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Research Article

A Generalization of a Class of Matrices: Analytic Inverse and Determinant

F. N. Valvi

Department of Mathematics, University of Patras, 26500 Patras, Greece

Correspondence should be addressed to F. N. Valvi, fvalvi@upatras.gr

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The aim of this paper is to present the structure of a class of matrices that enables explicit inverse to be obtained. Starting from an already known class of matrices, we construct a Hadamard product that derives the class under consideration. The latter are defined by $4n-2$ parameters, analytic expressions of which provide the elements of the lower Hessenberg form inverse. Recursion formulae of these expressions reduce the arithmetic operations in evaluating the inverse to $\mathcal{O}(n^2)$.

1. Introduction

In [1], a class of matrices $K_n = [a_{ij}]$ with elements

$$a_{ij} = \begin{cases} 1, & i \leqslant j, \\ a_j, & i > j \end{cases} \tag{1.1}$$

is treated. A generalization of this class is presented in [2] by the matrix $G_n = [b_{ij}]$, where

$$b_{ij} = \begin{cases} b_j, & i \leqslant j, \\ a_j, & i > j. \end{cases} \tag{1.2}$$

In this paper, we consider a more extended class of matrices, M, and we deduce in analytic form its inverse and determinant. The class under consideration is defined by the Hadamard product of G_n and a matrix L, which results from G_n first by assigning the values

$a_i = l_{n-i+1}$ and $b_i = k_{n-i+1}$ to the latter in order to get a matrix K, say, and then by the relation $L = PK^T P$, where $P = [p_{ij}]$ is the permutation matrix with elements

$$p_{ij} = \begin{cases} 1, & i = n - j + 1, \\ 0, & \text{otherwise}. \end{cases} \tag{1.3}$$

The so-constructed class is defined by $4n - 2$ parameters, and its inverse has a lower Hessenberg analytic expression. By assigning particular values to these parameters, a great variety of test matrices occur.

It is worth noting that the classes L and G_n that produce the class $M = L \circ G_n$ belong to the extended DIM classes presented in [3] as well as to the categories of the upper and lower Brownian matrices, respectively, as they have been defined in [4].

2. The Class of Matrices and Its Inverse

Let $M = [m_{ij}]$ be the matrix with elements

$$m_{ij} = \begin{cases} k_i b_j, & i \leqslant j, \\ l_i a_j, & i > j, \end{cases} \tag{2.1}$$

that is,

$$M = \begin{bmatrix} k_1 b_1 & k_1 b_2 & k_1 b_3 & \cdots & k_1 b_{n-1} & k_1 b_n \\ l_2 a_1 & k_2 b_2 & k_2 b_3 & \cdots & k_2 b_{n-1} & k_2 b_n \\ l_3 a_1 & l_3 a_2 & k_3 b_3 & \cdots & k_3 b_{n-1} & k_3 b_n \\ \cdots & & & & & \\ l_{n-1} a_1 & l_{n-1} a_2 & l_{n-1} a_3 & \cdots & k_{n-1} b_{n-1} & k_{n-1} b_n \\ l_n a_1 & l_n a_2 & l_n a_3 & \cdots & l_n a_{n-1} & k_n b_n \end{bmatrix}. \tag{2.2}$$

If $M^{-1} = [\mu_{ij}]$ is its inverse, then the following expressions give its elements

$$\mu_{ij} = \begin{cases} \dfrac{k_{i+1} b_{i-1} - l_{i+1} a_{i-1}}{c_{i-1} c_i}, & i = j = 2, 3, \ldots, n-1, \\ \dfrac{k_2}{c_0 c_1}, & i = j = 1, \\ \dfrac{b_{n-1}}{c_{n-1} c_n}, & i = j = n, \\ (-1)^{i+j} \dfrac{d_{j-1} g_i \prod_{\nu=j+1}^{i-1} f_\nu}{\prod_{\nu=j-1}^{i} c_\nu}, & i > j, \\ -\dfrac{1}{c_i}, & i = j - 1, \\ 0, & i < j - 1, \end{cases} \tag{2.3}$$

where

$$\begin{aligned}
&c_i = k_{i+1}b_i - l_{i+1}a_i, \quad i = 1,2,\dots,n-1, \qquad c_0 = k_1, \qquad c_n = b_n,\\
&d_i = a_{i+1}b_i - a_ib_{i+1}, \quad i = 1,2,\dots,n-2, \qquad d_0 = a_1,\\
&f_i = l_ia_i - k_ib_i, \quad i = 2,3,\dots,n-1,\\
&g_i = k_{i+1}l_i - k_il_{i+1}, \quad i = 2,3,\dots,n-1, \qquad g_n = l_n,
\end{aligned} \tag{2.4}$$

