MAXIMUM PRINCIPLES FOR A CLASS OF NONLINEAR SECOND-ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS IN DIVERGENCE FORM

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For a class of nonlinear elliptic boundary value problems in divergence form, we construct some general elliptic inequalities for appropriate combinations of $u(\mathbf{x})$ and $|\nabla u|^2$, where $u(\mathbf{x})$ are the solutions of our problems. From these inequalities, we derive, using Hopf's maximum principles, some maximum principles for the appropriate combinations of $u(\mathbf{x})$ and $|\nabla u|^2$, and we list a few examples of problems to which these maximum principles may be applied.

1. Introduction

Let $u(\mathbf{x})$ be the classical solution of the following nonlinear boundary value problems:

$$\left(g(u,|\nabla u|^2)u_{,i}\right)_{,i}+h(\mathbf{x})f(u,|\nabla u|^2)=0,\quad \mathbf{x}\in\Omega, \tag{1.1}$$

$$u=0,\quad \mathbf{x}\in\partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 2$, with smooth boundary $\partial\Omega\in C^{2,\varepsilon}$, and f, g, and h are given functions assumed to satisfy the following conditions:

$$\begin{aligned} f,h\geq 0,&\qquad g>0,\\ f,h\in C^1,&\qquad g\in C^2. \end{aligned} \tag{1.3}$$

Moreover, we assume that (1.1) is uniformly elliptic, that is, we impose throughout the strong ellipticity condition

$$G(u,s):=g(u,s)+2s\frac{\partial g}{\partial s}>0,\quad s>0,\quad \mathbf{x}\in\Omega. \tag{1.4}$$

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Under these assumptions, a minimum principle for the solutions $u(\mathbf{x})$ of the nonlinear equation (1.1) follows immediately, that is, $u(\mathbf{x})$ must assume its minimum value on $\partial\Omega$.

Sufficient conditions on the data, for the existence of classical solutions of the nonlinear equation (1.1), are known and have been well studied in the literature. See, for instance, Ladyženskaja and Ural'ceva [5] for an account on this topic. Consequently, we will tacitly assume the existence of classical solutions of the problems considered in this paper.

Maximum principles for some particular cases of the boundary value problems (1.1)-(1.2) have been considered and investigated by various authors. For references on these topics we refer, for instance, to Payne and Philippin [6, 7], to Enache and Philippin [2], or to the book of Sperb [10]. In this paper, we will focus our attention on the following two particular cases, which do not seem to have been considered in the literature: the case $g = g(u)$, $f = f(u)$, in Section 2, respectively, the case $g = g(|\nabla u|^2)$, $f = f(|\nabla u|^2)$, in Section 3. In both cases, we will derive some maximum principles for appropriate combinations of u and $|\nabla u|^2$. These combinations will be of the following form:

$$\Phi(\mathbf{x},a,b) := g^2(u)|\nabla u|^2 + 2a\int_0^u f(s)g(s)\,ds + 2b\int_0^u sg(s)\,ds, \tag{1.5}$$

in Section 2, where a and b are some real positive parameters to be appropriately chosen, respectively,

$$\Psi(\mathbf{x},\alpha,\beta) := \int_0^{|\nabla u|^2} \frac{G(s)}{f(s)}\,ds + 2\alpha u + \beta u^2, \tag{1.6}$$

in Section 3, with $G(s) := g(s) + 2sg'(s) > 0$, where α and β are also some real positive parameters to be appropriately chosen.

Here and in the rest of the paper, we adopt the following notations:

$$u_{,i} := \frac{\partial u}{\partial x_i}, \qquad u_{,ij} := \frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{1.7}$$

Moreover, we adopt the summation convention, that is, summation from 1 to N is understood on repeated indices. Using these notations, we have, for example,

$$u_{,ij}u_{,i}u_{,j} = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \tag{1.8}$$

2. Derivation of maximum principles for Φ

In this section, we focus our attention on the boundary value problems (1.1)-(1.2), with $g = g(u)$ and $f = f(u)$. Since the particular case $h \equiv \text{const}$ has already been treated by Payne and Philippin in [7], we consider only the general case when $h(\mathbf{x})$ is a nonconstant function.

