Hermitian scalar product we define an L^2-scalar product

$$(\psi, \varphi) = \int_M \langle \psi, \varphi \rangle \mathrm{v}_g, \tag{2.2}$$

for any spinors ψ and φ in $\Gamma_c(\Sigma M)$. Additionally, given a connection 1-form A on $\mathbb{S}^1 M$, $A : T(\mathbb{S}^1 M) \to i\mathbb{R}$ and the connection 1-form ω^M on SOM for the Levi-Civita connection ∇^M,

we consider the associated connection on the principal bundle $SOM \times_M \mathbb{S}^1 M$, and hence a covariant derivative ∇ on $\Gamma(\Sigma M)$ [3].

The curvature of A is an imaginary valued 2-form denoted by $F_A = dA$, that is, $F_A = i\Omega$, where Ω is a real valued 2-form on $\mathbb{S}^1 M$. We know that Ω can be viewed as a real-valued 2-form on M [3]. In this case, $i\Omega$ is the curvature form of the associated line bundle L. It is the complex line bundle associated with the $\mathbb{S}^1$-principal bundle via the standard representation of the unit circle. For any spinor ψ and any real 2-form Ω, we have [1]

$$\langle i\Omega \cdot \psi, \psi \rangle \geqslant -\frac{c_n}{2} |\Omega|_g |\psi|^2, \tag{2.3}$$

where $|\Omega|_g$ is the norm of Ω given by $|\Omega|_g^2 = \sum_{i<j} (\Omega_{ij})^2$. Moreover, equality holds in (2.3) if and only if

$$\Omega \cdot \psi = i\frac{c_n}{2} |\Omega|_g \psi. \tag{2.4}$$

The Dirac operator is a first-order elliptic operator locally given by

$$D = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}. \tag{2.5}$$

It is an elliptic and formally self-adjoint operator with respect to the L^2-scalar product; that is, for all spinors ψ, φ, at least one of which is compactly supported on M, we have $(D\psi, \varphi) = (\psi, D\varphi)$. An important tool when examining the Dirac operator is the Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^* \nabla + \frac{1}{4} S \operatorname{Id}_{\Gamma(\Sigma M)} + \frac{i}{2} \Omega\cdot, \tag{2.6}$$

where ∇^* is the adjoint of ∇ and $\Omega\cdot$ is the extension of the Clifford multiplication to differential forms given by $(e_i^* \wedge e_j^*) \cdot \psi = e_i \cdot e_j \cdot \psi$. For the Friedrich connection $\nabla_X^f \psi = \nabla_X \psi + (f/n) X \cdot \psi$, where f is real valued function one gets a Schrödinger-Lichnerowicz type formula similar to the one obtained by Friedrich in [2]

$$(D - f)^2 \psi = \Delta^f \psi + \left(\frac{S}{4} + \frac{n-1}{n} f^2 \right) \psi + \frac{i}{2} \Omega \cdot \psi - \frac{n-1}{n} (2fD\psi + \nabla f \cdot \psi), \tag{2.7}$$

where Δ^f is the spinorial Laplacian associated with the connection ∇^f.

A complex number λ is an eigenvalue of D if there exists a nonzero eigenspinor $\psi \in \Gamma(\Sigma M) \cap L^2(\Sigma M)$ with $D\psi = \lambda\psi$. The set of all eigenvalues is denoted by $\sigma_p(D)$, the point spectrum. We know that if M is closed, the Dirac operator has a pure point spectrum but on open manifolds, the spectrum might have a continuous part. In general, the spectrum of the Dirac operator $\sigma(D)$ is composed of the point, the continuous and the residual spectrum. For complete manifolds, the residual spectrum is empty and $\sigma(D) \subset \mathbb{R}$. Thus, for complete manifolds, the spectrum can be divided into point and continuous spectrum.

But often another decomposition of the spectrum is used: the one into discrete spectrum $\sigma_d(D)$ and essential spectrum $\sigma_{\mathrm{ess}}(D)$.

A complex number λ lies in the essential spectrum of D if there exists a sequence of smooth compactly supported spinors ψ_i which are orthonormal with respect to the L^2-product and

$$\left\| (D - \lambda)\psi_i \right\|_{L^2} \longrightarrow 0. \tag{2.8}$$

The essential spectrum contains all eigenvalues of infinite multiplicity. In contrast, the discrete spectrum $\sigma_d(D) := \sigma_p(D) \setminus \sigma_{\mathrm{ess}}(D)$ consists of all eigenvalues of finite multiplicity. The proof of the next property can be found in [8]: on a Spin^c complete Riemannian manifold, 0 is in the essential spectrum of $D - \lambda$ if and only if 0 is in the essential spectrum of $(D-\lambda)^2$, and in this case, there is a normalized sequence $\psi_i \in \Gamma_c(\Sigma M)$ such that ψ_i converges L^2-weakly to 0 with $\|(D - \lambda)\psi_i\|_{L^2} \to 0$ and $\|(D - \lambda)^2 \psi_i\|_{L^2} \to 0$.

