Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 571950, 17 pages
doi:10.1155/2009/571950

Research Article

The Stochastic Ising Model with the Mixed Boundary Conditions

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Received 16 December 2008; Revised 12 April 2009; Accepted 19 June 2009

Recommended by Veli Shakhmurov

We estimate the spectral gap of the two-dimensional stochastic Ising model for four classes of mixed boundary conditions. On a finite square, in the absence of an external field, two-sided estimates on the spectral gap for the first class of (weak positive) boundary conditions are given. Further, at inverse temperatures $\beta > \beta_c$, we will show lower bounds of the spectral gap of the Ising model for the other three classes mixed boundary conditions.

1. Introduction and Definitions

We consider the most popular ferromagnetic model of statistical physics, which is the Ising model, see [1–5]. The property of ferromagnetism comes from the quantum mechanical spinning of electrons. Because a small magnetic dipole moment is associated with the spin, the electron acts like a magnet with one north pole and one south pole. Both the spin and the magnetic moment can be represented by an arrow which defines the direction of the electron's magnetic field. The spin can point up (spin value +1) or down (spin value −1), and it flips between the two orientations. Ferromagnetic models were invented in order to describe the ferromagnetic phase transition via a simple model. Considering the Ising model on the two-dimensional integer lattice $\mathbb{Z}^2$, at sufficiently low temperatures, we know that the model exhibits a phase transition, that is, there is a critical point $\beta_c > 0$, such that if $\beta > \beta_c$, the Ising model exhibits spontaneous magnetization, as is testified by the occurrence of more than one Gibbs measure in the infinite-volume limit. For example, see Aizenman and Higuchi's research work in this field. Especially the cases of free, plus, and minus boundary conditions for finite-volume Gibbs measures have been studied, see; [1–10] for more details. Beside the above three kinds of boundary conditions, it is also interesting for us to discuss other kinds of boundary conditions, for example Dobrushin boundary conditions and some mixed boundary conditions as we will consider in this paper. Dobrushin boundary conditions are the two-component boundary conditions, which are defined by

$$\tau^{\varphi}(x) = \begin{cases} 1, & \text{if } x_2 > x_1 \tan\varphi, \\ -1, & \text{otherwise}, \end{cases} \tag{1.1}$$

where $\varphi \in (-\pi/2, \pi/2)$ and $x = (x_1, x_2) \in \mathbb{Z}^2$. And the corresponding properties of the phase boundary fluctuations for the two-dimensional Ising model have been studied; see [2]. The research work on the Ising model with other mixed boundary conditions has also made some progress, this can be found in Abraham's review in Domb-Lebowitz (Vol 10); see [4, 11–15]. The object of the present paper is to study the spectral gap of the Ising model; the rate at which the Ising model converges to the equilibrium and the spectral gap of the model are closed linked, see [1, Chapter 9] for more details. So this work originates in an attempt to understand the relaxation phenomena of the model with some kind of Dobrushin boundary conditions.

In this paper, we study the Ising model with four classes mixed boundary conditions in a finite square of side $L+1$ in the absence of an external field. The first class consists of free boundary conditions with a small number of plus sites added; the second class consists of a kind of generalized Dobrushin boundary conditions; the third class consists of two minus droplets (wetting) on the left and right sides; and the fourth class consists of the sites on the bottom side which are mostly plus, and with free boundary conditions on the other three sides. Theorem 3.1 of this paper shows that certain upper and lower bounds on the gap in the case of free boundary conditions essentially remain unchanged if replacing the free boundary conditions with a suitable "weak mixing" boundary condition. Theorem 5.1 shows that, in the phase transition regime (i.e., $\beta > \beta_c$), for a certain class of "strong mixing" boundary conditions one has basically the same lower bound on the spectral gap as in the case of, for example, all "+" on one boundary edge and free boundary conditions elsewhere.

