ENTIRE POSITIVE SOLUTION TO THE SYSTEM OF NONLINEAR ELLIPTIC EQUATIONS

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The second-order nonlinear elliptic system $-\Delta u = f_1(x)u^{\alpha} + g_1(x)u^{-\beta} + h_1(x)u^{\gamma}P(v)$, $-\Delta v = f_2(x)v^{\alpha} + g_2(x)v^{-\beta} + h_2(x)v^{\gamma}P(u)$ with $0 < \alpha,\beta,\gamma < 1$, is considered in $\mathbb{R}^N$. Under suitable hypotheses on functions f_i, g_i, h_i $(i = 1,2)$, and P, it is shown that this system possesses an entire positive solution $(u,v) \in \mathbb{C}^{2,\theta}_{\mathrm{loc}}(\mathbb{R}^N) \times \mathbb{C}^{2,\theta}_{\mathrm{loc}}(\mathbb{R}^N)$ $(0 < \theta < 1)$ such that both u and v are bounded below and above by positive constant multiples of $|x|^{2-N}$ for all $|x| \geq 1$.

1. Introduction

This paper is concerned with the second-order nonlinear elliptic system

$$\begin{aligned} -\Delta u &= f_1(x)u^{\alpha} + g_1(x)u^{-\beta} + h_1(x)u^{\gamma}P(v), \\ -\Delta v &= f_2(x)v^{\alpha} + g_2(x)v^{-\beta} + h_2(x)v^{\gamma}P(u), \end{aligned} \qquad x \in \mathbb{R}^N \ (N \geq 3), \tag{1.1}$$

where Δ is the Laplacian operator, $0 < \alpha,\beta,\gamma < 1$ are constants, the functions f_i, g_i, h_i $(i = 1,2)$ are nonnegative and locally Hölder continuous with exponent $\theta \in (0,1)$ in $\mathbb{R}^N$, and $P : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ is a continuous differentiable function, where $\mathbb{R}_+ = (0,+\infty)$, $\overline{\mathbb{R}}_+ = [0,+\infty)$.

We are interested in the study of the existence of entire positive solutions $(u(x),v(x))$ to (1.1) which satisfy the condition that each of its elements decays between two positive multiples of $|x|^{2-N}$ as x tends to infinity. By an entire solution of (1.1) is meant a pair of functions $(u,v) \in \mathbb{C}^{2,\theta}_{\mathrm{loc}}(\mathbb{R}^N) \times \mathbb{C}^{2,\theta}_{\mathrm{loc}}(\mathbb{R}^N)$ which satisfies (1.1) at every point x in $\mathbb{R}^N$.

The existence of entire positive solutions of the equation

$$\Delta u + f(x,u) = 0, \quad x \in \mathbb{R}^N,\ N \geq 3, \tag{1.2}$$

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has been proved under various hypotheses, see [6, 7, 10–12, 14, 17]. In Particular, for the generalized Emden-Fowler equation

$$\Delta u + K(x)u^{\lambda} = 0, \quad x \in \mathbb{R}^N,\ N \geq 3, \tag{1.3}$$

where λ is a constant, and K is a positive locally θ-Hölder continuous function in $\mathbb{R}^N$, Fukagai [7] has proved for $\lambda \in (0,1)$ that if

$$\int_1^{+\infty} s^{N-1-\lambda(N-2)} K^*(s)\,ds < +\infty, \quad K^*(s) = \max_{|x|=s} K(x), \tag{1.4}$$

then there is an entire positive solution of (1.3) that is minimal, that is, bounded below and above, respectively, by a positive constant times $|x|^{2-N}$ as x tends to infinity.

Equation (1.3) with $\lambda \in (0,1)$ is said to be of sublinear type; if λ is negative, then (1.3) is said to be of singular type, and such equations arise from the boundary layer theory of viscous fluids, see [3, 13]. In this paper, we focus on elliptic systems of mixed type.