Spinor Bundles Associated with Conformally Related Metrics

The conformal class of g is the set of metrics $\overline{g} = e^{2u}g$, for a real function u on M. At a given point x of M, we consider a g-orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x M$. The corresponding $\overline{g}$-orthonormal basis is denoted by $\{\overline{e}_1 = e^{-u}e_1, \ldots, \overline{e}_n = e^{-u}e_n\}$. This correspondence extends to the Spin^c level to give an isometry between the associated spinor bundles. We put a $"{-}"$ above every object which is naturally associated with the metric $\overline{g}$. Then, for any spinor field ψ and φ, one has $\langle \overline{\psi}, \overline{\varphi} \rangle = \langle \psi, \varphi \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the natural Hermitian scalar products on $\Gamma(\Sigma M)$, and on $\Gamma(\Sigma \, \overline{M})$. The corresponding Dirac operators satisfy

$$\overline{D}\left(e^{-((n-1)/2)u}\overline{\psi} \right) = e^{-((n+1)/2)u}\overline{D\psi}. \tag{2.9}$$

The norms of any real 2-form Ω with respect to g and $\overline{g}$ are related by

$$\left. |\Omega| \right._{\overline{g}} = e^{-2u}\left. |\Omega| \right._g. \tag{2.10}$$

Hijazi [6] showed that on a spin manifold the energy-momentum tensor verifies

$$\left| \ell^{\overline{\varphi}} \right|_{\overline{g}}^2 = e^{-2u}|\ell^{\varphi}|_g^2 = e^{-2u}|\ell^{\psi}|_g^2, \tag{2.11}$$

where $\varphi = e^{-((n-1)/2)u}\psi$. We extend the result to a Spin^c manifold and get the same relation.

Refined Kato Inequalities

On a Riemannian manifold (M, g), the Kato inequality states that away from the zeros of any section φ of a Riemannian or Hermitian vector bundle E endowed with a metric connection ∇, we have

$$\left| d(|\varphi|) \right| \le |\nabla \varphi|. \tag{2.12}$$

This could be seen as follows $2|\varphi||d(|\varphi|)| = |d(|\varphi|)^2| = 2|\langle\nabla\varphi,\varphi\rangle| \leq 2|\varphi||\nabla\varphi|$. In [17], refined Kato inequalities were obtained for sections in the kernel of first order elliptic differential operators P. They are of the form $|d(|\varphi|)| \leq k_P|\nabla\varphi|$, where k_P is a constant depending on the operator P and $0 < k_P < 1$. Without the assumption that $\varphi \in \ker P$, we get away from the zero set of φ

$$\left|d|\varphi|\right| \leq |P\varphi| + k_P|\nabla\varphi|. \tag{2.13}$$

A proof of (2.13) can be found in [8, 17, 18] or [9]. In [17], the constant k_P is determined in terms of the conformal weights of the differential operator P. For the Dirac operator D and for $D - \lambda$, where $\lambda \in \mathbb{R}$, we have $k_D = k_{D-\lambda} = \sqrt{(n-1)/n}$.

3. Proof of the Hijazi Type Inequalities

First, we follow the main idea of the proof of the original Hijazi inequality in the compact case [6, 7], and its proof on spin noncompact case obtained by Grosse [9]. We choose the conformal factor with the help of an eigenspinor and we use cutoff functions near its zero set and near infinity to obtain compactly supported test functions.

Proof of Theorem 1.2. Let $\psi \in C^\infty(M,S) \cap L^2(M,S)$ be a normalized eigenspinor; that is, $D\psi = \lambda\psi$ and $\|\psi\| = 1$. Its zero set Υ is closed and lies in a closed countable union of smooth $(n-2)$-dimensional submanifolds which has locally finite $(n-2)$-dimensional Hausdorff measure [19]. We can assume without loss of generality that Υ is itself a countable union of $(n-2)$-submanifolds described above. Fix a point $p \in M$. Since M is complete, there exists a cutoff function $\eta_i : M \to [0,1]$ which is zero on $M \setminus B_{2i}(p)$ and equal 1 on $B_i(p)$, where $B_l(p)$ is the ball of center p and radius l. In between, the function is chosen such that $|\nabla\eta_i| \leq 4/i$ and $\eta_i \in C_c^\infty(M)$. While η_i cuts off ψ at infinity, we define another cutoff near the zeros of ψ. Let $\rho_{a,\epsilon}$ be the function

$$\rho_{a,\epsilon}(x) = \begin{cases} 0 & \text{for } r < a\epsilon, \\ 1 - \delta\ln\dfrac{\epsilon}{r} & \text{for } a\epsilon \leq r \leq \epsilon, \\ 1 & \text{for } \epsilon < r, \end{cases} \tag{3.1}$$

