Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2012, Article ID 309398, 12 pages
doi:10.1155/2012/309398

Research Article

Resonances for Perturbed Periodic Schrödinger Operator

Mouez Dimassi

Institut de Mathématiques de Bordeaux, Université Bordeaux 1, 351, Cours de la Libération, 33405 Talence, France

Correspondence should be addressed to Mouez Dimassi, dimassi@math.univ-paris13.fr

Received 29 September 2011; Accepted 27 November 2011

Academic Editor: Ali Mostafazadeh

In the semiclassical regime, we obtain a lower bound for the counting function of resonances corresponding to the perturbed periodic Schrödinger operator $P(h) = -\Delta + V(x) + W(hx)$. Here V is a periodic potential, W a decreasing perturbation and h a small positive constant.

1. Introduction

The quantum dynamics of a Bloch electron in a crystal subject to external electric field, which varies slowly on the scale of the crystal lattice, is governed by the Schrödinger equation

$$P(h) = -\Delta + V(x) + W(hx). \tag{1.1}$$

Here V is periodic with respect to the crystal lattice $\Gamma \subset \mathbb{R}^n$, and it models the electric potential generated by the lattice of atoms in the crystal. The potential W is a decreasing perturbation and h a small positive constant.

There has been a growing interest in the rigorous study of the spectral properties of Bloch electrons in the presence of slowly varying external perturbations (see [1–11]).

Since the work of Peierls [10] and Slater [11], it is well known that, if h is sufficiently small, then solutions of $P(h)$ are governed by the "semiclassical" Hamiltonian

$$H(y, \eta) = \lambda(\eta + A(y)) + V(y). \tag{1.2}$$

Here $\lambda(k)$ is one of the "band functions" describing the Floquet spectrum of the unperturbed Hamiltonian

$$P_0 = -\Delta_x + V(x). \tag{1.3}$$

One argues that for suitable wave packets, which are spread over many lattice spacings, the main effect of a periodic potential on the electron dynamics consists in changing the dispersion relation from the free kinetic energy $E_{\text{free}}(k) = |k|^2$ to the modified kinetic energy $\lambda(k)$ given by the Bloch band.

The problem of resonances has been examined in [12] for the one-dimensional case and in [13] for the general case. In particular, a similar reduction to (1.2) for resonances has been obtained in [13].

This paper continues our previous works [13, 14] on the resonances and the eigenvalues counting function for $P(h)$. In [14], Dimassi and Zerzeri obtained a local trace formula for resonances. As a consequence, they obtained an upper bound for the number of resonances of $P(h)$ in any h-independent complex neighborhood of some energy E. The purpose of this paper is to give a lower bound for the number of resonances of $P(h)$.

In the case where $V = 0$, it is known that, for $0 < E$ in the analytic singular support (from now on sing supp_a for short) of the distribution $d\rho_0 * \mu$, then the operator $P(h) = -\Delta + W(hx)$ has at least $C_\Omega h^{-n}$ resonances in any h-independent complex neighborhood Ω of E (see, e.g., [15]). Here

$$\mu(t) = \int_{\{x\in\mathbb{R}^n;\, W(x)>t\}} dx,$$
$$\rho_0(t) = (2\pi)^{-n}\operatorname{vol}(B(0,1))(\max(t,0))^{n/2}. \tag{1.4}$$

Using the explicit formula of ρ_0 we see that the analytic singular support of the distributions μ and $d\rho_0 * \mu$ coincide.

In the case where $V \neq 0$ the situation is different. Following Theorem 1.6 in [14] and Lemma 2.1 of the next section, we have to change ρ_0 by

$$\rho(\lambda) := \frac{1}{(2\pi)^n}\sum_{j\geq 1}\int_{\{k\in E^*;\, \lambda_j(k)\leq\lambda\}} dk, \tag{1.5}$$

which is the integrated density of states corresponding to the nonperturbed Hamiltonian P_0 (see Section 2).

If $\lambda_j(k)$ is a simple eigenvalue near some point e_0, then $\lambda_j(k)$ is a smooth function, and if $e_0 = \lambda_j(k)$ is a critical value, we expect in general that e_0 will belong to the analytic singular support of $\rho(\lambda)$. In particular, we expect that near every point $e \in e_0 + \text{sing supp}_a(\mu)$ there exists at least Ch^{-n}, $C > 0$, resonances.

Multiple eigenvalues ($\lambda_j(k_0) = \lambda_{j+1}(k_0) = e_0$) can also give rise to singularities of $\rho(\lambda)$ and then lead to the existence of resonances near $e_0 + \text{sing supp}_a(\mu)$.

The purpose of this paper is to describe all these situations. Some results of this paper are announced without proofs in [16].