It is well known that some reaction-diffusion equations have been investigated in connection with models of population dynamics [2, 5, 9, 15]. To mention some, in [15], the equation $\partial u/\partial t - d\Delta u^m = f(x,u)$ is studied. For some mutualistic symbiosis population models of two species, it may be necessary to study equation systems such as

$$\begin{aligned} \frac{\partial u}{\partial t} - d_1 \Delta u^m &= f_1(x)u^{\rho} + g_1(x)u^{\sigma} + h_1(x)u^{\mu}P(v), \\ \frac{\partial v}{\partial t} - d_2 \Delta v^m &= f_2(x)v^{\rho} + g_2(x)v^{\sigma} + h_2(x)v^{\mu}P(u), \end{aligned} \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $0 < \rho$, $\mu < m$, $-m < \sigma < 0$, and $d_1, d_2 > 0$. Obviously, the positive equilibrium solutions to system (1.5) in $\mathbb{R}^N$ are corresponding to the entire positive solutions of a system in the form of (1.1).

Some existence results of elliptic system

$$\begin{aligned} \Delta u + F_1(x,u,v) &= 0, \\ \Delta v + F_2(x,u,v) &= 0 \end{aligned} \tag{1.6}$$

have been established in [4, 11, 16, 19–21]. In particular, in [20], the existence of the equilibrium solutions is established for the Volterra-Lotka mutualistic symbiosis model in the case of equal linear birth rates, using the method of upper and lower solutions. However, for so-called mixed type in which F_1 and F_2 involve both singular and sublinear terms, results regarding the existence of positive entire solutions cannot be derived from those in the literature.

The aim of this article is to develop the theory of existence of positive solutions for nonlinear elliptic systems. Based on a comparison principle, using the Schauder-Tychonoff fixed point theorem, we establish one main theorem regarding the existence of entire positive solutions for the system (1.1). Our results are applicable to systems such as

$$\begin{aligned} -\Delta u &= f_1(x)u^{\alpha} + g_1(x)u^{-\beta} + h_1(x)u^{\gamma}v^{\delta}, \\ -\Delta v &= f_2(x)v^{\alpha} + g_2(x)v^{-\beta} + h_2(x)v^{\gamma}u^{\delta}, \end{aligned} \quad x \in \mathbb{R}^N, \tag{1.7}$$

or

$$\begin{aligned} -\Delta u &= f_1(x)u^{\alpha} + g_1(x)u^{-\beta} + h_1(x)u^{\gamma}(c_0+v)^{-\delta}, \\ -\Delta v &= f_2(x)v^{\alpha} + g_2(x)v^{-\beta} + h_2(x)v^{\gamma}(c_0+u)^{-\delta}, \end{aligned} \quad x \in \mathbb{R}^N, \tag{1.8}$$

with $0 < \alpha, \beta, \gamma < 1$, $c_0 \geq 0$, $0 < \delta < 1 - \gamma$, and some other kinds of systems even more general (see Remark 2.2). Moreover, our method can be used to deal with similar systems on a bounded domain.

2. Main results

First, we denote by ϕ the function defined on $\mathbb{R}$:

$$\phi(t) = 1, \quad \text{if } 0 \leq t < 1; \qquad \phi(t) = t^{2-N}, \quad \text{if } t \geq 1. \tag{2.1}$$

A solution $(u(x), v(x))$ for equation system (1.1) is usually called a minimal positive entire solution if both $u(x)$ and $v(x)$ are between two positive constant multiples of function $\phi(|x|)$ in whole $\mathbb{R}^N$. This term comes from the fact that no positive solution of $\Delta u \leq 0$ in an exterior domain can decay more rapidly than a constant multiple of $|x|^{2-N}$, see [18].