where $r = d(x,\Upsilon)$ is the distance from x to Υ. The constant $0 < a < 1$ is chosen such that $\rho_{a,\epsilon}(a\epsilon) = 0$, that is, $a = e^{-1/\delta}$. Then, $\rho_{a,\epsilon}$ is continuous, constant outside a compact set and Lipschitz. Hence, for $\varphi \in \Gamma(\Sigma M)$ the spinor $\rho_{a,\epsilon}\varphi$ is an element in $H_1^r(\Sigma M)$ for all $1 \leq r \leq \infty$. Now, consider $\Psi := \eta_i\rho_{a,\epsilon}\psi \in H_1^r(\Sigma M)$. These spinors are compactly supported on $M \setminus \Upsilon$. Furthermore, $\overline{g} = e^{2u}g = h^{4/(n-2)}g$ with $h = |\psi|^{(n-2)/(n-1)}$ is a metric on $M \setminus \Upsilon$. Setting $\overline{\Phi} := e^{-((n-1)/2)u}\overline{\Psi}$ ($\varphi = e^{-((n-1)/2)u}\psi$), (2.3), (2.10), (2.11), and the Schrödinger-Lichnerowicz formula imply

$$\begin{aligned}
\left\|\overline{\nabla}^{\ell^{\overline{\Phi}}}\overline{\Phi}\right\|_{\overline{g}}^2 &= \left\|\overline{D}\,\overline{\Phi}\right\|_{\overline{g}}^2 - \frac{1}{4}\int_{M-\Upsilon}\overline{S}\left|\overline{\Phi}\right|^2 v_{\overline{g}} - \int_{M-\Upsilon}\left|\ell^{\overline{\Phi}}\right|^2\left|\overline{\Phi}\right|^2 v_{\overline{g}} - \int_{M-\Upsilon}\left\langle\frac{i}{2}\Omega \cdot \overline{\Phi}, \overline{\Phi}\right\rangle v_{\overline{g}} \\
&\leqslant \left\|\overline{D}\,\overline{\Phi}\right\|_{\overline{g}}^2 - \frac{1}{4}\int_M\left(\overline{S}e^{2u} - c_n|\Omega|_g\right)|\Psi|^2 e^{-u}v_g - \int_M\left|\ell^{\Psi}\right|^2|\Psi|^2 e^{-u}v_g \qquad (3.2) \\
&= \left\|\overline{D}\,\overline{\Phi}\right\|_{\overline{g}}^2 - \frac{1}{4}\int_M\left(h^{-1}L^\Omega h\right)|\Psi|^2 e^{-u}v_g - \int_M\left|\ell^{\Psi}\right|^2|\Psi|^2 e^{-u}v_g,
\end{aligned}$$

where $\nabla_X^{\ell^\varphi}\varphi$ is the spinor field defined in [6] by $\nabla_X^{\ell^\varphi}\varphi := \nabla_X\varphi + \ell^\varphi(X)\cdot\varphi$, and where we used $|\overline{\Phi}|^2 v_{\overline{g}} = e^u|\Psi|^2 v_g$ and $\overline{S}e^{2u} - c_n|\Omega|_g = h^{-1}L^\Omega h$ (see [5]). Using $\overline{D}\,\overline{\varphi} = \lambda e^{-u}\overline{\varphi}$ and $\langle\nabla(\eta_i\rho_{a,\epsilon})\cdot\overline{\varphi},\overline{\varphi}\rangle \in C^\infty(M,i\mathbb{R})$, we calculate

$$\left\|\overline{D}\,\overline{\Phi}\right\|_{\overline{g}}^2 = \left\|\nabla(\eta_i\rho_{a,\epsilon})\cdot\overline{\varphi}\right\|_{\overline{g}}^2 + \lambda^2\int_M \eta_i^2\rho_{a,\epsilon}^2 e^{-(n+2)u}|\varphi|^2 v_g. \tag{3.3}$$

Inserting (3.3) and $\|\overline{\nabla}^{\ell^{\overline{\Phi}}}\overline{\Phi}\|_{\overline{g}}^2 \geqslant 0$ in the above inequality, we get

$$\left\|\nabla(\eta_i\rho_{a,\epsilon})\cdot\overline{\varphi}\right\|_{\overline{g}}^2 \geq \frac{1}{4}\int_M \left(h^{-1}L^\Omega h\right)|\Psi|^2 e^{-u} v_g + \int_M \left|\ell^\Psi\right|^2|\Psi|^2 e^{-u} v_g - \lambda^2\int_M \eta_i^2\rho_{a,\epsilon}^2|\varphi|^2 e^{-(n+2)u} v_g. \tag{3.4}$$

Moreover, we have $\|\nabla(\eta_i\rho_{a,\epsilon})\cdot\overline{\varphi}\|_{\overline{g}}^2 = \int_M |\nabla(\eta_i\rho_{a,\epsilon})\cdot\psi|^2 e^{-u} v_g$. Thus, with $e^u = |\psi|^{2/(n-1)}$ the above inequality reads

$$\begin{aligned}
\int_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2((n-2)/(n-1))} v_g &\geq \frac{1}{4}\int_M \eta_i\rho_{a,\epsilon}|\psi|^{(n-2)/(n-1)} L^\Omega\left(\eta_i\rho_{a,\epsilon}|\psi|^{(n-2)/(n-1)}\right) v_g \\
&\quad - \lambda^2\int_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2((n-2)/(n-1))} v_g \\
&\quad - \frac{n-1}{n-2}\int_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2((n-2)/(n-1))} v_g \\
&\quad + \int_M |\ell^\psi|^2|\psi|^{2((n-2)/(n-1))}\eta_i^2\rho_{a,\epsilon}^2 v_g.
\end{aligned} \tag{3.5}$$