The paper is organized as follows: in the next section, we introduce some notations and state some technical lemmas. In Section 3 we give an upper bound for resonances near singularities of the density of states measure ρ generated by a band crossing. In Section 4 we give an upper bound for resonances near the edge of bands.

2. Preliminaries

Let $\Gamma = \oplus_{i=1}^{n}\mathbb{Z}a_i$ be the lattice generated by the basis $a_1, a_2, \ldots, a_n$, $a_i \in \mathbb{R}^n$. The dual lattice Γ^* is defined as the lattice generated by the dual basis $\{a_1^*, a_2^*, \ldots, a_n^*\}$ determined by $a_i \cdot a_j^* = 2\pi\delta_{ij}$, $i, j = 1, 2, \ldots, n$. Let E be a fundamental domain for Γ, and let E^* be a fundamental domain for Γ^*. If we identify opposite edges of E (resp., E^*), then it becomes a flat torus denoted by $\mathbb{T} = \mathbb{R}^n/\Gamma$ (resp., $\mathbb{T}^* = \mathbb{R}^n/\Gamma^*$).

Let V be a real valued potential, C^∞ and Γ-periodic. For k in $\mathbb{R}^n$, we define

$$P_0(k) = (D_x + k)^2 + V(x) \tag{2.1}$$

as an unbounded operator on $L^2(\mathbb{T})$ with domain $H^2(\mathbb{T})$. The Hamiltonian $P_0(k)$ is semi-bounded and self-adjoint. Since the resolvent of $(D_x+k)^2$ is compact, the resolvent of $P_0(k)$ is also compact, and therefore $P_0(k)$ has a complete set of (normalized) eigenfunctions $\Phi_n(\cdot, k) \in H^2(\mathbb{T}^*)$, $n \in \mathbb{N}$, called Bloch functions. The corresponding eigenvalues accumulate at infinity, and we enumerate them according to their multiplicities:

$$\lambda_1(k) \le \lambda_2(k) \le \cdots . \tag{2.2}$$

Since $e^{-ix\gamma^*}H_0(k)e^{ix\gamma^*} = H_0(\gamma^* + k)$, the band function $\lambda_n(k)$ is periodic with respect to Γ^*. The function $\lambda_n(k)$ is called a band function, and the closed intervals $\Lambda_n := \lambda_n(\mathbb{T}^*)$ are called bands.

Standard perturbation theory shows that $\lambda_n(k)$ is a continuous function of k and is real analytic in a neighborhood of any k such that

$$\lambda_{n-1}(k) < \lambda_n(k) < \lambda_{n+1}(k). \tag{2.3}$$

We fix λ in the spectrum of the unperturbed operator P_0. We make the following hypothesis on the spectrum of the unperturbed Schrödinger operator.

(H1) For all k_0 with $\lambda_i(k_0) = \lambda$, the eigenvalue $\lambda_i(k_0)$ is simple and $d_k\lambda_i(k_0) \neq 0$.

Now, let us recall some well-known facts about the density of states associated with P_0. The density of states measure ρ is defined as follows:

$$\rho(\lambda) := \frac{1}{(2\pi)^n}\sum_{j\ge 1}\int_{\{k\in E^*;\ \lambda_j(k)\le\lambda\}} dk, \tag{2.4}$$

where E^* is a fundamental domain of $\mathbb{R}^n/\Gamma^*$. Since the spectrum of P_0 is absolutely continuous, the measure ρ is absolutely continuous with respect to the Lebesgue measure $d\lambda$. Thus, the density of states of P_0, $\partial\rho/\partial\lambda$ is locally integrable.

We now consider the perturbed periodic Schrödinger operator:

$$P(h) := P_0 + W(hx), \tag{2.5}$$

where $W \in C^\infty(\mathbb{R}^n;\mathbb{R})$. We assume that there exist positive constants a and C such that W extends analytically to $\Gamma(a) := \{z \in \mathbb{C}^n;\ |\Im(z)| \le a\langle \Re(z)\rangle\}$ and

$$|W(z)| \le C\langle z\rangle^{-\tilde{n}}, \quad \text{uniformly on } z \in \Gamma(a),\ \tilde{n} > n, \tag{2.6}$$

where $\langle z\rangle = (1+|z|^2)^{1/2}$. Here $\Re(z)$,$\Im(z)$ denote, respectively, the real part and the imaginary part of z.

This assumption allows us to define the resonances of $P(h)$ by the spectral deformation method (see [17]). We follow essentially the presentation of [13].