Theorem 2.1. *Suppose that $0 < \alpha$, β, $\gamma < 1$ are constants and the functions g_i, h_i $(i = 1, 2)$, and P satisfy the following conditions:*

(T) *f_i, g_i, h_i are locally Hölder continuous with exponent $\theta \in (0, 1)$ in $\mathbb{R}^N$ and*

$$\int_1^{+\infty} s^{N-1-\alpha(N-2)} f_i^*(s)\,ds < +\infty, \quad f_i^*(s) = \max_{|x|=s} f_i(x),$$

$$\int_1^{+\infty} s^{N-1+\beta(N-2)} g_i^*(s)\,ds < +\infty, \quad g_i^*(s) = \max_{|x|=s} g_i(x),$$

$$\int_1^{+\infty} s^{N-1-\gamma(N-2)} h_i^*(s)P^*(s)\,ds < +\infty, \quad h_i^*(s) = \max_{|x|=s} h_i(x),\ P^*(s) = \max_{|x|=s} P(\phi(|x|)),$$

$$f_{i*}(s) + g_{i*}(s) + h_{i*}(s) \not\equiv 0 \quad \text{for } s \geq 0,$$

$$f_{i*}(s) = \min_{|x|=s} f_i(x), \qquad g_{i*}(s) = \min_{|x|=s} g_i(x); \qquad h_{i*}(s) = \min_{|x|=s} h_i(x); \tag{2.2}$$

(P) *$P : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ is a continuous differentiable function satisfying that there exists a $\lambda \in (0, 1-\gamma)$ such that for all $k \geq 1$ and $c \in [k^{-1}, k]$,*

$$P(cs) \leq k^{\lambda} P(s), \quad \forall s > 0. \tag{2.3}$$

Then the system (1.1) possesses a positive entire solution $(u,v) \in \mathbb{C}^{2,\theta}_{\mathrm{loc}}(\mathbb{R}^N) \times \mathbb{C}^{2,\theta}_{\mathrm{loc}}(\mathbb{R}^N)$ *such that each of* u *and* v *decays between two positive constant multiples of* $\phi(|x|)$ *as* x *tends to infinity, that is, the solution is minimal.*

Remark 2.2. Examples of function $P(s)$ satisfying the condition (P) are

$$P(s) = (c_0 + s)^{-\delta}, \quad (c_0 > 0,\ 0 < \delta \le \lambda) \tag{2.4}$$

as suggested in (1.8),

$$P(s) = s^{\delta} + s^{-\sigma}, \quad (0 < \delta \le \lambda,\ 0 < \sigma \le \lambda) \tag{2.5}$$

or

$$P(s) = \frac{s^{\delta}}{(c_0 + s)^{\sigma}}, \quad (c_0 > 0,\ \delta > 0,\ \sigma > 0,\ 0 < \delta + \sigma \le \lambda) \tag{2.6}$$

and so on.

3. Proof of results

Lemma 3.1. *Consider the equation*

$$-\Delta u = f(x)u^{\alpha} + g(x)u^{-\beta} + h(x)u^{\gamma}. \tag{3.1}$$

Suppose that f, g, h *are nonnegative functions defined on* $\mathbb{R}^N$, *and* $0 < \alpha, \beta, \gamma < 1$ *are constants. If* f, g, h *are locally Hölder continuous with exponent* $\theta \in (0,1)$ *in* $\mathbb{R}^N$ *and*
(T')

$$\begin{gathered}
\int_1^{+\infty} s^{N-1-\alpha(N-2)} f^*(s)\,ds < +\infty, \quad f^*(s) = \max_{|x|=s} f(x), \\
\int_1^{+\infty} s^{N-1+\beta(N-2)} g^*(s)\,ds < +\infty, \quad g^*(s) = \max_{|x|=s} g(x), \\
\int_1^{+\infty} s^{N-1-\gamma(N-2)} h^*(s)\,ds < +\infty, \quad h^*(s) = \max_{|x|=s} h(x), \\
f_*(s) + g_*(s) + h_*(s) \not\equiv 0, \quad \text{for } s \ge 0, \\
f_*(s) = \min_{|x|=s} f(x), \qquad g_*(s) = \min_{|x|=s} g(x), \qquad h_*(s) = \min_{|x|=s} h(x),
\end{gathered} \tag{3.2}$$

then (3.1) possesses a unique positive entire solution $u \in \mathbb{C}^{2,\theta}_{\mathrm{loc}}(\mathbb{R}^N)$ *such that* u *decays between two positive constant multiples of* $\phi(|x|)$ *as* x *tends to infinity, that is, the solution is minimal.*