Let $\mathbb{Z}^2$ be the usual two-dimensional square lattice with sites $x = (x_1, x_2)$, equipped with the l_1-norm: $\|x\| = |x_1| + |x_2|$. We consider the standard two-dimensional Ising model in a finite square Λ, which is defined by

$$\Lambda(L) = \left\{ x \in \mathbb{Z}^2 : 0 \le x_i \le L,\ i = 1, 2 \right\} \tag{1.2}$$

for an integer L. Let $\Omega_\Lambda = \{-1, +1\}^\Lambda$ be the configuration space, an element of Ω_Λ will usually be denoted by σ_Λ. Whenever confusion does not arise we will also omit the subscript Λ in the notation σ_Λ. Given $\Lambda \subset \mathbb{Z}^2$, we define the interior and exterior boundaries of Λ as

$$\begin{aligned} \partial_{\text{int}} \Lambda &\equiv \{ x \in \Lambda : \exists y \notin \Lambda,\ \|x - y\| = 1 \}, \\ \partial_{\text{ext}} \Lambda &\equiv \{ x \notin \Lambda : \exists y \in \Lambda,\ \|x - y\| = 1 \}, \end{aligned} \tag{1.3}$$

and the edge boundary $\partial\Lambda$ as

$$\partial\Lambda = \{ (x, y) : x \in \partial_{\text{int}}\Lambda,\ y \in \partial_{\text{ext}}\Lambda,\ \|x - y\| = 1 \}. \tag{1.4}$$

We also denote by $|\Lambda|$ the cardinality of Λ. Given a boundary condition $\tau \in \Omega = \{-1, 0, +1\}^{\mathbb{Z}^2}$, we consider the Hamiltonian

$$H_\Lambda^\tau(\sigma) = -\frac{1}{2} \sum_{x,y\in\Lambda, \|x-y\|=1} (\sigma(x)\sigma(y) - 1) - \sum_{(x,y)\in\partial\Lambda} (\sigma(x)\tau(y) - 1). \tag{1.5}$$

If we set $\tau(y) = +1$ for all $y \in \mathbb{Z}^2$, the boundary condition is called the plus boundary condition, if $\tau(y) = -1$ for all y, then the resulting boundary condition is called the minus boundary condition, and if $\tau(y) = 0$ for all y, then we call the resulting boundary condition the free or open boundary condition. The Gibbs measure associated with the Hamiltonian is defined as

$$\mu_\Lambda^{\beta,\tau}(\sigma) = Z^{\beta,\tau}(\Lambda)^{-1} \exp\{-\beta H_\Lambda^\tau(\sigma)\}, \tag{1.6}$$

and the partition function is given by

$$Z^{\beta,\tau}(\Lambda) = \sum_{\sigma\in\Omega_\Lambda} \exp\{-\beta H_\Lambda^\tau(\sigma)\}, \tag{1.7}$$

where $\beta > 0$ is a parameter.

We are interested in the case where β is greater than the critical value β_c. In this case, the Gibbs measures $\mu_\Lambda^{\beta,+}$ and $\mu_\Lambda^{\beta,-}$ corresponding to + and − boundary conditions respectively, will converge to different limits μ^+ and μ^- as Λ expands to the whole plane $\mathbb{Z}^2$, and the famous Aizenman-Higuchi result shows that the plus and the minus state are the only extreme Gibbs measures. Let $\mu_\Lambda^{\beta,\emptyset}$ denote the Gibbs measure with free boundary conditions, it is known that the free boundary condition state converges to the symmetric mixture of the plus and minus states. The stochastic dynamics which we want to study is defined by the Markov generator

$$\left(L_\Lambda^{\beta,\tau} f\right)(\sigma) = \sum_{x\in\Lambda} c(x,\sigma,\tau)\left[f(\sigma^x) - f(\sigma)\right] \tag{1.8}$$

acting on $L^2(\Omega, d\mu_\Lambda^{\beta,\tau})$, where the $c(x,\sigma,\tau)$ are the transition rates for the process which satisfy the detailed balance condition

$$c(x,\sigma,\tau)\mu_\Lambda^{\beta,\tau}(\sigma) = c(x,\sigma^x,\tau)\mu_\Lambda^{\beta,\tau}(\sigma^x) \tag{1.9}$$

for any integer L, $x \in \Lambda, \sigma \in \Omega_\Lambda$, where

$$\sigma^x(y) = \begin{cases} +\sigma(y), & \text{if } y \neq x, \\ -\sigma(y), & \text{if } y = x. \end{cases} \tag{1.10}$$

Also the rates satisfy a boundedness condition: there exist $c_m(\beta)$ and $c_M(\beta)$ such that

$$0 < c_m(\beta) \leq \inf_{x,\sigma} c(x,\sigma,\tau) \leq \sup_{x,\sigma} c(x,\sigma,\tau) \leq c_M(\beta) < \infty. \tag{1.11}$$