Differentiating (1.5), we successively obtain

$$\Phi_{,k} = 2gg'|\nabla u|^2 u_{,k} + 2g^2 u_{,ik}u_{,i} + 2afgu_{,k} + 2bugu_{,k}, \tag{2.1}$$

$$\begin{aligned}\frac{1}{2}(g(u)\Phi_{,k})_{,k} = {} & g(g')^2|\nabla u|^4 + g^2g''|\nabla u|^4 - gg'hf|\nabla u|^2 \\ & + 4g^2g'u_{,ik}u_{,i}u_{,k} + g^2(gu_{,ik})_{,k}u_{,i} + g^3u_{,ik}u_{,ik} \\ & + a(f'g + fg')g|\nabla u|^2 - af^2gh + bg^2|\nabla u|^2 \\ & + bgg'u|\nabla u|^2 - bufgh.\end{aligned} \tag{2.2}$$

Next, we differentiate (1.1) to obtain

$$(g'u_{,i}u_{,k} + gu_{,ki})_{,k} = (gu_{,k})_{,ki} = -h_{,i}f - hf'u_{,i}, \tag{2.3}$$

from which we compute

$$(gu_{,ik})_{,k}u_{,i} = -f\nabla h\nabla u - hf'|\nabla u|^2 - g''|\nabla u|^4 - g'u_{,ik}u_{,k}u_{,i} - g'|\nabla u|^2\Delta u. \tag{2.4}$$

Making use of the Cauchy-Schwarz inequality in the following form:

$$|\nabla u|^2 u_{,ik}u_{,ik} \geq u_{,ik}u_{,k}u_{,ij}u_{,j}, \tag{2.5}$$

and of (2.1), we obtain

$$u_{,ik}u_{,ik} \geq \frac{1}{g^2}\left[g'|\nabla u|^2 + (af + bu)\right]^2 + \dots, \quad \text{in } \Omega\setminus\omega. \tag{2.6}$$

In (2.6), $\omega := \{\mathbf{x} \in \Omega : \nabla u(\mathbf{x}) = 0\}$ is the set of critical points of u and dots stand for terms containing $\Phi_{,k}$. We also make use of (2.1) to obtain the following identity:

$$u_{,ik}u_{,i}u_{,k} = -\frac{1}{g}\left[g'|\nabla u|^2 + (af + bu)\right]|\nabla u|^2 + \dots, \tag{2.7}$$

where dots have the same meaning as above.

Next, using the differential equation (1.1) in the equivalent form

$$\Delta u = -\frac{hf}{g} - \frac{g'}{g}|\nabla u|^2, \tag{2.8}$$

and inserting (2.4), (2.6), (2.7), and (2.8) in (2.2), we obtain after some reductions that the second-order differential operator

$$L\Phi := \frac{1}{2}(g(u)\Phi_{,k})_{,k} \tag{2.9}$$

satisfies the following inequality:

$$
\begin{aligned}
& L\Phi + |\nabla u|^{-2} W_k \Phi_{,k} \\
& \quad \geq g^2 \left\{ [(a-h)f' + b]|\nabla u|^2 - f h_{,i} u_{,i} + \frac{1}{g}[(af+bu)^2 - fh(af+bu)] \right\}, \quad \text{in } \Omega \diagdown \omega,
\end{aligned}
\tag{2.10}
$$

where W_k is the kth component of a vector field regular throughout Ω.

Now, we consider the following two inequalities:

$$
\begin{aligned}
(af+bu)^2 - fh(af+bu) &\geq \left[\left(a - \frac{h}{2}\right)^2 - \frac{h^2}{2} \right] f^2, \\
g|\nabla u|^2 - f h_{,i} u_{,i} &\geq -\frac{|\nabla h|^2 f^2}{4g}.
\end{aligned}
\tag{2.11}
$$