Lemma 3.2. *Suppose that* $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ *is a continuous function such that one of the following assumptions is satisfied:*

(F_1) $s^{-1}f(x,s)$ *is strictly decreasing in* s *for each* $x \in \mathbb{R}^N$,

(F_2) *$s^{-1}f(x,s)$ is strictly decreasing in s for each x in a subset Ω_0 of $\mathbb{R}^N$ and both $f(x,s)$ and $s^{-1}f(x,s)$ are nonincreasing in s for all x in the remainder part $\mathbb{R}^N - \Omega_0$.*

Let $w,v \in \mathbb{C}^2(\mathbb{R}^N)$ satisfy

(a) $\Delta w + f(x,w) \le 0 \le \Delta v + f(x,v)$ *in* $\mathbb{R}^N$,

(b) $w,v > 0$ *in* $\mathbb{R}^N$ *and* $\liminf_{|x|\to\infty}(w(x) - v(x)) \ge 0$,

(c) Δv *in* $L^1(\mathbb{R}^N)$.

Then $w \ge v \in \mathbb{R}^N$.

The proof of Lemma 3.1 is given for completeness in the appendix of this article. Lemma 3.2 is an extension of [17, Lemma 1], so the proof is omitted here for briefness.

Proof of Theorem 2.1. Consider the equation

$$-\Delta u = f_1(x)u^{\alpha} + g_1(x)u^{-\beta} + h_1(x)u^{\gamma}P(\phi(|x|)), \quad x \in \mathbb{R}^N. \tag{3.3}$$

In view of (T) and Lemma 3.1, we find that there exists, for (3.3), a unique entire positive solution $u_0(x) \in \mathbb{C}^{2,\theta}_{\text{loc}}(\mathbb{R}^N)$. With the same argument, for the equation

$$-\Delta v = f_2(x)v^{\alpha} + g_2(x)v^{-\beta} + h_2(x)v^{\gamma}P(\phi(|x|)), \quad x \in \mathbb{R}^N, \tag{3.4}$$

there exists a unique entire positive solution $v_0(x) \in \mathbb{C}^{2,\theta}_{\text{loc}}(\mathbb{R}^N)$. Moreover, it is obvious that there is a constant $c_0 > 1$ such that for any $x \in \mathbb{R}^N$,

$$\begin{aligned} c_0^{-1}\phi(|x|) &\le u_0(x) \le c_0\phi(|x|), \\ c_0^{-1}\phi(|x|) &\le v_0(x) \le c_0\phi(|x|). \end{aligned} \tag{3.5}$$

For any constant $E \ge 1$, denote

$$\begin{gathered} U_E \equiv \{u \in \mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N) \mid E^{-1}u_0(x) \le u(x) \le Eu_0(x),\ x \in \mathbb{R}^N\}, \\ V_E \equiv \{v \in \mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N) \mid E^{-1}v_0(x) \le v(x) \le Ev_0(x),\ x \in \mathbb{R}^N\}, \\ Q \equiv U_E \times V_E. \end{gathered} \tag{3.6}$$

Obviously, Q is closed and convex. For each $(u,v) \in U_E \times V_E$, by Poisson equations theory and (T), the problem

$$-\Delta\tilde{u} = f_1(x)u^{\alpha} + g_1(x)u^{-\beta} + h_1(x)u^{\gamma}P(v), \quad x \in \mathbb{R}^N, \tag{3.7}$$

has a unique solution $\tilde{u} \in \mathbb{C}^{2,\theta}_{\text{loc}}(\mathbb{R}^N) \subset \mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N)$, and the problem

$$-\Delta\tilde{v} = f_2(x)v^{\alpha} + g_2(x)v^{-\beta} + h_2(x)v^{\gamma}P(u), \quad x \in \mathbb{R}^N, \tag{3.8}$$