Hence, we obtain

$$\frac{2n-3}{n-2}\int_M |\nabla(\eta_i\rho_{a,\epsilon})|^2|\psi|^{2((n-2)/(n-1))} v_g \geq \left(\frac{\mu_1}{4} + \inf_M|\ell^\varphi|^2 - \lambda^2\right)\int_M \eta_i^2\rho_{a,\epsilon}^2|\psi|^{2((n-2)/(n-1))} v_g, \tag{3.6}$$

where μ_1 is the infimum of the spectrum of the perturbed conformal Laplacian. With $|\eta_i\nabla\rho_{a,\epsilon} + \rho_{a,\epsilon}\nabla\eta_i|^2 \leq 2\eta_i^2|\nabla\rho_{a,\epsilon}|^2 + 2\rho_{a,\epsilon}^2|\nabla\eta_i|^2$, we have

$$k\int_M \left(\eta_i^2|\nabla\rho_{a,\epsilon}|^2 + \rho_{a,\epsilon}^2|\nabla\eta_i|^2\right)|\psi|^{2((n-2)/(n-1))} v_g \geq \left(\frac{\mu_1}{4} + \inf_M|\ell^\varphi|^2 - \lambda^2\right)\left\|\eta_i\rho_{a,\epsilon}|\psi|^{(n-2)/(n-1)}\right\|^2, \tag{3.7}$$

where $k = 2((2n-3)/(n-2))$. Next, we examine the limits when a goes to zero. Recall that $\Upsilon \cap \overline{B_{2i}(p)}$ is bounded, closed $(n-2)$-C^∞-rectifiable and has still locally finite $(n-2)$-dimensional Hausdorff measure. For fixed i, we estimate

$$\int_M |\nabla \rho_{a,\epsilon}|^2 \eta_i^2 |\psi|^{2((n-2)/(n-1))} v_g \leq \sup_{B_{2i}(p)} |\psi|^{2((n-2)/(n-1))} \int_{B_{2i}(p)} |\nabla \rho_{a,\epsilon}|^2 v_g. \tag{3.8}$$

Furthermore, we set $\mathcal{B}_{\epsilon,p} := \{x \in B_\epsilon \mid d(x,p) = d(x,\Upsilon)\}$ with $B_\epsilon := \{x \in M \mid d(x,\Upsilon) \leq \epsilon\}$. For ϵ sufficiently small each $\mathcal{B}_{\epsilon,p}$ is star shaped. Moreover, there is an inclusion $\mathcal{B}_{\epsilon,p} \hookrightarrow B_\epsilon(0) \subset \mathbb{R}^2$ via the normal exponential map. Then, we can calculate

$$\int_{B_\epsilon \cap B_{2i}(p)} |\nabla \rho_{a,\epsilon}|^2 v_g \leq \mathrm{vol}_{n-2}(\Upsilon \cap B_{2i}(p)) \sup_{x \in \Upsilon \cap B_{2i}(p)} \int_{\mathcal{B}_{\epsilon,x} \setminus \mathcal{B}_{ae,x}} |\nabla \rho_{a,\epsilon}|^2 v_{g'}$$

$$\leq c \, \mathrm{vol}_{n-2}(\Upsilon \cap B_{2i}(p)) \int_{B_\epsilon(0) \setminus B_{ae}(0)} |\nabla \rho_{a,\epsilon}|^2 v_{g_E} \tag{3.9}$$

$$\leq c' \int_{ae}^\epsilon \frac{\delta^2}{r} dr = -c'\delta^2 \ln a = c'\delta \longrightarrow 0 \quad \text{for } a \longrightarrow 0,$$

where vol_{n-2} denotes the $(n-2)$-dimensional volume and $g' = g_{|\mathcal{B}_{\epsilon,p}}$. The positive constants c and c' arise from $\mathrm{vol}_{n-2}(\Upsilon \cap B_{2i}(p))$ and the comparison of $v_{g'}$ with the volume element of the Euclidean metric. Furthermore, for any compact set $K \subset M$ and any positive function f it holds $\rho_{a,\epsilon}^2 f \nearrow f$, and thus by the monotone convergence theorem, we obtain when $a \to 0$,

$$\int_K \rho_{a,\epsilon}^2 f v_g \longrightarrow \int_K f v_g. \tag{3.10}$$

When applied to the functions $\rho_{a,\epsilon}^2 |\nabla \eta_i|^2 |\psi|^{2((n-2)/(n-1))}$, with $K = B_{2i}(p)$, we get

$$\int_{B_{2i}(p)} \rho_{a,\epsilon}^2 |\nabla \eta_i|^2 |\psi|^{2((n-2)/(n-1))} v_g \longrightarrow \int_{B_{2i}(p)} |\nabla \eta_i|^2 |\psi|^{2((n-2)/(n-1))} v_g, \tag{3.11}$$

as $a \to 0$ and thus,

$$k \int_M |\nabla \eta_i|^2 |\psi|^{2((n-2)/(n-1))} v_g \geq \left(\frac{\mu_1}{4} + \inf_M |\ell^\psi|^2 - \lambda^2 \right) \int_M \eta_i^2 |\psi|^{2((n-2)/(n-1))} v_g. \tag{3.12}$$