Let $v \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be Γ^*-periodic. For $t \in \mathbb{R}$, we introduce the spectral deformation family $\mathcal{U}_t$ defined by for all $u \in \mathcal{S}$:

$$\mathcal{U}_t u(r) := \mathcal{F}_h^{-1}\left\{\left(J_t^{1/2}(\mathcal{F}_h u)(v_t(k))\right)\right\}(r), \quad \forall x \in \mathbb{R}^n, \tag{2.7}$$

where $v_t(k) = k - tv(k)$ and $J_t(k)$ its Jacobian. Here $\mathcal{F}_h$ is the semiclassical Fourier transform:

$$[\mathcal{F}_h u](\xi) := \int_{\mathbb{R}^n} e^{-(i/h)x\xi} u(x)dx, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \tag{2.8}$$

Consider, for $t \in \mathbb{R}$, the family of unitarily equivalent operators

$$P_1(t,h) := \mathcal{U}_t P_1(h)\mathcal{U}_t^{-1}. \tag{2.9}$$

It was established in [13, Proposition 2.8] that $P_1(t,h)$ extends to an analytic type-$\mathcal{A}$ family of operators on $D(t_0) := \{t \in \mathbb{C};\ |t| < t_0\}$ with domain $H^2(\mathbb{R}^n)$. Moreover, under the assumptions (H1) and (2.6), there exists a neighborhood $\widetilde{\Omega}$ of z_0 and a small positive constant η such that, for $t \in D(t_0)$ with $\Im t > 0$, the spectrum of $P_1(t,h)$ in $\Omega_t := \{z \in \widetilde{\Omega}; \Im z > -\eta \Im t\}$ consists of discrete eigenvalues of finite multiplicities that lie in the lower half plane (see [13, formula (4.9)]). These eigenvalues are t-independent under small variations of $\Im t > 0$ and are called resonances. We will denote the set of resonances by $\operatorname{Res}(P(h))$.

For $f \in C_0^\infty(\mathbb{R})$, we set

$$\langle \mu, f\rangle = \int [f(W(x)) - f(0)]dx, \tag{2.10}$$

$$\langle \omega, f\rangle = \frac{1}{(2\pi)^n}\sum_{j\ge 1}\int_{E^*}\int_{\mathbf{R}_x^n}[f(W(x)+\lambda_j(k)) - f(\lambda_j(k))]dk\,dx, \tag{2.11}$$

For $E > 0$, let

$$\nu_+(E) := \int_{\{x\in\mathbb{R}^n;\ W(x)\geq E\}} dx. \tag{2.12}$$

Similarly, for $E < 0$, we set

$$\nu_-(E) := \int_{\{x\in\mathbb{R}^n;\ W(x)\leq E\}} dx. \tag{2.13}$$

Clearly, $\nu_+(E)$ (resp., $\nu_-(E)$) is a decreasing function of E (resp., an increasing function of E) and

$$\mu_{|_{\mathbb{R}^\pm}} = -\frac{d}{dE}\nu_\pm(E). \tag{2.14}$$

Lemma 2.1. *The distributions ω and μ are real valued of order ≤ 1. Moreover, in $\mathfrak{D}'(\mathbb{R})$, one has*

$$\omega = d\rho * \mu. \tag{2.15}$$

Proof. Applying Taylor's formula to the right-hand side of (2.10), we obtain

$$\left|\langle \mu, f\rangle\right| \leq \sup\left|f'\right| \int |W(x)| dx, \tag{2.16}$$

which together with (2.6) imply that μ is a distribution of order ≤ 1, with

$$\operatorname{supp}\ \mu \subset \left[\inf\ W(x), \sup\ W(x)\right]. \tag{2.17}$$

Consequently, $d\rho * \mu$ is well defined in $\mathfrak{D}'(\mathbb{R})$ and for all $f \in C_0^\infty(\mathbf{R})$, we have

$$\begin{aligned}
\langle d\rho * \mu, f\rangle &= \langle d\rho(t), \langle \mu, f(\cdot + t)\rangle\rangle \\
&= -\left\langle \rho(t), \int \left[f'(W(x) + t) - f'(t)\right] dx \right\rangle \\
&= -\frac{1}{(2\pi)^n} \sum_j \int_{E^*} \int_{\lambda_j(k)}^{\infty} \int_{\mathbf{R}_x^n} \left[f'(W(x) + t) - f'(t)\right] dx\, dt\, dk \\
&= \frac{1}{(2\pi)^n} \sum_j \int_{E^*} \int_{\mathbf{R}_x^n} \left[f\left(W(x) + \lambda_j(k)\right) - f\left(\lambda_j(k)\right)\right] dx\, dk \\
&= \langle \omega, f\rangle.
\end{aligned} \tag{2.18}$$

This ends the proof of the lemma. □

Let Ω be an open-bounded set in $\mathbb{R}^n$, and let $\widetilde{\Omega}$ be a complex neighborhood of Ω. Let $x \longrightarrow \varphi(x)$ be analytic on $\widetilde{\Omega}$ and real valued for all x in Ω. Let us introduce the real function

$$I(e) := \int_{\{x\in\Omega;\ \varphi(x)\le e\}} dx. \tag{2.19}$$

For $e \in \varphi(\Omega)$, we set

$$\Sigma(e) := \{x \in \Omega;\ \varphi(x) = e\}. \tag{2.20}$$

Lemma 2.2. *Let $e_0 \in \varphi(\Omega)$, and let $\Sigma(e)$, $I(e)$ be as above. One assumes that*

(i) $\nabla\varphi(x) \neq 0$ *for all* $x \in \Sigma(e_0)$,

(ii) $\partial\Omega \cap \Sigma(e_0) = \emptyset$.