has a unique solution $\tilde{v} \in \mathbb{C}^{2,\theta}_{\text{loc}}(\mathbb{R}^N) \subset \mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N)$. Defining the mappings $A_1 : Q \to \mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N)$ by $A_1(u,v) = \tilde{u}$ and $A_2 : Q \to \mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N)$ by $A_2(u,v) = \tilde{v}$, we have that $A_i(u,v) \in \mathbb{C}^{2,\theta}_{\text{loc}}(\mathbb{R}^N)$, $i = 1,2$, and hence

$$\Phi(Q) \subset \mathbb{C}^{2,\theta}_{\text{loc}}(\mathbb{R}^N) \times \mathbb{C}^{2,\theta}_{\text{loc}}(\mathbb{R}^N). \tag{3.9}$$

We claim that if E is a positive constant large enough, then

$$E^{-1}u_0(x) \le \widetilde{u}(x) \le Eu_0(x), \quad x \in \mathbb{R}^N,$$
$$E^{-1}v_0(x) \le \widetilde{v}(x) \le Ev_0(x), \quad x \in \mathbb{R}^N, \tag{3.10}$$

hence we have $A_1(Q) \subset U_E$ and $A_2(Q) \subset V_E$. In fact, we have

$$\begin{aligned}
(Ec_0)^{-1}\phi(|x|) &\le E^{-1}u_0(x) \le u(x) \le Eu_0(x) \le (Ec_0)\phi(|x|),\\
-\Delta\widetilde{u} &= f_1(x)u^{\alpha} + g_1(x)u^{-\beta} + h_1(x)u^{\gamma}P(v)\\
&\le f_1(x)E^{\alpha}u_0^{\alpha} + g_1(x)E^{\beta}u_0^{-\beta} + h_1(x)u_0^{\gamma}E^{\gamma}P\left(\frac{v(x)}{\phi(|x|)}\phi(|x|)\right)\\
&\le f_1(x)Eu_0^{\alpha} + g_1(x)Eu_0^{-\beta} + h_1(x)u_0^{\gamma}E^{\gamma}(Ec_0)^{\lambda}P(\phi(|x|)),
\end{aligned} \tag{3.11}$$

while on the other hand, we have

$$-\Delta(Eu_0) = f_1(x)Eu_0^{\alpha} + g_1(x)Eu_0^{-\beta} + h_1(x)u_0^{\gamma}EP(\phi(|x|)). \tag{3.12}$$

Thus, if E is so large that $E^{(1-r-\lambda)/\lambda} \ge c_0$, then we have $\Delta\widetilde{u} \ge \Delta(Eu_0)$. It follows from the maximum principle for the operator $-\Delta$ that

$$\widetilde{u}(x) \le Eu_0(x), \quad x \in \mathbb{R}^N. \tag{3.13}$$

Similarly, we have

$$\widetilde{u}(x) \ge E^{-1}u(x), \quad x \in \mathbb{R}^N. \tag{3.14}$$

With the same argument, we conclude that

$$E^{-1}v_0(x) \le \widetilde{v}(x) \le Ev_0(x), \quad x \in \mathbb{R}^N. \tag{3.15}$$

Fix this E and define $\Phi : Q \to Q$ by

$$\Phi(u,v) = (A_1(u,v), A_2(u,v)), \quad \forall (u,v) \in Q, \tag{3.16}$$

and now we only need to prove that Φ has a fixed point in Q.

In order to use the Schauder-Tychonoff fixed point theorem, we will prove that the operator Φ satisfies the conditions through three steps.

(1) $\Phi(Q) \subset Q$. This is a direct conclusion of (3.10).

(2) $\Phi : Q \to Q$ is continuous. Obviously, it suffices to prove that A_1 and A_2 are both continuous in the sense that for $w_n \to w$ in Q, it holds true that $\|A_i w_n - A_i w\|_{0,\theta} \to 0$, $n \to \infty$, $i = 1,2$, here, for any sequence $\{u_n\} \subset \mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N)$, by writing $\|u_n\|_{0,\theta} \to 0$, $n \to \infty$, we mean that, for any closed bounded domain $G \subset \mathbb{R}^N$, $\|u_n\|_{\mathbb{C}^{0,\theta}(G)} \to 0$, $n \to \infty$.