Various choices of the transition rates $c(x,\sigma,\tau)$ are possible for the process. In the present paper, we take

$$c(x,\sigma,\tau) = \exp\left\{-\beta\sigma(x)\left[\sum_{y\in\Lambda,\|x-y\|=1}\sigma(y) + \sum_{(x,y)\in\partial\Lambda}\tau(y)\right]\right\}. \tag{1.12}$$

Finally, we define the spectral gap of this dynamics

$$\text{gap }(\Lambda,\beta,\tau) = \text{ gap }\left(L_\Lambda^{\beta,\tau}\right) = \inf_{f\in L^2\left(\Omega,d\mu_\Lambda^{\beta,\tau}\right)} \frac{\mathcal{E}_\Lambda^{\beta,\tau}(f,f)}{\text{Var}_\Lambda^{\beta,\tau}(f)}, \tag{1.13}$$

where

$$\begin{aligned}\mathcal{E}_\Lambda^{\beta,\tau}(f,f) &= \frac{1}{2}\sum_{\sigma\in\Omega_\Lambda}\sum_{x\in\Lambda}\mu_\Lambda^{\beta,\tau}(\sigma)c(x,\sigma,\tau)\left[f(\sigma^x)-f(\sigma)\right]^2,\\ \text{Var}_\Lambda^{\beta,\tau}(f) &= \frac{1}{2}\sum_{\sigma,\eta\in\Omega_\Lambda}\mu_\Lambda^{\beta,\tau}(\sigma)\mu_\Lambda^{\beta,\tau}(\eta)\left[f(\sigma)-f(\eta)\right]^2,\end{aligned} \tag{1.14}$$

where $\mathcal{E}_\Lambda^{\beta,\tau}(f,f)$ is the Dirichlet form associated with the generator $L_\Lambda^{\beta,\tau}$, and $\text{Var}_\Lambda^{\beta,\tau}$ is the variance relative to the probability measure $\mu_\Lambda^{\beta,\tau}$.

2. The Four Classes of Boundary Conditions for the Ising Model

In this section, we give the definitions of four classes boundary conditions for the Ising model, and give some descriptions of them. The estimates on the gap in the spectrum of the generator of the dynamics with plus, minus, open and mixed boundary conditions have made some progress in recent years. For example, for a finite volume Ising model, with zero external field and at sufficiently low temperature (i.e., $\beta \gg \beta_c$), Higuchi and Yoshida [4] show that for a certain class of boundary conditions in which neither "+" nor "−" predominates the other, the spectral gap on a square shrinks exponentially fast in the side-length L. In the present paper, we discuss 4 classes of mixed boundary conditions $\tau_1,\tau_2,\tau_3,\tau_4$, and study the corresponding spectral gap of the Ising model in the absence of an external field on a finite square of side-length L. Next, we define the mixed boundary conditions $\tau_1,\tau_2,\tau_3,\tau_4$ as follows.

(I) First we consider the boundary condition τ_1 as follows:

$$\tau_1(y) = 1 \text{ or } 0 \quad \text{for any } y\in\partial_{\text{ext}}\Lambda,\ \left|\left\{y\in\partial_{\text{ext}}\Lambda : \tau_1(y)=1\right\}\right| \le C_1(L\ln L)^{1/2} \tag{2.1}$$

where C_1 is a positive constant, and $\tau_1(y)=0$ means that there is no spin on the site y, or the site y is open.

Remark 2.1. From the definition of the boundary condition τ_1, it means that the number of "+" spins on the outer boundary sites of $\Lambda(L)$ is about $C_1(L\ln L)^{1/2}$, the overwhelming part of the boundary sites of $\Lambda(L)$ is free or open, and we call τ_1 the "weak boundary condition". In this case, we can show that the spectral gaps for the Ising model with τ_1 boundary condition (or other weak boundary conditions) are similar to those for the Ising model with the free boundary condition.

(II) The boundary condition τ_2 is defined as follows. For any $y \in \partial_{\text{ext}}\Lambda(L)$ and any l_1, l_2 such that $-1 \le l_1 < l_2 \le L+1$,

$$\tau_2(y) = \begin{cases} -1, & y_2 \ge l_2, \\ 0, & l_1 < y_2 < l_2, \\ 1, & y_2 \le l_1, \end{cases} \tag{2.2}$$

where $\tau_2(y) = 0$ means that there is no spin on the site y, or the site y is open.