Then the function

$$I(e) := \int_{\{x\in\Omega;\ \varphi(x)\le e\}} dx \tag{2.21}$$

is analytic near e_0.

Proof. Let ϵ be a small positive constant such that $\nabla\varphi(x) \neq 0$ when $x \in \Sigma_\epsilon(e_0) := \varphi^{-1}(]e_0 - \epsilon, e_0 + \epsilon[)$. Without any loss of generality we may assume that $\partial_{x_1}\varphi \neq 0$ for all $x \in \Sigma_\epsilon(e_0)$. By the change of variable $H : x \longrightarrow (\varphi(x), x_2, \dots, x_n)$ we have

$$\int_{\{x\in\Sigma_\epsilon(e_0):;\ \varphi(x)\le e\}} dx = \int_{\{x\in H(\Sigma_\epsilon(e_0):);\ x_1\le e\}} \mathrm{Jac}\Big(H^{-1}(x)\Big) dx. \tag{2.22}$$

Clearly the right-hand side of the above equality is analytic. Combining this with the fact that $\int_{\{x\in\Omega\setminus\Sigma_\epsilon(e_0):;\ \varphi(x)\le e\}} dx$ is constant for e near e_0 we get the lemma. □

Lemma 2.3. *If φ has a nondegenerate extremum at x_0 with $\varphi(x_0) = e_0$ and if $\nabla\varphi(x) \neq 0$ for all $x \in \Sigma_{e_0} \setminus \{x_0\}$, then*

$$I(e) = f(e - e_0) + H(\pm(e - e_0))\, g\left(\sqrt{\pm(e - e_0)}\right), \tag{2.23}$$

where f and g are analytic near zero and

$$g(t) \sim_{t\to 0} \frac{\mathrm{vol}(S^{n-1})}{n\sqrt{\det\varphi''(x_0)}} 2^{n/2} t^n. \tag{2.24}$$

Here $H(t)$ is the Heaviside function and $+$ $(-)$ corresponds to a minimum (maximum, resp.).

Proof. Here we only give a sketch of the proof. For the details we refer to [18]. Without any loss of generality, we only consider the case of minimum. By Morse lemma there exist a neighborhood U of x_0, $\epsilon > 0$ and a local analytic diffeomorphism $D : \Omega \to B(0,\epsilon)$ such that

$$\int_{\{x\in U;\, \varphi(x)\le e\}} dx = \int_{\{x\in B(0,\epsilon);\, |x|^2\le e-e_0\}\{\ \}} \mathrm{Jac}\Big(D^{-1}(x)\Big)dx. \tag{2.25}$$

By a simple calculus we show, using polar coordinates, that the integral of the r.h.s. is equal to $H(e-e_0)g(\sqrt{e-e_0})$. On the other hand, since $\nabla\varphi(x)\neq 0$ for $x\in\Sigma_{e_0}\setminus\{x_0\}$, it follows from Lemma 2.2 that

$$\int_{\{x\in O\setminus U;\, \varphi(x)\le e\}} dx \tag{2.26}$$

is analytic near e_0. This ends the proof of the lemma. □

3. Lower-Bound Near Singularities due to Band Crossing

Here we are interested in the C^∞ singular support (which will be denoted by sing supp). Recall that $x_0 \notin$ sing supp μ if and only if μ is C^∞ near x_0. The case of analytic singular support can be treated similarly.

In this section we study resonances near singularities of $\rho(\lambda)$ generated by a band crossing. We will only consider the two-dimensional case. With similar assumptions, one can treat the case $n\ge 2$.

We assume that $\lambda_j(k)$ is double eigenvalues $\lambda_{j-1}(k_0) < \lambda_j(k_0) = e_0 = \lambda_{j+1}(k_0) < \lambda_{j+2}(k_0)$ and that for all $k\neq k_0$ such that $\lambda_i(k) = e_0$, $\lambda_i(k)$ is simple and $\nabla\lambda_i(k)\neq 0$.