Using (2.11), we obtain, in $\Omega \diagdown \omega$, the following inequality:

$$
L\Phi + |\nabla u|^{-2} W_k \Phi_{,k} \geq g f^2 \left\{ \left(a - \frac{h}{2}\right)^2 - \frac{h^2}{2} - \frac{|\nabla h|^2}{4} \right\},
\tag{2.12}
$$

if $b + (a-h)f' \geq g$. Consequently,

$$
L\Phi + |\nabla u|^{-2} W_k \Phi_{,k} \geq 0, \quad \text{in } \Omega \diagdown \omega,
\tag{2.13}
$$

if the positive constants a and b are chosen to satisfy the following two conditions:

$$
a \geq \max_{\Omega} \left(\frac{h(\mathbf{x})}{2} + \sqrt{\frac{h^2(\mathbf{x})}{2} + \frac{|\nabla h|^2}{4}} \right) := a_1,
\tag{2.14}
$$

$$
b + (a-h)f' \geq g.
\tag{2.15}
$$

The following result is now a direct consequence of Hopf's first maximum principle [1, 3, 8, 9].

Theorem 2.1. *Let $u(\mathbf{x})$ be a classical solution of (1.1), with $g = g(u)$ and $f = f(u)$, in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and let $\Phi(\mathbf{x}, a, b)$ be the function defined in (1.5). If the positive parameters a and b are chosen to satisfy (2.14)-(2.15), then the function $\Phi(\mathbf{x}, a, b)$ takes its maximum value either on $\partial\Omega$ or at a critical point of u (i.e., a point in Ω where $\nabla u = 0$).*

Remark 2.2. (i) In the case $N = 2$, we may replace the inequality (2.5) by the following identity:

$$
u_{,ik} u_{,ik} |\nabla u|^2 = |\nabla u|^2 (\Delta u)^2 + 2 u_{,i} u_{,ij} u_{,k} u_{,kj} - 2 \Delta u u_{,ij} u_{,i} u_{,j}.
\tag{2.16}
$$

This identity leads to the same result if we replace the condition (2.14) by the following one:

$$a \geq \max_{\Omega}\left(\frac{3h(\mathbf{x})}{4} + \sqrt{\frac{10h^2(\mathbf{x})}{16} + \frac{|\nabla h|^2}{4}}\right) := a_2. \tag{2.17}$$

(ii) The parameter b, satisfying (2.15), may be difficult to compute if g is not a bounded function. However, there are situations when b could be taken to be 0. For instance when $f' > 0$ and $g/f' \leq M$, with M a positive constant, the following choice for the real parameter a will be sufficient for the conclusion of Theorem 2.1:

$$a \geq \max\left\{\max_{\Omega}\{h+M\}, \max_{\Omega}\left\{\frac{h}{2} + \sqrt{\frac{h^2}{2} + \frac{|\nabla h|^2}{4}}\right\}\right\}. \tag{2.18}$$

(iii) Theorem 2.1 holds independently of the boundary conditions for $u(\mathbf{x})$. However, in what follows, we will show that the maximum value of $\Phi(\mathbf{x}, a, b)$ must occur at a critical point of u, if Ω is a *convex* domain in $\mathbb{R}^N$.

Suppose that $\Phi(\mathbf{x}, a, b)$ takes its maximum value at $\mathbf{P}$ on $\partial\Omega$. Then, by Hopf's second maximum principle [4, 8], we must have $\Phi \equiv$ cte in Ω or $\partial\Phi/\partial n > 0$ at $\mathbf{P}$. We now compute the outward normal derivative $\partial\Phi/\partial n$ at an arbitrary point of $\partial\Omega$. Since $u = 0$ on $\partial\Omega$, we obtain

$$\frac{\partial\Phi}{\partial n} = 2gg'u_n^3 + 2g^2u_{nn}u_n + 2afgu_n. \tag{2.19}$$

From the differential equation (1.1), evaluated on $\partial\Omega \in C^{2,\varepsilon}$, we have

$$g'u_n^2 + g[u_{nn} + (N-1)Ku_n] + hf = 0. \tag{2.20}$$

In (2.19) and (2.20), u_n and u_{nn} are the first and second outward normal derivatives of u on $\partial\Omega$, and K is the average curvature of $\partial\Omega$. The insertion of (2.20) in (2.19) leads to

$$\frac{\partial\Phi}{\partial n} = 2fg(a-h)u_n - 2(N-1)Kg^2u_n^2, \quad \text{on } \partial\Omega. \tag{2.21}$$

Clearly, if a satisfies (2.14) or (2.17), we have $\partial\Phi/\partial n \leq 0$ on $\partial\Omega$, so that Φ cannot take its maximum value on $\partial\Omega$. Note that $\nabla u \neq 0$ on $\partial\Omega$ in view of Hopf's second principle [1, 4, 8, 9]. We formulate these results in the following theorem.