with

$$\prod_{\nu=j+1}^{i-1} f_\nu = 1 \qquad \text{whenever } i = j+1, \tag{2.5}$$

and with the obvious assumption

$$c_i \neq 0, \quad i = 0,1,2,\dots,n. \tag{2.6}$$

3. The Proof

We prove that the expressions (2.3) give the inverse matrix M^{-1}. To that purpose, we reduce M to the identity matrix by applying elementary row operations. Then the product of the corresponding elementary matrices gives the inverse matrix. In particular, adopting the conventions (2.4), we apply the following sequence of row operations:

Operation 1. row $i - (k_i/k_{i+1}) \times \text{row}(i+1)$, $i = 1,2,\dots,n-1$, which gives the lower triangular matrix

$$\begin{bmatrix}
\frac{k_1}{k_2}c_1 & 0 & \cdots & 0 & 0\\
\frac{a_1}{k_3}g_2 & \frac{k_2}{k_3}c_2 & \cdots & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\frac{a_1}{k_n}g_{n-1} & \frac{a_2}{k_n}g_{n-1} & \cdots & \frac{k_{n-1}}{k_n}c_{n-1} & 0\\
g_na_1 & g_na_2 & \cdots & g_na_{n-1} & g_nb_n
\end{bmatrix}. \tag{3.1}$$

Operation 2. row $i - (k_ig_i/k_{i+1}g_{i-1}) \times \text{row}(i-1)$, $i = n, n-1,\dots,3$, $k_{n+1} = 1$, which results in a bidiagonal matrix with main diagonal

$$\left(\frac{k_1c_1}{k_2}, \frac{k_2c_2}{k_3}, \dots, \frac{k_{n-1}c_{n-1}}{k_n},\ k_nb_n\right) \tag{3.2}$$

and lower first diagonal

$$\left(\frac{a_1 g_2}{k_3}, \frac{k_3 g_3 f_2}{k_4 g_2}, \ldots, \frac{k_{n-1} g_{n-1} f_{n-2}}{k_n g_{n-2}}, \frac{k_n g_n f_{n-1}}{g_{n-1}}\right). \tag{3.3}$$

Operation 3. row $2 - (k_2 a_1 g_2 / k_1 k_3 c_1) \times \text{row}\, 1$, and row $i - (k_i k_i g_i f_{i-1} / k_{i-1} k_{i+1} g_{i-1} c_{i-1}) \times \text{row}(i-1)$, $i = 3, 4, \ldots, n$, which gives the diagonal matrix

$$\begin{bmatrix} \frac{k_1 c_1}{k_2} & \frac{k_2 c_2}{k_3} & \cdots & \frac{k_{n-1} c_{n-1}}{k_n} & k_n b_n \end{bmatrix}. \tag{3.4}$$

Operation 4. $k_{i+1}/k_i c_i \times \text{row}\, i$, $i = 1, 2, \ldots, n$, which gives the identity matrix.

Operations 1–4 transform the identity matrix to the following forms, respectively:

Form 1. The upper bidiagonal matrix consisting of the main diagonal

$$(1, 1, \ldots, 1) \tag{3.5}$$

and the upper first diagonal

$$\left(-\frac{k_1}{k_2}, -\frac{k_2}{k_3}, \ \ldots, -\frac{k_{n-1}}{k_n}\right). \tag{3.6}$$

Form 2. The tridiagonal matrix

$$\begin{bmatrix} 1 & -\frac{k_1}{k_2} & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\frac{k_2}{k_3} & \cdots & 0 & 0 \\ 0 & -\frac{k_3 g_3}{k_4 g_2} & \frac{k_3(k_4 l_2 - k_2 l_4)}{k_4 g_2} & \cdots & 0 & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & \frac{k_{n-1}(k_n l_{n-2} - k_{n-2} l_n)}{k_n g_{n-2}} & -\frac{k_{n-1}}{k_n} \\ 0 & 0 & 0 & \cdots & -\frac{k_n g_n}{g_{n-1}} & \frac{k_n l_{n-1}}{g_{n-1}} \end{bmatrix}. \tag{3.7}$$