Next, we have to study the limit when $i \to \infty$: Since M has finite volume and $\|\psi\| = 1$, the Hölder inequality ensures that $\int_M |\psi|^{2((n-2)/(n-1))} v_g$ is bounded. With $|\nabla \eta_i| \leq 4/i$, we

get the result. Equality is attained if and only if $\|\overline{\nabla}^{\ell\overline{\Phi}}\overline{\Phi}\|^2_{\overline{g}} \to 0$ for $i \to \infty$, $a \to 0$ and $\Omega \cdot \psi = i(c_n/2)|\Omega|_g\psi$. But we have

$$
0 \longleftarrow \left\|\overline{\nabla}^{\ell\overline{\Phi}}\overline{\Phi}\right\|^2_{\overline{g}} = \left\|\eta_i\rho_{a,\epsilon}\overline{\nabla}^{\ell\overline{\Phi}}\overline{\varphi} + \nabla(\eta_i\rho_{a,\epsilon}) \cdot \overline{\varphi}\right\|_{\overline{g}} \geq \left\|\eta_i\rho_{a,\epsilon}\overline{\nabla}^{\ell\overline{\varphi}}\overline{\varphi}\right\|_{\overline{g}} - \left\|\nabla(\eta_i\rho_{a,\epsilon}) \cdot \overline{\varphi}\right\|_{\overline{g}}.
$$

(3.13)

Since $\|\nabla(\eta_i\rho_{a,\epsilon}) \cdot \overline{\varphi}\|_{\overline{g}} \to 0$, we conclude that $\overline{\nabla}^{\ell\overline{\varphi}}\overline{\varphi}$ has to vanish on $M \setminus \Upsilon$.   □

Remark 3.1. By the Cauchy-Schwarz inequality, we have

$$
|\ell^{\psi}|^2 \geqslant \frac{(\mathrm{tr}(\ell^{\psi}))^2}{n} = \frac{\lambda^2}{n},
$$

(3.14)

where tr denotes the trace of ℓ^{ψ}. Hence, the Hijazi type (1.6) can be derived. Equality is achieved if and only if the eigenspinor associated with the first eigenvalue λ_1 is a Spin^c Killing spinor. In fact, if equality holds then $\lambda^2 = (n/4(n-1))\mu_1 = (1/4)\mu_1+|\ell^{\psi}|^2$ and equality in (3.14) is satisfied. Hence, it is easy to check that

$$
T^{\psi}(e_i, e_j) = 0 \quad \text{for } i \neq j, \quad T^{\psi}(e_i, e_i) = \pm\frac{\lambda}{n}.
$$

(3.15)

Finally, $\ell^{\psi}(X) = \pm(\lambda/n)X$ and $\ell^{\overline{\varphi}}(X) = e^{-u}\ell^{\psi}(X) = \pm(\lambda/n)e^{-u}X$. By (1.8) we get that $\overline{\varphi}$ is a generalized Killing Spin^c spinor and hence a Killing Spin^c spinor for $n \geqslant 4$ [1, Theorem 1.1]. The function e^{-u} is then constant and ψ is a Killing Spin^c spinor. For $n = 3$, we follow the same proof as in [1]. First we suppose that $\lambda_1 \neq 0$, because if $\lambda_1 = 0$, the result is trivial. We consider the Killing vector $\overline{\xi}$ defined by

$$
i\overline{g}\left(\overline{\xi}, X\right) = \left\langle X \cdot \overline{\varphi}, \overline{\varphi}\right\rangle_{\overline{g}} \quad \text{for every } X \in \Gamma(TM).
$$

(3.16)

In [1], it is shown that $d\overline{\xi} = 2\lambda_1 e^{-u}(*\overline{\xi}), \nabla|\overline{\xi}|^2 = 0$ and $\overline{\xi} \cdot \overline{\varphi} = i|\overline{\xi}|^2\overline{\varphi}$, where $*$ is the Hodge operator defined on differential forms. Since $*\overline{\xi}(\overline{\xi}, \cdot) = 0$, the 2-form Ω can be written $\Omega = F\overline{\xi} + \overline{\xi} \wedge \alpha$, where α is a real 1-form and F a function. We have [1]

$$
\Omega\left(\overline{\xi}, \cdot\right) = \left|\overline{\xi}\right|^2 \alpha(\cdot) = -4\lambda_1 d(e^{-u})(\cdot),
$$

$$
\Omega \cdot \overline{\varphi} = -iF\overline{\varphi} - i\left|\overline{\xi}\right|^2 \alpha \cdot \overline{\varphi}.
$$

(3.17)