Theorem 2.3. *Let $u(\mathbf{x})$ be a classical solution of (1.1)-(1.2), with $g = g(u)$ and $f = f(u)$ in a bounded convex domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and let $\Phi(\mathbf{x}, a, b)$ be the function defined in (1.5) with a and b as in Theorem 2.1. Then the function $\Phi(\mathbf{x}, a, b)$ takes its maximum value at a critical point of u.*

Remark 2.4. (i) Theorems 2.1 and 2.3 also hold in the case $f(s) \le 0$, $s > 0$.

(ii) Theorem 2.3 requires that Ω be a convex domain. This restriction can, of course, be relaxed requiring that at each point of $\partial\Omega$, the average curvature is nonnegative.

3. Derivation of maximum principles for Ψ

In this section, we focus our attention on the boundary value problems (1.1)-(1.2), with $g = g(|\nabla u|^2)$ and $f = f(|\nabla u|^2)$. Since the particular case $h \equiv$ const has already been treated by Payne and Philippin in [6], we consider only the general case when $h(\mathbf{x})$ is a nonconstant function.

From (1.6), we successively compute

$$\Psi_{,k} = 2\frac{G}{f}u_{,ik}u_{,i} + 2\alpha u_{,k} + 2\beta u u_{,k}, \tag{3.1}$$

$$\begin{aligned}\Psi_{,kj} = 4\left[\frac{G'}{f} - \frac{f'}{f^2}G\right]u_{,ik}u_{,i}u_{,lj}u_{,l} + 2\frac{G}{f}\left[u_{,ikj}u_{,i} + u_{,ik}u_{,ij}\right] \\ + 2\alpha u_{,kj} + 2\beta u_{,j}u_{,k} + 2\beta u u_{,kj},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\Delta\Psi = 4\left[\frac{G'}{f} - \frac{f'}{f^2}G\right]u_{,ik}u_{,i}u_{,lk}u_{,l} + 2\frac{G}{f}\left[(\Delta u)_{,i}u_{,i} + u_{,ik}u_{,ik}\right] \\ + 2\alpha\Delta u + 2\beta|\nabla u|^2 + 2\beta u\Delta u.\end{aligned} \tag{3.3}$$

Next, we replace Δu and $(\Delta u)_{,i}u_{,i}$ in (3.3) using the differential equation (1.1) in the equivalent form

$$\Delta u = -2\frac{g'}{g}u_{,lk}u_{,l}u_{,k} - \frac{hf}{g}. \tag{3.4}$$

Differentiating (3.4), we obtain

$$\begin{aligned}(\Delta u)_{,i}u_i = -4\left(\frac{g'}{g}\right)'\left(u_{,lk}u_{,l}u_{,k}\right)^2 - 2\frac{g'}{g}\left(u_{,ilk}u_{,l}u_{,k}u_{,i} + 2u_{,lk}u_{,li}u_{,k}u_{,i}\right) \\ - \frac{f}{g}h_{,i}u_{,i} - h\frac{f'}{g}2u_{,ik}u_{,k}u_{,i} + 2\frac{g'}{g^2}hf u_{,ik}u_{,k}u_{,i}.\end{aligned} \tag{3.5}$$

Now, we would like to construct a second-order elliptic differential inequality for Ψ that contains no third-order derivatives of u. This will be achieved if we consider the following operator:

$$L\Psi := \Delta\Psi + 2\frac{g'}{g}\Psi_{,kj}u_{,k}u_{,j}, \tag{3.6}$$

for which we obtain after some reductions

$$
\begin{aligned}
L\Psi = {} & 2\frac{G}{f}u_{,ik}u_{,ik} + 4\left[\frac{G'}{f} - \frac{f'}{f^2}G - \frac{G}{f}\frac{g'}{g}\right]u_{,ik}u_{,i}u_{,lk}u_{,l} \\
& + 8\left[\frac{g'}{g}\left(\frac{G'}{f} - \frac{f'}{f^2}G\right) - \frac{G}{f}\left(\frac{g'}{g}\right)'\right]\left(u_{,lk}u_{,l}u_{,k}\right)^2 + 4h\frac{G}{g}\left[\frac{g'}{g} - \frac{f'}{f}\right]u_{,ik}u_{,i}u_{,k} \\
& - 2\frac{G}{g}h_{,i}u_{,i} - 2(\alpha+\beta u)\frac{hf}{g} + 2\beta\frac{G}{g}|\nabla u|^2.
\end{aligned}
\tag{3.7}
$$