Form 3. The lower Hessenberg matrix

$$\begin{bmatrix} 1 & -\dfrac{k_1}{k_2} & \cdots & 0 & 0 \\ -\dfrac{a_1 g_2 k_2}{k_1 k_3 c_1} & \dfrac{k_2}{k_3 c_1}(k_3 b_1 - l_3 a_1) & \cdots & 0 & 0 \\ \dfrac{a_1 g_3 k_3 f_2}{k_1 k_4 c_1 c_2} & -\dfrac{d_1 g_3 k_3}{k_4 c_1 c_2} & \cdots & 0 & 0 \\ \cdots & & & & \\ s\dfrac{a_1 g_{n-1} k_{n-1} f_2 \cdots f_{n-2}}{k_1 k_n c_1 \cdots c_{n-2}} & s\dfrac{d_1 g_{n-1} k_{n-1} f_3 \cdots f_{n-2}}{k_n c_1 c_2 \cdots c_{n-2}} & \cdots & \dfrac{k_{n-1}}{k_n c_{n-2}}(k_n b_{n-2} - l_n a_{n-2}) & -\dfrac{k_{n-1}}{k_n} \\ s\dfrac{a_1 g_n k_n f_2 \cdots f_{n-1}}{k_1 c_1 \cdots c_{n-1}} & s\dfrac{d_1 g_n k_n f_3 \cdots f_{n-1}}{c_1 c_2 \cdots c_{n-1}} & \cdots & -\dfrac{d_{n-2} g_n k_n}{c_{n-2} c_{n-1}} & \dfrac{b_{n-1} k_n}{c_{n-1}} \end{bmatrix}, \quad (3.8)$$

where the symbol s stands for the quantity $(-1)^{i+j}$.

Form 4. The matrix whose elements are given by the expressions (2.4).

The determinant of M takes the form

$$\det(M) = k_1 b_n (k_2 b_1 - l_2 a_1) \cdots (k_n b_{n-1} - l_n a_{n-1}). \quad (3.9)$$

Evidently, M is singular if $c_i = 0$ for some $i \in \{0, 1, 2, \ldots, n\}$.

4. Numerical Complexity

The inverse of the matrix M is given explicitly by the expressions (2.3). However, a careful reader could easily derive the recursive algorithm that gives the elements under the main diagonal of M^{-1}. In particular,

$$\begin{aligned} \mu_{i,i-1} &= -\frac{d_{i-2} g_i}{c_{i-2} c_{i-1} c_i}, \quad i = 2, 3, \ldots, n, \\ \mu_{i,i-s-1} &= -\frac{d_{i-s-2} f_{i-s}}{d_{i-s-1} c_{i-s-2}} \mu_{i,i-s}, \quad i = 3, 4, \ldots, n, \ s = 1, 2, \ldots, i-2, \end{aligned} \quad (4.1)$$

or, alternatively,

$$\begin{aligned} \mu_{j+1,j} &= -\frac{d_{j-1} g_{j+1}}{c_{j-1} c_j c_{j+1}}, \quad j = 1, 2, \ldots, n-1, \\ \mu_{j+s+1,j} &= -\frac{g_{j+s+1} f_{j+s}}{g_{j+s} c_{j+s+1}} \mu_{j+s,j}, \quad j = 1, 2, \ldots, n-2, \ s = 1, 2, \ldots, n-j-1, \end{aligned} \quad (4.2)$$

where the c_i, d_i, f_i, and g_i are given by the relations (2.4). By use of the above algorithms, the estimation of the whole inverse of the matrix M is carried out in $2n^2 + 11n - 19$

multiplications/divisions, since the coefficient of μ_{ij} depends only on the second (first) subscript, respectively, and in $5n - 9$ additions/subtractions.

5. Remarks

When replacing k_i, b_i, and a_i by a_i, k_i, and b_i, respectively, the matrix M (see (2.2)) is transformed into the transpose C^T of the matrix C [5, Section 2]. However, the primary fact for a test matrix is the structure of its particular pattern, which succeeds in yielding the analytic expression of its inverse. In the present case, the determinants of the minors of the elements m_{ij}, $i > j + 1$, vanish to provide the lower Hessenberg type inverse. In detail, each minor of M, that occurs after having removed the ith row and jth column, $i > j+1$, has the determinant of its $(i-1) \times (i-1)$ upper left minor equal to zero, since the last two columns of the latter are linearly dependent. Accordingly, by using induction, it can be proved that all the remaining upper left minors of order $i, i+1, \ldots, n-1$ vanish.

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