But equality in (1.1) is achieved so $\Omega \cdot \overline{\varphi} = i(c_n/2)|\Omega|_{\overline{g}}\overline{\varphi}$, which implies that $\Omega \cdot \overline{\varphi}$ is collinear to $\overline{\varphi}$ and hence $\alpha \cdot \overline{\varphi}$ is collinear to $\overline{\varphi}$. Moreover, $d(e^{-u})(\overline{\xi}) = -(1/4\lambda_1)\Omega(\overline{\xi}, \overline{\xi}) = 0$ so $\alpha(\overline{\xi}) = 0$. It is easy to check that $\langle \alpha \cdot \overline{\varphi}, \overline{\varphi}\rangle_{\overline{g}} = 0$ which gives $\alpha \cdot \overline{\varphi} \perp \overline{\varphi}$. Because of $\alpha \cdot \overline{\varphi} \perp \overline{\varphi}$ and $\alpha \cdot \overline{\varphi}$ is collinear to $\overline{\varphi}$, we have $\alpha \cdot \overline{\varphi} = 0$ and finally $\alpha = 0$. Using (3.17), we obtain $d(e^{-u}) = 0$, that is, e^{-u} is constant, hence $\overline{\varphi}$ is a Killing Spin^c spinor and finally ψ is also a Spin^c Killing spinor.

Proof of Theorem 1.1. The proof of Theorem 1.1 is similar to Theorem 1.2. It suffices to take $\overline{g} = g$, that is, $e^u = 1$. The Friedrich type (1.1) is obtained from the Hijazi type (1.6).   □

Next, we want to prove Theorem 1.3 using the refined Kato inequality.

Proof of Theorem 1.3. We may assume $\mathrm{vol}(M, g) = 1$. If λ is in the essential spectrum of D, then 0 is in the essential spectrum of $D - \lambda$ and of $(D - \lambda)^2$. Thus, there is a sequence $\psi_i \in \Gamma_c(\Sigma M)$ such that $\|(D - \lambda)^2 \psi_i\| \to 0$ and $\|(D - \lambda)\psi_i\| \to 0$, while $\|\psi_i\| = 1$. We may assume that $|\psi_i| \in C_c^\infty(M)$. That can always be achieved by a small perturbation. Now let, $1/2 \leq \beta \leq 1$. Then $|\psi_i|^\beta \in H_1^2(M)$. First, we will show that the sequence $\|d(|\psi_i|^\beta)\|$ is bounded: by the Cauchy-Schwarz inequality, we have

$$\left| \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} \left\langle (D - \lambda)^2 \psi_i, \psi_i \right\rangle \mathrm{v}_g \right| \leq \left\| |\psi_i|^{2\beta-1} \right\|_{\{|\psi_i| \neq 0\}} \left\| (D - \lambda)^2 \psi_i \right\|$$

$$\leq \|\psi_i\|^{2\beta-1} \left\| (D - \lambda)^2 \psi_i \right\| \tag{3.18}$$

$$= \left\| (D - \lambda)^2 \psi_i \right\|.$$

Using (2.3) and the Schrödinger-Lichnerowicz type (2.7), we obtain

$$\left\| (D - \lambda)^2 \psi_i \right\| \geq \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} \left| \nabla^\lambda \psi_i \right|^2 \mathrm{v}_g + 2(\beta - 1) \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-3} \left\langle d|\psi_i| \cdot \psi_i, \nabla^\lambda \psi_i \right\rangle \mathrm{v}_g$$

$$+ \int \left(\frac{S}{4} - \frac{c_n}{4}|\Omega| - \frac{n-1}{n}\lambda^2 \right) |\psi_i|^{2\beta} \mathrm{v}_g - 2\frac{n-1}{n}\lambda \left\| |\psi_i|^{2\beta-1} \right\|_{\{|\psi_i| \neq 0\}} \left\| (D - \lambda)\psi_i \right\|. \tag{3.19}$$

The Cauchy-Schwarz inequality and the refined Kato (2.12) for the connection ∇^λ imply

$$\int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} \left| \nabla^\lambda \psi_i \right|^2 \mathrm{v}_g + 2(\beta - 1) \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-3} \left\langle d|\psi_i| \cdot \psi_i, \nabla^\lambda \psi_i \right\rangle \mathrm{v}_g$$

$$\geq (2\beta - 1) \int_{|\psi_i| \neq 0} |\psi_i|^{2\beta-2} |d(|\psi_i|)|^2 \mathrm{v}_g = (2\beta - 1)\frac{1}{\beta^2} \int_{|\psi_i| \neq 0} \left| d\left(|\psi_i|^\beta \right) \right|^2 \mathrm{v}_g. \tag{3.20}$$