Making use of (3.1), we now compute

$$
\begin{aligned}
u_{,ik}u_{,i}u_{,k} &= -(\alpha+\beta u)\frac{f}{G}|\nabla u|^2 + \dots, \\
\left(u_{,ik}u_{,i}u_{,k}\right)^2 &= (\alpha+\beta u)^2\frac{f^2}{G^2}|\nabla u|^4 + \dots,
\end{aligned}
\tag{3.8}
$$

$$
u_{,ik}u_{,i}u_{,lk}u_{,l} = (\alpha+\beta u)^2\frac{f^2}{G^2}|\nabla u|^2 + \dots, \tag{3.9}
$$

where dots stand for terms containing $\Psi_{,k}$. Combining (3.9) with (2.5), we obtain the inequality

$$
u_{,ik}u_{,ik} \geq (\alpha+\beta u)^2\frac{f^2}{G^2} + \dots, \quad \text{in } \Omega\setminus\omega, \tag{3.10}
$$

where $\omega := \{\mathbf{x} \in \Omega : \nabla u(\mathbf{x}) = 0\}$ is the set of critical points of u and dots have the same meaning as above.

It then follows from (3.7), (3.8), (3.9), and (3.10) that the following inequality holds:

$$
\begin{aligned}
L\Psi + |\nabla u|^{-2}W_k\Psi_{,k} \geq {} & \frac{2G}{g}\left\{\left[\beta - 2\frac{f'}{G}\left[(\alpha+\beta u)^2 - (\alpha+\beta u)h\right]\right]|\nabla u|^2\right. \\
& \left. - h_{,i}u_{,i} + \frac{f}{g}\left[(\alpha+\beta u)^2 - (\alpha+\beta u)h\right]\right\}, \quad \text{in } \Omega\setminus\omega,
\end{aligned}
\tag{3.11}
$$

where W_k is the kth component of a vector field regular throughout Ω.

Now, we consider the following two inequalities:

$$
\begin{aligned}
(\alpha+\beta u)^2 - h(\alpha+\beta u) &\geq \left[\left(\alpha - \frac{h}{2}\right)^2 - \frac{h^2}{2}\right], \\
\frac{g}{f}|\nabla u|^2 - \nabla h\nabla u &\geq -\frac{f}{4g}|\nabla h|^2.
\end{aligned}
\tag{3.12}
$$

Inserting (3.12) in (3.11), we obtain, in $\Omega\setminus\omega$, the following inequality:

$$
L\Psi + |\nabla u|^{-2}W_k\Psi_{,k} \geq \frac{2G^2}{g^2}f\left\{\left(\alpha - \frac{h}{2}\right)^2 - \frac{h^2}{2} - \frac{|\nabla h|^2}{4}\right\}, \tag{3.13}
$$

valid if $\beta \geq g/f$ and $f' \leq 0$. Consequently,

$$L\Psi + |\nabla u|^{-2} W_k \Psi_{,k} \geq 0, \quad \text{in } \Omega \setminus \omega, \tag{3.14}$$

if the positive constants α and β are chosen to satisfy the following two conditions:

$$\alpha \geq \max_{\Omega} \left(\frac{h(\mathbf{x})}{2} + \sqrt{\frac{h^2(\mathbf{x})}{2} + \frac{|\nabla h|^2}{4}} \right) := \alpha_1, \tag{3.15}$$

$$\beta \geq \max_{\Omega} \left(\frac{g}{f} + \frac{f'}{G} \frac{|\nabla h|^2}{2} \right), \tag{3.16}$$

and the function f satisfies

$$f' \leq 0. \tag{3.17}$$

The following result is now a direct consequence of Hopf's first maximum principle [1, 3, 8, 9].