Since $P_0(k)$ is analytic in k, this implies that, for $|k-k_0|\le\delta$ (with δ small enough), the span $V(k)$, of the eigenvectors of $P_0(k)$ corresponding to eigenvalues in the set $\{e; |e-e_0|\le\delta\}$, has a basis $\varphi_j(x,k)$, $\varphi_{j+1}(x,k)$, which is orthonormal and real analytic in k. The restriction of $P_0(k)$ to $V(k)$ has the matrix

$$\begin{pmatrix} \alpha(k) & \overline{b(k)} \\ b(k) & \beta(k) \end{pmatrix}, \tag{3.1}$$

which can be written

$$\begin{pmatrix} a(k)+c(k) & b_1(k)-ib_2(k) \\ b_1(k)+ib_2(k) & a(k)-c(k) \end{pmatrix}, \tag{3.2}$$

where $a(k) = \alpha(k)+\beta(k)/2$, $c(k) = \alpha(k)-\beta(k)/2$, $b_1(k)$ and $b_2(k)$ are real valued. Next the periodic potential is assumed to have the symmetry $V(x) = V(-x)$. This symmetry is typical of metals. This symmetry forces $b(k)$ to be real valued (i.e., $b_2(k) = 0$), (see [19]). Consequently, near k_0 we have

$$\lambda_j(k) = a(k) - \sqrt{c^2(k)+b_1^2(k)}, \qquad \lambda_{j+1}(k) = a(k) + \sqrt{c^2(k)+b_1^2(k)}. \tag{3.3}$$

We assume that $\nabla b_1(k_0)$, $\nabla c(k_0)$ are independent. Since $n = 2$, $(\nabla b_1(k_0), \nabla c(k_0))$ is a basis in $\mathbb{R}^2$. Set $\nabla a(k_0) = \alpha_1 \nabla b_1(k_0) + \alpha_2 \nabla c(k_0)$.

Lemma 3.1. *Let $\nabla a(k_0) = \alpha_1 \nabla b_1(k_0) + \alpha_2 \nabla c(k_0)$ be as above. One assumes that*

$$\alpha_1^2 + \alpha_2^2 < 1. \tag{3.4}$$

Then there exist an open connected neighborhood J of e_0 and analytic functions f and g such that

$$\rho(e) = f(e) + (H(e - e_0) - H(e_0 - e))g(e), \tag{3.5}$$

with

$$g''(e_0) \neq 0, \quad \forall e \in J. \tag{3.6}$$

Proof. To simplify the notation we assume that $k_0 = 0$ and $e_0 = 0$.

Let Ω be a neighborhood of $k_0 = 0$. We introduce

$$(2\pi)^n \rho_1(e) = \int_{\{k\in\Omega;\ \lambda_n(k)\le e\}} dk + \int_{\{k\in\Omega;\ \lambda_{n+1}(k)\le e\}} dk, \tag{3.7}$$

so that

$$(2\pi)^n \left(\rho(e) - \rho_1(e)\right) = \sum_{j\notin\{n,n+1\}} \int_{\{k\in E^*;\ \lambda_j(k)\le e\}} dk + \int_{\{k\in E^*\setminus\Omega;\ \lambda_n(k)\le e\}} dk + \int_{\{k\in E^*\setminus\Omega;\ \lambda_{n+1}(k)\le e\}} dk. \tag{3.8}$$

Due to Lemma 2.2, the right-hand side of the above equalities is analytic near 0.

Since $\nabla b_1(k_0)$, $\nabla c(k_0)$ are independent, there exist a neighborhood Ω of $k_0 = 0$, $\epsilon > 0$ and a local analytic diffeomorphism $\kappa : \Omega \to B(0, \epsilon)$ such that, with the change of variable $k \to \kappa(k)$, we obtain

$$(2\pi)^n \rho_1(e) = \int_{\{|k|\le\epsilon;\ G(k)+|k|\le e\}} F(k)dk + \int_{\{|k|\le\epsilon;\ G(k)-|k|\le e\}} F(k)dk, \tag{3.9}$$

where $G(k) = a(\kappa^{-1}(k))$ and $F(k) = \mathrm{Jac}(\kappa(k))$ are analytic near $k = 0$ and $\nabla G(0) = (\alpha_1, \alpha_2)$.