Hence, we have

$$\left\| (D - \lambda)^2 \psi_i \right\| \geq (2\beta - 1)\frac{1}{\beta^2} \int_{|\psi_i| \neq 0} \left| d\left(|\psi_i|^\beta \right) \right|^2 \mathrm{v}_g + \int \left(\frac{S}{4} - \frac{c_n}{4}|\Omega| - \frac{n-1}{n}\lambda^2 \right) |\psi_i|^{2\beta} \mathrm{v}_g$$

$$- 2\frac{n-1}{n}\lambda \left\| (D - \lambda)\psi_i \right\|. \tag{3.21}$$

Since $S-c_n|\Omega|$ is bounded from below, $\int(S-c_n|\Omega|)|\psi_i|^{2\beta}v_g \geq \inf(S-c_n|\Omega|)\ \|\psi_i\|_{2\beta}^{2\beta} \geq \min\{\inf(S-c_n|\Omega|), 0\}$ is also bounded. Thus, with $\|(D-\lambda)\psi_i\| \to 0$, we see that $\|d|\psi_i|^{\beta}\|$ is also bounded. Next, we fix $\alpha = (n-2)/(n-1)$ and obtain

$$
\begin{aligned}
\frac{\mu_1}{4} - \frac{n-1}{n}\lambda^2 &\leq \left(\frac{\mu_1}{4} - \frac{n-1}{n}\lambda^2\right)\||\psi_i|^\alpha\|^2 \\
&\leq \frac{1}{4}\int |\psi_i|^\alpha L^\Omega |\psi_i|^\alpha v_g - \frac{n-1}{n}\lambda^2 \||\psi_i|^\alpha\|^2 \\
&= \int |\psi_i|^{2((n-2)/(n-1))-2}\left[\frac{n}{n-1}|d(|\psi_i|)|^2 + \frac{1}{2}d^*d\left(|\psi_i|^2\right)\right. \\
&\qquad\left. + \left(\frac{S}{4} - \frac{c_n}{4}|\Omega| - \frac{n-1}{n}\lambda^2\right)|\psi_i|^2\right]v_g,
\end{aligned}
\tag{3.22}
$$

where we used the definition of μ_1 as the infimum of the spectrum of L^Ω and $|\psi_i|^\alpha d^*d(|\psi_i|^\alpha) = (\alpha/2)|\psi_i|^{2\alpha-2}d^*d(|\psi_i|^2) - \alpha(\alpha-2)|\psi_i|^{2\alpha-2}|d(|\psi_i|)|^2$. Next, using the following:

$$
\begin{aligned}
\frac{1}{2}d^*d\langle\psi_i,\psi_i\rangle &\leqslant \left\langle D^2\psi_i,\psi_i\right\rangle - \frac{1}{4}(S-c_n|\Omega|)|\psi_i|^2 - |\nabla\psi_i|^2, \\
\left|\nabla^\lambda\psi_i\right|^2 &= |\nabla\psi_i|^2 - 2\frac{\lambda}{n}\mathrm{Re}\langle(D-\lambda)\psi_i,\psi_i\rangle - \frac{\lambda^2}{n}|\psi_i|^2,
\end{aligned}
\tag{3.23}
$$

we have

$$
\begin{aligned}
\frac{\mu_1}{4} - \frac{n-1}{n}\lambda^2 &\leq \int |\psi_i|^{2((n-2)/(n-1))-2}\left(\frac{n}{n-1}|d(|\psi_i|)|^2 - \left|\nabla^\lambda\psi_i\right|^2\right)v_g \\
&\quad + \int |\psi_i|^{2((n-2)/(n-1))-2}\left\langle(D-\lambda)^2\psi_i,\psi_i\right\rangle v_g \\
&\quad + \int 2\left(1-\frac{1}{n}\right)\lambda|\psi_i|^{2((n-2)/(n-1))-2}\mathrm{Re}\langle(D-\lambda)\psi_i,\psi_i\rangle v_g.
\end{aligned}
\tag{3.24}
$$

The limit of the last two summands vanish since

$$
\begin{aligned}
\left|\int |\psi_i|^{2((n-2)/(n-1))-2}\left\langle(D-\lambda)^2\psi_i,\psi_i\right\rangle v_g\right| &\leq \left\|(D-\lambda)^2\psi_i\right\|\left\||\psi_i|^{(n-3)/(n-1)}\right\| \longrightarrow 0, \\
\left|\int |\psi_i|^{2((n-2)/(n-1))-2}\mathrm{Re}\langle(D-\lambda)\psi_i,\psi_i\rangle v_g\right| &\leq \|(D-\lambda)\psi_i\|\left\||\psi_i|^{(n-3)/(n-1)}\right\| \longrightarrow 0.
\end{aligned}
\tag{3.25}
$$

For the other summand, we use the Kato type (2.13)

$$
|d(|\psi|)| \leq |(D-\lambda)\psi| + k\left|\nabla^\lambda\psi\right|,
\tag{3.26}
$$

which holds outside the zero set of ψ, and where $k = \sqrt{(n-1)/n}$. Thus, for $n \geq 5$, we can estimate

$$\int |\psi_i|^{2\alpha-2}\left(\frac{n}{n-1}|d(|\psi_i|)|^2 - \left|\nabla^\lambda \psi_i\right|^2\right)v_g$$

$$= \int |\psi_i|^{2\alpha-2}\left(k^{-1}|d(|\psi_i|)| - \left|\nabla^\lambda \psi_i\right|\right)\left(k^{-1}|d(|\psi_i|)| + \left|\nabla^\lambda \psi_i\right|\right)v_g$$

$$\leq k^{-1}\int_{\{|d(|\psi_i|)|\geq k|\nabla^\lambda \psi_i|\}} |\psi_i|^{2\alpha-2}\left|(D-\lambda)\psi_i\right|\left(k^{-1}|d(|\psi_i|)| + \left|\nabla^\lambda \psi_i\right|\right)v_g \qquad (3.27)$$

$$\leq 2k^{-2}\int_{\{|d(|\psi_i|)|\geq k|\nabla^\lambda \psi_i|\}} |\psi_i|^{2\alpha-2}\left|(D-\lambda)\psi_i\right||d(|\psi_i|)|v_g$$

$$\leq 2k^{-2}\frac{n-1}{n-3}\left\|(D-\lambda)\psi_i\right\|\left\|d\left(|\psi_i|^{(n-3)/(n-1)}\right)\right\|.$$