(III) The boundary condition τ_3 is defined as follows. For any $y \in \partial_{\text{ext}}\Lambda(L)$ and any l_1, l_2 such that $-1 \le l_1 < l_2 \le L+1$ and $|l_2 - l_1| < C_3(L\ln L)^{1/2}$,

$$\tau_3(y) = \begin{cases} -1, & l_1 < y_2 < l_2, \\ +1, & \text{otherwise}. \end{cases} \tag{2.3}$$

(IV) The boundary condition τ_4 is defined as follows. Let A_i ($i = 1, 2, \ldots$) be the connected subsets of $\{y \in \partial_{\text{ext}}\Lambda(L) : y_2 = -1\}$, such that $|\bigcup_i A_i| < C_4(L\ln L)^{1/2}$ and for any $y \in \partial_{\text{ext}}\Lambda(L)$,

$$\tau_4(y) = \begin{cases} 0, & y_2 \ge 0 \\ 0, & y \in \bigcup_i A_i \\ 1, & y_2 = -1,\ y \notin \bigcup_i A_i \end{cases} \tag{2.4}$$

where $\tau_4(y) = 0$ means that there is no spin on the site y.

Remark 2.2. In the above three classes of mixed boundary conditions τ_i, $i = 2, 3, 4$, we see that there are many "+" and "−" spins on the outer boundary sites of $\Lambda(L)$. In this case, the boundary conditions may have a "strong effect" on the spectral gap of the Ising model.

3. Probability Estimates of Ising Model for the Boundary Condition τ_1

In this section, we consider the Gibbs measure and the corresponding spectral gap of the Ising model with the weak boundary condition τ_1, and we will show upper bounds and lower bounds in terms of the corresponding Gibbs measure and the spectral gap of the Ising model with free boundary conditions.

Theorem 3.1. *Let the boundary condition τ_1 be given by* (2.1)*, then for any $\beta > 0$, we have*

$$\exp\left\{-2\beta C_1(L\ \ln\ L)^{1/2}\right\}\mu_\Lambda^{\beta,\emptyset}(\sigma) \le \mu_\Lambda^{\beta,\tau_1}(\sigma) \le \exp\left\{+2\beta C_1(L\ \ln\ L)^{1/2}\right\}\mu_\Lambda^{\beta,\emptyset}(\sigma), \tag{3.1}$$

$$\exp\left\{-8\beta C_1(L\ \ln\ L)^{1/2}\right\}\ \text{gap}\ \left(L_\Lambda^{\beta,\emptyset}\right) \le\ \text{gap}\ \left(L_\Lambda^{\beta,\tau_1}\right) \le \exp\left\{8\beta C_1(L\ \ln\ L)^{1/2}\right\}\ \text{gap}\ \left(L_\Lambda^{\beta,\emptyset}\right). \tag{3.2}$$

Proof of Theorem 3.1. Let $\mu_\Lambda^{\beta,\tau_1}(\sigma)$ denote the Gibbs measure with the boundary condition τ_1, then by the definition of Gibbs measure, we have

$$\mu_\Lambda^{\beta,\tau_1}(\sigma) = \frac{\exp\{-\beta H_\Lambda^{\tau_1}(\sigma)\}}{\sum_\sigma \exp\{-\beta H_\Lambda^{\tau_1}(\sigma)\}} = \frac{\exp\left\{-\beta H_\Lambda^{\emptyset}(\sigma)\right\}B(\sigma)}{\sum_\sigma \exp\left\{-\beta H_\Lambda^{\emptyset}(\sigma)\right\}B(\sigma)}, \tag{3.3}$$

where $B(\sigma) = \exp\{\beta \sum_{(x,y)\in\partial\Lambda} \delta(y)(\sigma(x)\tau_1(y) - 1)\}$, and $\delta(y) = \tau_1(y)$. So we have

$$\exp\left\{-2\beta C_1(L\ \ln\ L)^{1/2}\right\} \le B(\sigma) \le 1. \tag{3.4}$$

By (3.3) and the computation of the Hamiltonian for the Ising model, we have

$$\exp\left\{-2\beta C_1(L\ \ln\ L)^{1/2}\right\}\mu_\Lambda^{\beta,\emptyset}(\sigma) \le \mu_\Lambda^{\beta,\tau_1}(\sigma) \le \exp\left\{+2\beta C_1(L\ \ln\ L)^{1/2}\right\}\mu_\Lambda^{\beta,\emptyset}(\sigma). \tag{3.5}$$