Using polar coordinates and making the change $r \to -r$, $\omega \to -\omega$ in the second integral, we get

$$(2\pi)^n \rho_1(e) = \int_{S^1} \int_{\{0\le r\le\delta;\ G(r\omega)+r\le e\}} F(r\omega) r\, dr\, d\omega - \int_{S^1} \int_{\{-\delta\le r\le 0;\ G(r\omega)+r\le e\}} F(r\omega) r\, dr\, d\omega, \tag{3.10}$$

which can be written

$$(2\pi)^n \rho_1(e) = \int_{S^1} \int_{\{0 \le r \le \delta;\, G(r\omega)+r \le e\}} F(r\omega) r \, dr \, d\omega + \int_{S^1} \int_{\{-\delta \le r \le 0;\, G(r\omega)+r \ge e\}} F(r\omega) r \, dr \, d\omega - c_0, \tag{3.11}$$

where $c_0 = \int_{S^1} \int_{\{-\delta \le r \le 0\}} F(r\omega) r dr d\omega$. Since

$$\partial_r (G(r\omega) + r)_{|_{r=0}} = \langle \nabla G(0), \omega \rangle + 1 \ge \eta > 0, \tag{3.12}$$

uniformly on $\omega \in S^1$, there exist $\delta_1, \delta_2 > 0$ (independent on $\omega \in S^1$) such that $\Upsilon : r \to \Upsilon(r) = G(r\omega) + r$ from $]-\delta_1, \delta_1[$ into $]-\delta_2, \delta_2[$ is an analytic diffeomorphism. Hence, for $|e|$ small enough

$$\begin{aligned}(2\pi)^n \rho_1(e) + c_0 &= \int_{S^1} \int_{\{t \ge 0;\, t \le e\}} F\left(\Upsilon^{-1}(t)\omega\right) \frac{\Upsilon^{-1}(t)}{\Upsilon'(t)} dt \, d\omega \\ &\quad + \int_{S^1} \int_{\{t \le 0;\, t \ge e\}} F\left(\Upsilon^{-1}(t)\omega\right) \frac{\Upsilon^{-1}(t)}{\Upsilon'(t)} dt \, d\omega \\ &= (H(e) - H(-e)) g(e),\end{aligned} \tag{3.13}$$

where

$$g(e) = \int_0^e \int_{S^1} F\left(\Upsilon^{-1}(t)\omega\right) \frac{\Upsilon'(t)}{\Upsilon^{-1}(t)} dt \, d\omega. \tag{3.14}$$

Using that

$$\Upsilon^{-1}(0) = 0 \tag{3.15}$$

we deduce $g''(0) = F(0) \int_{S^1} (\langle \nabla G(0), \omega \rangle + 1)^{-2} d\omega \ne 0$. □

We denote by $\#A$ the number of elements of A, counted with their multiplicity. The main result of this section is the following.

Theorem 3.2. *Let $\lambda, e_0 \in \sigma(P_0)$ with $\lambda \in (e_0 + \text{sing supp}(\mu))$. One assumes the following.*

(i) *The periodic potential V satisfies $V(x) = V(-x)$.*

(ii) *There exists $k_0 \in \mathbb{R}^n/\Gamma^*$ such that $\lambda_{j-1}(k_0) < \lambda_j(k_0) = e_0 = \lambda_{j+1}(k_0) < \lambda_{j+2}(k_0)$.*

(iii) *For all $k \notin k_0 + \Gamma^*$ such that $\lambda_i(k) = e_0$, the eigenvalue $\lambda_i(k)$ is simple and $\nabla \lambda_i(k) \ne 0$.*

(iv) *The numbers (α_1, α_2) satisfy* (3.4), *and $(\lambda - \text{supp}(\mu)) \subset J$. Here J is the interval given by Lemma 3.1.*

(v) *λ satisfies (H1).*

Then for all h-independent complex neighborhoods Ω of λ, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that, for $h \in]0, h_0[$,

$$\#\{z \in \Omega;\ z \in \operatorname{Res}(P(h))\} \geq C_\Omega h^{-n}. \tag{3.16}$$

Proof. Without any loss of generality we may assume that $e_0 = 0$. Set

$$K(\cdot) := (H(\cdot) - H(-\cdot))g(\cdot), \tag{3.17}$$

where $g(\cdot)$ is the function given in Lemma 3.1.

The assumption that $(\lambda - \operatorname{supp}(\mu)) \subset J$ ensures that, in the study of $d\rho * \mu$ near λ, one only needs the value of ρ in J given by (3.4). More precisely, it implies that

$$\omega(t) = d\rho * \mu(t) = \rho * d\mu(t) = f * d\mu + K(\cdot) * d\mu = (1) + (2), \tag{3.18}$$

for t near λ.

Since f is smooth, the first term of the right-hand side of the above equation is also smooth.

Clearly, it follows from assumption (2.6) and Lemma 2.2 that the sing $\operatorname{supp}(\mu)$ is a discrete set. Thus, the point λ is isolated in sing $\operatorname{supp}(\mu)$. We recall that we have assumed that $e_0 = 0$.