For $n \geq 5$, we have $1 \geq (n-3)/(n-1) \geq 1/2$ and, thus, $\||d|\psi_i|^{(n-3)/(n-1)}\|$ is bounded. Together with $\|(D-\lambda)\psi_i\| \to 0$, we obtain the following: for all $\epsilon > 0$, there is an i_0 such that for all $i \geq i_0$, we have

$$\int |\psi_i|^{2((n-2)/(n-1))-2}\left(\frac{n}{n-1}|d|\psi_i||^2 - \left|\nabla^\lambda \psi_i\right|^2\right)v_g \leq \epsilon. \qquad (3.28)$$

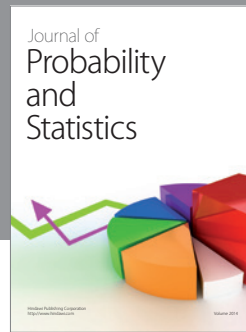Hence, we have $\mu_1/4 \leq ((n-1)/n)\lambda^2$. $\qquad \square$

Acknowledgment

References

[1] M. Herzlich and A. Moroianu, "Generalized Killing spinors and conformal eigenvalue estimates for Spin^c manifolds," *Annals of Global Analysis and Geometry*, vol. 17, no. 4, pp. 341–370, 1999.

[2] Th. Friedrich, "Der erste eigenwert des Dirac-operators einer kompakten, riemannschen mannig-faltigkeit nichtnegativer skalarkrümmung," *Mathematische Nachrichten*, vol. 97, pp. 117–146, 1980.

[3] T. Friedrich, *Dirac Operator's in Riemannian Geometry*, vol. 25 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2000.

[4] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry*, vol. 38 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 1989.

[5] R. Nakad, "Lower bounds for the eigenvalues of the Dirac operator on Spin^c manifolds," *Journal of Geometry and Physics*, vol. 60, no. 10, pp. 1634–1642, 2010.

[6] O. Hijazi, "Lower bounds for the eigenvalues of the Dirac operator," *Journal of Geometry and Physics*, vol. 16, no. 1, pp. 27–38, 1995.

[7] O. Hijazi, "A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors," *Communications in Mathematical Physics*, vol. 104, no. 1, pp. 151–162, 1986.

[8] N. Grosse, *On a spin conformal invariant on open manifolds*, Ph.D. thesis, Universität Leipzig, Leipzig, Germany, 2008.

[9] N. Grosse, "The Hijazi inequality inconformally parabolic manifolds," *Differential Geometry*. In press. http://arxiv.org/abs/0804.3878.

[10] C. Bär, "Spectral bounds for Dirac operators on open manifolds," *Annals of Global Analysis and Geometry*, vol. 36, no. 1, pp. 67–79, 2009.

[11] C. Bär, P. Gauduchon, and A. Moroianu, "Generalized cylinders in semi-Riemannian and Spin geometry," *Mathematische Zeitschrift*, vol. 249, no. 3, pp. 545–580, 2005.

[12] E. C. Kim and T. Friedrich, "The Einstein-Dirac equation on Riemannian spin manifolds," *Journal of Geometry and Physics*, vol. 33, no. 1-2, pp. 128–172, 2000.

[13] R. Nakad, "The energy-momentum tensor on spin^c manifolds," to appear in *International Journal of Geometric Methods in Modern Physics*.

[14] B. Morel, *Tenseur d'impulsion d'énergie et géométrie spinorielle extrinsèque*, Ph.D. thesis, Institut Èlie Cartan, Nancy, France, 2002.

[15] T. Friedrich, "On the spinor representation of surfaces in Euclidean 3-space," *Journal of Geometry and Physics*, vol. 28, no. 1-2, pp. 143–157, 1998.

[16] R. Nakad and J. Roth, "Isometric immersions of hypersurfaces into Spin^c manifolds via spinors," In preparation.

[17] D. M. J. Calderbank, P. Gauduchon, and M. Herzlich, "Refined Kato inequalities and conformal weights in Riemannian geometry," *Journal of Functional Analysis*, vol. 173, no. 1, pp. 214–255, 2000.

[18] T. Branson, "Kato constants in Riemannian geometry," *Mathematical Research Letters*, vol. 7, no. 2-3, pp. 245–262, 2000.

[19] C. Bär, "Zero sets of solutions to semilinear elliptic systems of first order," *Inventiones Mathematicae*, vol. 138, no. 1, pp. 183–202, 1999.