This completes the proof of (3.1). Next we show the spectral gap inequality of (3.2). From the definition of (1.13) and (3.1), we have the following estimates:

$$\begin{aligned} \text{Var}_\Lambda^{\beta,\tau_1}(f) &= \frac{1}{2}\sum_{\sigma,\eta\in\Omega_\Lambda} \mu_\Lambda^{\beta,\tau_1}(\sigma)\mu_\Lambda^{\beta,\tau_1}(\eta)\left[f(\sigma) - f(\eta)\right]^2 \\ &\le \frac{1}{2}\exp\left\{+4\beta C_1(L\ \ln\ L)^{\frac{1}{2}}\right\}\sum_{\sigma,\eta\in\Omega_\Lambda} \mu_\Lambda^{\beta,\emptyset}(\sigma)\mu_\Lambda^{\beta,\emptyset}(\eta)\left[f(\sigma) - f(\eta)\right]^2, \end{aligned} \tag{3.6}$$

and similarly

$$\begin{aligned} \varepsilon_\Lambda^{\beta,\tau_1}(f,f) &= \frac{1}{2}\sum_{\sigma\in\Omega_\Lambda}\sum_{x\in\Lambda}\mu_\Lambda^{\beta,\tau_1}(\sigma)c(x,\sigma,\tau_1)\left[f(\sigma^x) - f(\sigma)\right]^2 \\ &\ge \frac{1}{2}\exp\left\{-4\beta C_1(L\ \ln\ L)^{\frac{1}{2}}\right\}\sum_{\sigma\in\Omega_\Lambda}\sum_{x\in\Lambda}\mu_\Lambda^{\beta,\emptyset}(\sigma)c(x,\sigma,\emptyset)\left[f(\sigma^x) - f(\sigma)\right]^2, \end{aligned} \tag{3.7}$$

where $c(x,\sigma,\emptyset)$ denote the transition rates for the Ising model with the free boundary condition, and the following estimate is used in the above last inequality (see (1.12)):

$$\begin{aligned} c(x,\sigma,\tau_1) &= c(x,\sigma,\emptyset)\exp\left\{-\beta\sigma(x)\sum_{(x,y)\in\partial\Lambda}\tau_1(y)\right\} \\ &\geq c(x,\sigma,\emptyset)\exp\left\{-\beta C_1(L\ \ln\ L)^{1/2}\right\}. \end{aligned} \tag{3.8}$$

So we have

$$\operatorname{gap}(\Lambda(L),\beta,\tau_1) \geq \exp\left\{-8\beta C_1(L\ \ln\ L)^{1/2}\right\}\ \operatorname{gap}(\Lambda(L),\beta,\emptyset). \tag{3.9}$$

This completes the proof of the lower bound for (3.2), and with the same method, we can prove the upper bound of (3.2). Then we finish the proof of Theorem 3.1. □

It should note that this first class of weak positive boundary conditions is weak in the sense that the gap is similar to the free one, but still not so weak, in that in contrast to the free boundary condition case, it will lead to convergence to the plus measure (not the mixed measure) in the thermodynamic limit. Next we introduce an important result which comes from [10], it plays an important role in proving Theorem 5.1 of the present paper. Let R be the rectangle

$$R = \left\{x \in \mathbb{Z}^2 : 0 \leq x_1 \leq L_1,\ 0 \leq x_2 \leq L_2\right\} \tag{3.10}$$

with $L_1 \geq L_2 \geq (L_1 \ln L_1)^{1/2}$. $\mu_R^{\beta,\eta_1,\eta_2,\eta_3,\eta_4}$ denote the probability Gibbs measure on the rectangle R with the boundary conditions $\eta_1,\eta_2,\eta_3,\eta_4$ on the outer boundary of its four sides ordered clockwise starting from the bottom side. If one of the boundary configurations η_i is identically equal to $+1$ or -1, then we replace it by a $+$ or $-$ sign. For example $\eta_1,-,+,-$ means η_1 boundary condition on the bottom side of R, minus boundary condition on the vertical ones and plus boundary condition on the top one. In particular, $[0]$ boundary condition means -1 on the top side of the rectangle and $+1$ on the remaining three sides. Thus by [10, Theorem 3], we have the following Lemma 3.2.