Theorem 3.1. *Let $u(\mathbf{x})$ be a classical solution of (1.1), with $g = g(|\nabla u|^2)$ and $f = f(|\nabla u|^2)$, in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and let $\Psi(\mathbf{x}, \alpha, \beta)$ be the function defined in (1.6). If the positive parameters α and β are chosen to satisfy (3.15)-(3.16) and f satisfies (3.17), then the function $\Psi(\mathbf{x}, \alpha, \beta)$ takes its maximum value either on $\partial\Omega$ or at a critical point of u (i.e., a point in Ω where $\nabla u = 0$).*

Remark 3.2. (i) The parameter β, satisfying (3.16), may be difficult to compute if g/f is not a bounded function.

(ii) Theorem 3.1 holds independently of the boundary conditions for $u(\mathbf{x})$. However, in what follows, we will show that the maximum value of $\Psi(\mathbf{x}, \alpha, \beta)$ must occur at a critical point of u, if Ω is a *convex* domain in $\mathbb{R}^N$.

Suppose that $\Psi(\mathbf{x}, \alpha, \beta)$ takes its maximum value at $\mathbf{P}$ on $\partial\Omega$. Then, by Hopf's second maximum principle [4, 8], we must have $\Psi \equiv \text{cte}$ in Ω or $\partial\Psi/\partial n > 0$ at $\mathbf{P}$. We now compute the outward normal derivative $\partial\Psi/\partial n$ at an arbitrary point of $\partial\Omega$. Since $u = 0$ on $\partial\Omega$, we obtain

$$\frac{\partial \Psi}{\partial n} = 2\frac{G}{f} u_n u_{nn} + 2\alpha u_n. \tag{3.18}$$

From the differential equation (1.1), evaluated on $\partial\Omega \in C^{2,\varepsilon}$, we have

$$G u_{nn} + g(N-1) K u_n + h f = 0. \tag{3.19}$$

In (3.18) and (3.19), u_n and u_{nn} are the first and second outward normal derivatives of u on $\partial\Omega$, and K is the average curvature of $\partial\Omega$. The insertion of (3.19) in (3.18) leads to

$$\frac{\partial \Psi}{\partial n} = -2\frac{g}{f}(N-1) K u_n^2 + 2(\alpha - h) u_n, \quad \text{on } \partial\Omega. \tag{3.20}$$

Clearly, if α satisfies (3.15), we have $\partial\Psi/\partial n \leq 0$ on $\partial\Omega$, so that Ψ cannot take its maximum value on $\partial\Omega$. Note that $\nabla u \neq 0$ on $\partial\Omega$ in view of Hopf's second principle [1, 4, 8, 9]. We formulate these results in the following theorem.

Theorem 3.3. *Let $u(\mathbf{x})$ be a classical solution of (1.1)-(1.2), with $g = g(|\nabla u|^2)$ and $f = f(|\nabla u|^2)$, in a bounded convex domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and let $\Psi(\mathbf{x}, \alpha, \beta)$ be the function defined in (1.6) with α and β as in Theorem 3.1. Then the function $\Psi(\mathbf{x}, \alpha, \beta)$ takes its maximum value at a critical point of u.*

4. Examples

In this section, we list a few examples of problems for which the maximum principles obtained in the Theorems 2.3 and 3.3 may be applied. In general, we would expect the maximum principle derived for $\Phi(\mathbf{x}, a, b)$, respectively, $\Psi(\mathbf{x}, \alpha, \beta)$, to yield upper bounds for solutions, for the magnitude of its gradient, or for the distance from a critical point of solution to the boundary of the domain Ω, assumed to be bounded and convex in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega \in C^{2,\varepsilon}$.

Example 4.1. Let $u(\mathbf{x})$ be the classical solution of the boundary value problem

$$\Delta u + p|\nabla u|^2 + h(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{4.1}$$

$$u = 0, \quad \mathbf{x} \in \partial\Omega, \tag{4.2}$$

where $p = \text{const} > 0$ (the case $p = 0$ was studied in [2]) and $h \in C^1(\Omega)$ is a nonnegative function satisfying the following condition:

$$a := \max\left\{\max_{\Omega}\left\{h + \frac{1}{p}\right\}, \max_{\Omega}\left\{\frac{h}{2} + \sqrt{\frac{h^2}{2} + \frac{|\nabla h|^2}{4}}\right\}\right\} < \frac{\pi}{4d^2 p}, \tag{4.3}$$

where d is the radius of the largest ball inscribed in Ω.