Let $\chi \in C_0^\infty(B(0,1))$ (resp., $\theta \in C_0^\infty(B(\lambda,1))$) be equal to one near zero (resp., λ). Here $B(y,r)$ is the disc of center y and radius r. Set $\chi_\epsilon = \chi(\cdot/\epsilon)$ and $\theta_\epsilon = \theta(\cdot/\epsilon)$. We choose $\epsilon > 0$ small enough such that

$$\text{sing supp}(\mu) \cap \operatorname{supp} \theta_\epsilon = \{\lambda\}. \tag{3.19}$$

To study the second term of the right-hand side of (3.18), we write it in the form

$$(2) = K(\cdot)(1 - \chi_\epsilon) * d\mu + K(\cdot)\chi_\epsilon * \theta_\epsilon d\mu + K(\cdot)\chi_\epsilon * (1 - \theta_\epsilon)d\mu = (3) + (4) + (5). \tag{3.20}$$

Since $K(\cdot)(1 - \chi_\epsilon)$ is smooth the term (3) is also smooth. Using (3.19) and the fact that the support of $K(\cdot)\chi_\epsilon$ is small for $\epsilon \ll 1$, we see that the term (5) is C^∞ near λ.

Now, we claim that

$$\text{sing supp}(4) = \{\lambda\}. \tag{3.21}$$

First, from a standard result on the singular support, we have

$$\text{sing supp}(4) \subset \text{sing supp}\big(K(\cdot)\chi_\epsilon\big) + \text{sing supp}\big(\theta_\epsilon d\mu\big) = \{\lambda\}. \tag{3.22}$$

Consequently, to prove the claim it suffices to show that $(4) \notin C_0^\infty(\mathbf{R})$. We recall that (4) has a compact support.

A simple calculus and Lemma 3.1 yield

$$c\left(1+|\xi|^2\right)^{-1} \le \left|\widehat{K(\cdot)\chi_\epsilon}(\xi)\right| \le C. \tag{3.23}$$

Here $\widehat{f}(\xi)$ is the Fourier transform of f. Consequently, $\widehat{\theta_\epsilon d\mu} \in S(\mathbb{R})$ if and only if $\widehat{(4)} \in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space of C^∞ function of rapid decrease.

On the other hand, (3.19) implies that $\widehat{\theta_\epsilon \mu} \notin S(\mathbb{R})$. Combining this with the above remarks we get the claim.

Summing up, we have proved that $\lambda \in \text{sing supp}(\omega = d\rho * \mu)$.

Now, applying the following result of [14] we obtain Theorem 3.2.

Theorem 3.3 (see [14]). *Let $\lambda \in$ sing supp$_a(\omega)$. Assume that λ satisfies (H1). Then for every h-independent complex neighborhood $\widetilde{\Omega}$ of λ, there exists $h_0 = h(\widetilde{\Omega})$ sufficiently small and $C = C(\widetilde{\Omega})$ large enough such that, for $h \in]0, h_0[$,*

$$\#\left\{z \in \widetilde{\Omega}; z \in Res(P(h))\right\} \ge C\left(\widetilde{\Omega}\right) h^{-n}. \tag{3.24}$$

□

Remark 3.4. Let e_0 be a singularity of the integrated density of states, generated by a band crossing. Theorem 3.2 shows that there is at least $\sim h^{-n}$ resonances near $e_0 + t$, where t is in the singular support of the distribution μ defined by

$$\mu(t) = \int_{\{x \in \mathbb{R}^n; W(x) > t\}} dx. \tag{3.25}$$

4. Lower Bound of the Counting Function near the Edges of Bands

In this section we study resonances generated by analytic singularities of ρ near the edge of bands. The following result is a consequence of Lemma 2.3.

Lemma 4.1. *Let $e_0 \in \sigma(P_0)$. One assumes the following.*

(i) *If $\lambda_j(k) = e_0$, then $\lambda_j(k)$ is a simple eigenvalue of $H_0(k)$.*

(ii) *There exist i_0 and k_0 such that $\lambda_{i_0}(k_0) = e_0$, $\nabla\lambda_{i_0}(k_0) = 0$, $\pm\partial^2\lambda_{i_0}(k_0) > 0$ and $\nabla\lambda_{i_0}(k) \neq 0$, for all $k \in E^*, k \neq k_0$.*

(iii) *For all $k \in \lambda_i^{-1}\{e_0\}$ with $i \neq i_0$, $\nabla\lambda_i(k) \neq 0$.*

Then there exists an open connected neighborhood J of e_0 such that

$$\rho(e) = f(e-e_0) + H(\pm(e-e_0))g\left(\sqrt{\pm(e-e_0)}\right), \quad \forall e \in J, \tag{4.1}$$

where f and g are analytic near zero and $g(0) = 0, \ldots, g^{(n-1)}(0) = 0$, $g^{(n)}(0) \neq 0$. Here, $+(-)$ corresponds to a local minimum (maximum, resp.).

Now, repeating the arguments in the proof of Theorem 3.2 and using Lemma 4.1, we obtain the following.