Denote

$$F(x) \equiv f_1(x)u^{\alpha} + g_1(x)u^{-\beta} + h_1(x)u^{\gamma}P(v),$$
$$F_n(x) \equiv f_1(x)u_n^{\alpha} + g_1(x)u_n^{-\beta} + h_1(x)u_n^{\gamma}P(v_n). \tag{3.17}$$

We have obviously that

$$||F_n - F||_{0,\theta} \longrightarrow 0, \text{ as } ||u_n - u||_{0,\theta} + ||v_n - v||_{0,\theta} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.18}$$

By Lemma 3.1, we may let $\tilde{u}$ be the unique solution of the equation $\Delta\tilde{u} = F(x)$ and let $\tilde{u}_n$ be the unique solution of the equation $\Delta\tilde{u}_n = F_n(x)$. Then by the Schauder estimation theory, we know that for any bounded domain $G \subset \mathbb{R}^N$, there exists a constant $\mathbb{C} > 0$ such that

$$||\tilde{u}_n - \tilde{u}||_{\mathbb{C}^{2,\theta}(\overline{G})} \leq \mathbb{C}||F_n - F||_{\mathbb{C}^{0,\theta}(\overline{G})}, \tag{3.19}$$

and hence

$$||\tilde{u}_n - \tilde{u}||_{\mathbb{C}^{0,\theta}(\overline{G})} \leq \mathbb{C}||F_n - F||_{\mathbb{C}^{0,\theta}(\overline{G})}. \tag{3.20}$$

Therefore,

$$||\tilde{u}_n - \tilde{u}||_{0,\theta} \longrightarrow 0, \quad \text{as } ||u_n - u||_{0,\theta} + ||v_n - v||_{0,\theta} \longrightarrow 0, \quad n \longrightarrow \infty, \tag{3.21}$$

that is, A_1 is a continuous mapping from Q to U_E. Similarly, A_2 is also a continuous mapping from Q to V_E.

(3) $\Phi(Q)$ is relatively compact in $\mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N) \times \mathbb{C}^{0,\theta}_{\text{loc}}(\mathbb{R}^N)$.

We first recall the gradient estimates for Poisson's equation (see [8]). For any bounded domain $\Omega \subset \mathbb{R}^N$, if $\Delta u = f$ in Ω, then

$$\sup_{\Omega} d_x |Du(x)| \leq \mathbb{C}\left(\sup_{\Omega}|u| + \sup_{\Omega} d_x^2 |f(x)|\right), \tag{3.22}$$

where $d_x = \text{dist}(x, \partial\Omega)$ and $\mathbb{C} = \mathbb{C}(N)$.

Denote $B_m \equiv \{x \in \mathbb{R}^N; \|x\| < m\}$, $m = 1, 2, \ldots$. For each $u \in A_1(Q)$, we have by (3.22) that

$$\begin{aligned}
\sup_{B_m} |Du(x)| &\leq \sup_{B_m} d_x |Du(x)| \leq \sup_{B_{m+1}} d_x |Du(x)| \\
&\leq \mathbb{C}\left(\sup_{B_{m+1}} |u(x)| + \sup_{B_{m+1}} d_x^2 |F(x)|\right) \\
&\leq \mathbb{C}\left(\sup_{B_{m+1}} |E\phi(x)| + (m+1)^2 \sup_{B_{m+1}} |F(x)|\right) \leq K_m,
\end{aligned} \tag{3.23}$$

where K_m depends only on m and N.