Multiplying (4.1) by e^{pu} we obtain

$$\left(e^{pu}u_{,i}\right)_{,i} + e^{pu}h(\mathbf{x}) = 0, \tag{4.4}$$

that is, (1.1) with $f(u) = g(u) = e^{pu}$. Theorem 2.3 implies that the auxiliary function

$$\Phi(\mathbf{x}, a, 0) = e^{2pu}|\nabla u|^2 + \frac{a}{p}\left(e^{2pu} - 1\right) \tag{4.5}$$

takes its maximum value at a critical point of u. This leads to the following inequality:

$$e^{2pu}|\nabla u|^2 \leq \frac{a}{p}\left(e^{2pu_m} - e^{2pu}\right), \tag{4.6}$$

where $u_m := \max_\Omega u(\mathbf{x})$. Inequality (4.6) may be used to derive an upper bound for u_m. To this end, let P be a point where $u = u_m$ and Q a point on $\partial\Omega$ nearest to P. Let r measure the distance from P along the ray connecting P and Q. Clearly, we have

$$-\frac{du}{dr} \le |\nabla u|. \tag{4.7}$$

Integrating (4.7) from Q to P and making use of (4.6), we obtain

$$\int_0^{u_m} \frac{e^{pu}du}{\sqrt{e^{2pu_m} - e^{2pu}}} \le \sqrt{\frac{a}{p}} \int_P^Q dr = \sqrt{\frac{a}{p}}\delta \le \sqrt{\frac{a}{p}} d, \tag{4.8}$$

where $\delta = d(P,Q)$, We obtain

$$u_m \le \frac{1}{p}\log\left(\frac{1}{\cos(\sqrt{ap}d)}\right), \tag{4.9}$$

and, consequently,

$$|\nabla u|^2 \le \frac{a}{p}\left(\frac{1}{\cos^2(\sqrt{ap}d)} - 1\right). \tag{4.10}$$

Example 4.2. Let $u(\mathbf{x})$ be the classical solution of the boundary value problems

$$u\Delta u + p|\nabla u|^2 + h(\mathbf{x})u^2 = 0, \quad \mathbf{x} \in \Omega, \tag{4.11}$$

$$u = 0, \quad \mathbf{x} \in \partial\Omega, \tag{4.12}$$

where $p = \text{const} \in (-1,1)$ and $h \in C^1(\Omega)$ is a nonnegative function.

Multiplying (4.11) by u^{p-1}, we obtain

$$(u^p u_{,i})_{,i} + h(\mathbf{x})u^{p+1} = 0, \tag{4.13}$$

that is, (1.1) with $f(u) = u^{p+1}, g(u) = u^p$. Theorem 2.3 implies that the auxiliary function

$$\Phi(\mathbf{x}, a, 0) = u^{2p}|\nabla u|^2 + \frac{a}{p+1}u^{2p+2}, \tag{4.14}$$

with

$$a := \max\left\{\max_\Omega\left\{h + \frac{1}{p+1}\right\}, \max_\Omega\left\{\frac{h}{2} + \sqrt{\frac{h^2}{2} + \frac{|\nabla h|^2}{4}}\right\}\right\} \tag{4.15}$$

takes its maximum value at a critical point of u. This leads to the following inequality:

$$u^{2p}|\nabla u|^2 \le \frac{a}{p+1}(u_m^{2p+2} - u^{2p+2}), \tag{4.16}$$

where $u_m := \max_\Omega u(\mathbf{x})$. Integrating (4.16) in the same way as in the previous examples, we obtain

$$\frac{\pi}{2(p+1)} = \int_0^{u_m} \frac{u^p\,du}{\sqrt{u_m^{2p+2} - u^{2p+2}}} \le \sqrt{\frac{a}{p+1}}\delta, \tag{4.17}$$

where $\delta = d(P,Q)$. This shows that the critical points of $u(\mathbf{x})$ are at distance $\delta \ge \pi/2\sqrt{(p+1)a}$ from the boundary.