Theorem 4.2. *Let $e_0, \lambda \in \sigma(P_0)$ with $\lambda \in (e_0 + sing\, supp_a(\mu))$. One assumes the following.*

(i) *λ satisfies (H1),*

(ii) *e_0 satisfies the assumptions of Lemma 4.1,*

(iii) *$(\lambda - \mathrm{supp}(\mu)) \subset J$. Here J is the interval given by Lemma 4.1.*

Then for all h-independent complex neighborhoods Ω of λ, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that, for $h \in]0, h_0[$,

$$\#\{z \in \Omega;\ z \in \mathrm{Res}(P(h))\} \geq C_\Omega h^{-n}. \tag{4.2}$$

Remark 4.3. Notice that the assumptions (iv) in Theorem 3.2 and (iii) in Theorem 4.2 are satisfied if $\|W\|_\infty$ is small.

References

[1] V. S. Buslaev, "Semi-classical approximation for equations with periodic coefficients," *Russian Mathematical Surveys*, vol. 42, pp. 97–125, 1987.
[2] M. C. Chang and Q. Niu, "Berry phase, hyperorbits, and the Hofstadter spectrum," *Physical Review Letters*, vol. 75, pp. 1348–1351, 1996.
[3] M. C. Chang and Q. Niu, "Berry phase, hyperorbits, and the Hofstadter spectrum: semiclassical in magnetic Bloch bands," *Physical Review B*, vol. 53, pp. 7010–7022, 1996.
[4] M. Dimassi, J. C. Guillot, and J. Ralston, "Semiclassical asymptotics in magnetic Bloch bands," *Journal of Physics*, vol. 35, no. 35, pp. 7597–7605, 2002.
[5] M. Dimassi, J.-C. Guillot, and J. Ralston, "On effective Hamiltonians for adiabatic perturbations of magnetic Schrödinger operators," *Asymptotic Analysis*, vol. 40, no. 2, pp. 137–146, 2004.
[6] C. Gérard, A. Martinez, and J. Sjöstrand, "A mathematical approach to the effective Hamiltonian in perturbed periodic problems," *Communications in Mathematical Physics*, vol. 142, no. 2, pp. 217–244, 1991.
[7] F. Hôvermann, H. Spohn, and S. Teufel, "Semiclassical limit for the Schrödinger equation with a short scale periodic potential," *Communications in Mathematical Physics*, vol. 215, no. 3, pp. 609–629, 2001.
[8] W. Horn, "Semi-classical constructions in solid state physics," *Communications in Partial Differential Equations*, vol. 16, no. 2-3, pp. 255–289, 1991.
[9] G. Nenciu, "Dynamics of band electrons in electric and magnetic fields rigorous justification of the effective Hamiltonians," *Reviews of Modern Physics*, vol. 63, no. 1, pp. 91–127, 1991.
[10] R. Peierls, "Zur Theorie des diamagnetimus von leitungselektronen," *Zeitschrift für Physikalische*, vol. 80, pp. 763–791, 1933.
[11] J.C. Slater, "Electrons in perturbed periodic lattices," *Physical Review*, vol. 76, pp. 1592–1600, 1949.
[12] N. E. Firsova, "Resonances of Hill operator perturbed by an exponentially decreasing additive potential," *Mathematical Notes*, vol. 36, pp. 854–861, 1984.
[13] M. Dimassi, "Resonances for slowly varying perturbations of a periodic Schrödinger operator," *Canadian Journal of Mathematics*, vol. 54, no. 5, pp. 998–1037, 2002.
[14] M. Dimassi and M. Zerzeri, "A local trace formula for resonances of perturbed periodic Schrödinger operators," *Journal of Functional Analysis*, vol. 198, no. 1, pp. 142–159, 2003.
[15] J. Sjöstrand, "A trace formula for resonances and application to semi-classical Schrödinger operators," in *Séminaire Équations aux Dérivées Partielles*, exposé no. 11 (1996-97), 1996.
[16] M. Dimassi and M. Mnif, "Lower bounds for the counting function of resonances for a perturbation of a periodic Schrödinger operator by decreasing potential," *Comptes Rendus Mathématique*, vol. 335, no. 12, pp. 1013–1016, 2002.
[17] E. Balslev and J. M. Combes, "Spectral properties of many-body Schrödinger operators with dilatation-analytic interactions," *Communications in Mathematical Physics*, vol. 22, pp. 280–294, 1971.
[18] L. Nédeléc, "Resonances for matrix Schrödinger operators," *Duke Mathematical Journal*, vol. 106, no. 2, pp. 209–236, 2001.
[19] J. Ralston, "Magnetic breakdown," *Astérisque*, no. 210, pp. 263–282, 1992.