Furthermore, by (3.23), we know that

$$\frac{|u(x)-u(y)|}{|x-y|} \le |Du(t_0x+(1-t_0)y)| \le K_m, \quad \forall x,y \in B_m. \tag{3.24}$$

This shows that $A_1(Q)$, restricted on $\overline{B}_m$, is a bounded subset of $\mathbb{C}^{0,1}(\overline{B}_m)$. By the compact embedding result (see [1]); $\mathbb{C}^{0,1}(\overline{\Omega}) \hookrightarrow \hookrightarrow \mathbb{C}^{0,\theta}(\overline{\Omega})$, for any bounded domain $\Omega \subset \mathbb{R}^N$, it is seen that $A_1(Q)$, restricted on $\overline{B}_m$, is a relative compact subset of $\mathbb{C}^{0,\theta}(\overline{B}_m)$. Therefore, for any arbitrary sequence $\{u_n\}_{n\ge1} \subset A_1(Q)$, there exists a subsequence $\{u_n^{(m)}\}_{n\ge1} \subset A_1(Q)$ which is convergent on $\overline{B}_m$ in the sense of the norm $\|\cdot\|_{\mathbb{C}^{0,\theta}(\overline{B}_m)}$. The case for $A_2(Q)$ is similar.

Considering $\bigcup_{m=1}^{\infty} B_m = \mathbb{R}^N$, by the diagonal method, we conclude, for $i=1$ and $i=2$, respectively, that for an arbitrary sequence $\{u_n\}_{n\ge1} \subset A_i(Q)$, there exists a subsequence, say, $\{u_n^{(n)}\}_{n\ge1} \subset A_i(Q)$, which is convergent in the sense of the norm $\|\cdot\|_{\mathbb{C}^{0,\theta}(K)}$ on any compact subset K of $\mathbb{R}^N$, that is, $A_i(Q)$ is relatively compact in $\mathbb{C}^{0,\theta}_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore, $\Phi(Q)=A_1(Q)\times A_2(Q)$ is a relatively compact subset of $\mathbb{C}^{0,\theta}_{\mathrm{loc}}(\mathbb{R}^N)\times\mathbb{C}^{0,\theta}_{\mathrm{loc}}(\mathbb{R}^N)$.

Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $(u,v) \in Q$ such that $\Phi(u,v)=(u,v)$, that is, (u,v) satisfies the system (1.1). This completes the proof of Theorem 2.1. □

Appendix

Proof of Lemma 3.1. Let

$$\begin{aligned} F(x,u) &= f(x)u^{\alpha} + g(x)u^{-\beta} + h(x)u^{\gamma}, \\ G(t,u) &= f^{*}(t)u^{\alpha} + g^{*}(t)u^{-\beta} + h^{*}(t)u^{\gamma}, \\ g(t,u) &= f_{*}(t)u^{\alpha} + g_{*}(t)u^{-\beta} + h_{*}(t)u^{\gamma}, \end{aligned} \tag{A.1}$$

then $g(|x|,u) \le F(x,u) \le G(|x|,u)$, $x \in \mathbb{R}^N$, $u>0$. It follows from (T') that

$$0 < \int_0^{+\infty} s^{N-1} g(s,\phi(s))\,ds < \int_0^{+\infty} s^{N-1} G(s,\phi(s))\,ds < +\infty. \tag{A.2}$$

Then we define two functions by

$$y(t) = \mathcal{J}[G(t,\phi(t))], \quad z(t) = \mathcal{J}[g(t,\phi(t))], \quad t>0, \tag{A.3}$$

where $\mathcal{J}$ is the integral operator defined by

$$\mathcal{J}[E](t) = \frac{1}{N-2}\left[\int_0^t \left(\frac{s}{t}\right)^{N-2} sE(s)\,ds + \int_t^{+\infty} sE(s)\,ds\right], \quad t>0. \tag{A.4}$$

By the simple calculation, we have

$$y'' + \frac{N-1}{t}y' = -G(t,\phi(t)), \quad z'' + \frac{N-1}{t}z' = -g(t,\phi(t)), \quad t > 0, \tag{A.5}$$

$$l_1\phi(t) \le y(t), \quad z(t) \le l_2\phi(t), \quad t > 0, \tag{A.6}$$

for some positive constants l_1 and l_2.