Example 4.3. Let $u(\mathbf{x})$ be the classical solution of the boundary value problems

$$\begin{gathered}\left(\frac{u_{,i}}{\sqrt{1+|\nabla u|^2}}\right)_{,i} + h(\mathbf{x})\frac{1}{\sqrt{1+|\nabla u|^2}} = 0, \quad \mathbf{x} \in \Omega,\\ u = 0, \quad \mathbf{x} \in \partial\Omega,\end{gathered} \tag{4.18}$$

where $h \in C^1(\Omega)$ is a nonnegative function satisfying the following conditions:

$$\begin{gathered}|\nabla h|^2 \ge 4,\\ \alpha := \max_\Omega \left\{\frac{h}{2} + \sqrt{\frac{h^2}{2} + \frac{|\nabla h|^2}{4}}\right\} < \frac{\pi}{2d}\end{gathered} \tag{4.19}$$

where d is the radius of the largest ball inscribed in Ω.

In this case, we have (1.1) with $g(|\nabla u|^2) = f(|\nabla u|^2) = (1+|\nabla u|^2)^{-1/2}$. Theorem 3.3 implies that the auxiliary function

$$\Psi(\mathbf{x},\alpha,0) = \log\left(1+|\nabla u|^2\right) + 2\alpha u, \tag{4.20}$$

takes its maximum value at a critical point of u. This leads to the following inequality:

$$\log\left(1+|\nabla u|^2\right) \le 2\alpha(u_m - u) \tag{4.21}$$

or

$$e^{2\alpha u}|\nabla u|^2 \le e^{2\alpha u_m} - e^{2\alpha u}, \tag{4.22}$$

where $u_m := \max_\Omega u(\mathbf{x})$. Integrating (4.22), as in the previous applications, we obtain

$$u_m \le \frac{1}{\alpha}\log\left(\frac{1}{\cos(\alpha d)}\right) \tag{4.23}$$

and, consequently,

$$|\nabla u|^2 \le \tan(\alpha d). \tag{4.24}$$

Example 4.4. Let $u(\mathbf{x})$ be the classical solution of the boundary value problems

$$\left(\exp\left(\frac{1}{1+|\nabla u|^2}\right)u_{,i}\right)_{,i} + h(\mathbf{x})\exp\left(\frac{1}{1+|\nabla u|^2}\right) = 0, \quad \mathbf{x}\in\Omega, \qquad u = 0, \quad \mathbf{x}\in\partial\Omega, \tag{4.25}$$

where $h \in C^1(\Omega)$ is a nonnegative function and d, the radius of the largest ball inscribed in Ω, satisfies

$$d < \frac{\pi}{2\sqrt{2}}. \tag{4.26}$$

In this case, we have (1.1) with $g(|\nabla u|^2) = f(|\nabla u|^2) = \exp(1/(1+|\nabla u|^2))$. Theorem 3.3 implies that the auxiliary function

$$\Psi(\mathbf{x},\alpha,1) = \int_0^{|\nabla u|^2} \frac{s^2+1}{(s+1)^2}\,ds + 2\alpha u + u^2 \tag{4.27}$$

takes its maximum value at a critical point of u if the parameter α is chosen to satisfy

$$\alpha \ge \max_{\Omega}\left(\frac{h(\mathbf{x})}{2} + \sqrt{\frac{h^2(\mathbf{x})}{2} + \frac{|\nabla h|^2}{4}}\right). \tag{4.28}$$

This leads to the following inequality:

$$\frac{1}{2}|\nabla u|^2 \le \int_0^{|\nabla u|^2} \frac{s^2+1}{(s+1)^2}\,ds \le 2\alpha(u_m - u) + (u_m^2 - u^2) = (u_m+\alpha)^2 - (u+\alpha)^2, \tag{4.29}$$

where $u_m := \max_{\overline{\Omega}} u(\mathbf{x})$. Integrating (4.29) in the same way as in the previous applications, we obtain the following upper bound for u_m:

$$u_m \le \alpha\left(\frac{1}{\cos(d\sqrt{2})} - 1\right). \tag{4.30}$$

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