Take $\lambda = \max\{\alpha,\beta,\gamma\}$, then for any $k \ge 1$, if $k^{-1} \le c \le k$, then

$$\begin{aligned} G(t,cu) &\le k^{\lambda}G(t,u), \quad t \ge 0,\ u > 0, \\ g(t,cu) &\ge k^{-\lambda}g(t,u), \quad t \ge 0,\ u > 0. \end{aligned} \tag{A.7}$$

Moreover, letting $y_*(x) = k_1^{\lambda}y(x)$, and k_1 is a number such that $l_1k_1^{\lambda} \ge 1$, we have

$$\begin{aligned} y_*'' + \frac{N-1}{t}y_*' &= -k_1^{\lambda}G(t,\phi(t)), \quad t > 0, \\ \phi(t) \le y_*(t) &\le k_1^{\lambda}l_2\phi(t), \quad t > 0, \end{aligned} \tag{A.8}$$

by (A.5) and (A.6). Hence,

$$\begin{aligned} G(t,y_*) &= f^*(t)y_*^{\alpha} + g^*(t)y_*^{-\beta} + h^*(t)y_*^{\gamma} \\ &\le f^*(t)(l_2k_1^{\lambda})^{\alpha}\phi^{\alpha} + g^*(t)\phi^{-\beta} + h^*(t)(l_2k_1^{\lambda})^{\gamma}\phi^{\gamma} \\ &\le k_1^{\lambda}[f^*(t)\phi^{\alpha} + g^*(t)\phi^{-\beta} + h^*(t)\phi^{\gamma}] = k_1^{\lambda}G(t,\phi), \end{aligned} \tag{A.9}$$

where we take k_1 so big that $k_1 \ge l_2^{\alpha/\lambda(1-\alpha)}$ and $k_1 \ge l_2^{\gamma/\lambda(1-\gamma)}$, that is, $(l_2k_1^{\lambda})^{\alpha} \le k_1^{\lambda}$ and $(l_2k_1^{\lambda})^{\gamma} \le k_1^{\lambda}$.

Therefore, it follows that

$$y_*'' + \frac{N-1}{t}y_*' \le -G(t,y_*). \tag{A.10}$$

Similarly, letting $z_*(x) = k_2^{-\lambda}z(x)$ with constant $k_2^{-\lambda}l_2 \le 1$, we obtain

$$z_*'' + \frac{N-1}{t}z_*' \ge -g(t,z_*), \tag{A.11}$$

such that $z_*(t) < \delta$ for any $t > 0$ and

$$z_*(t) \le y_*(t), \quad t > 0. \tag{A.12}$$

Define $y_*(0)$ and $z_*(0)$ by continuity with (A.3), and let $v(x) = y_*(|x|)$ and $w(x) = z_*(|x|)$ for $x \in \mathbb{R}^N$, then from (A.10) and (A.11) v and w are, respectively, a supersolution and a subsolution of (3.1), with $v(x) \ge w(x)$ satisfied. Therefore, by super-subsolution principle, (3.1) has a positive entire solution u such that

$$w(x) \le u(x) \le v(x), \quad x \in \mathbb{R}^N. \tag{A.13}$$

For proving the uniqueness of such solutions, we suppose that v and w are two positive entire solutions of (3.1), then it is easily seen that the conditions (a) and (b) in Lemma 3.2 are satisfied even when v interchanges with w.

Moreover, since $|\Delta v|$ and $|\Delta w|$ are, respectively, given by $f(x,v)$ and $f(x,w)$, and both v and w are between two constant multiples of $\phi(|x|)$, we have, by (A.5) for some $k>0$,

$$|\Delta v(x)|, |\Delta w(x)| \leq k^{\lambda} G(x, \phi(|x|)), \tag{A.14}$$

hence it follows from (A.2) that both Δv and Δw are in $L^1(\mathbb{R}^N)$.

Therefore, using Lemma 3.2, we have $v \geq w$ as well as $w \geq v$ in $\mathbb{R}^N$, and hence $v \equiv w$. □

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