Lemma 3.2. *Let $\beta > \beta_c$ and $L_1 = L$, there exists a $m = m(\beta) > 0$, for all $x \in R$ with $x_2 \leq (3/4)L_2$, we have*

$$\mu_R^{\beta,+}(\sigma(x) = 1) - \mu_R^{\beta,[0]}(\sigma(x) = 1) \leq \exp\{-m \ln L\}. \tag{3.11}$$

Since a lot of research work has been done to investigate the statistical properties of the Ising model with the free boundary condition, see [1–4, 7], the results of Theorem 3.1 can be extended by invoking known results about the free boundary Ising model. For example, by above Lemma 3.2 and following the parallel proof of [7, Theorem 4.1], when β large enough, there exist $C' > 0$, such that for any large integer L, we can show that

$$\operatorname{gap}(\Lambda(L),\beta,\emptyset) \geq \exp\left\{-\beta\tau_\beta L - C'\beta(L\ \ln\ L)^{1/2}\right\} \tag{3.12}$$

where τ_β is the surface tension. We denote by $\tau_\beta(\theta)$ the surface tension at angle θ (for the details see [2]), which measures the free energy of an interface in the direction orthogonal to the vector $n_\theta = (\cos\theta, \sin\theta)$. Let θ $(0 \le \theta \le \pi/4)$ and L be a positive integer, and let $Z^{\beta}_{\Lambda(L)}(\theta)$ be the partition function on $\Lambda(L)$ with the boundary condition $\eta^{(\theta)}$, where $\eta^{(\theta)}(u) = -1$ if $u_2 > u_1 \tan\theta$, and $\eta^{(\theta)}(u) = +1$ if $u_2 < u_1 \tan\theta$. Then the surface tension $\tau_\beta(\theta)$ is defined by

$$\tau_\beta(\theta) = \lim_{L\to\infty} \frac{\cos\theta}{\beta L} \log\left(\frac{Z^{\beta}_{\Lambda(L)}(\theta)}{Z^{\beta,+}_{\Lambda(L)}}\right) \tag{3.13}$$

where $Z^{\beta,+}_{\Lambda(L)}$ is the partition function corresponding to the + boundary condition on $\Lambda(L)$. And let τ_β denote the surface tension at zero degrees. Then by Theorem 3.1 and (3.12), for β large enough, we have

$$\begin{aligned}\text{gap }(\Lambda(L),\beta,\tau_1) &\ge \exp\left\{-8\beta C_1 (L \ln L)^{1/2}\right\} \exp\left\{-\beta\tau_\beta L - C'\beta(L \ln L)^{1/2}\right\} \\ &\ge \exp\left\{-\beta\tau_\beta L - C\beta(L\ln L)^{1/2}\right\},\end{aligned} \tag{3.14}$$

where C is a positive constant. In fact, the existence of (3.12) and (3.14) in the supercritical case $\beta > \beta_c$ can be shown by the theory and methods in [7, 10], here we omit this part.

4. The Block Updates for the Ising Model

In this section, we will briefly introduce the notations for the block dynamics, for the details, see [2, 6]. The lattice system phase interfaces in two dimensions are known to fluctuate widely, see for example [16] for the W-R model and [2] for the Ising model. Dobrushin et al. [2] did a deep research work on the fluctuations of phase interfaces for the Ising model at a sufficiently large parameter β. The theory of the cluster expansions is applied to investigate the behaviors of interfaces fluctuations. Because the statistical analysis on the fluctuations of the interfaces is very important for us to estimate the spectral gap of the Ising model, we introduce a block dynamics to control and estimate the fluctuations of the interfaces. Let $V \subset \mathbb{Z}^2$ be a given finite set, $\tau \in \Omega_{\mathbb{Z}^2}$ be the boundary condition, and let $\mu^{\beta,\tau}_V$ the corresponding Gibbs measure which is given in Section 1. Let $\mathfrak{D} = \{V_1, \ldots, V_n\}$ be a covering of V, i.e., $V = \bigcup_i V_i$. Then we will denote by block dynamics with blocks $\{V_1, \ldots, V_n\}$ the continuous time Markov chain in which each block waits an exponential time of mean one and the configuration inside the block is replaced by a new configuration distributed according to the Gibbs measure of the block given the previous configuration outside the block. More precisely, the generator of the Markov process corresponding to $\mathfrak{D}$ is defined as (for details see [6])

$$\left(L^{\{V_i\},\beta,\tau}\right) f(\sigma_V) = \sum_{i=1}^{n} \sum_{\eta\in\Omega_{V_i}} \mu^{\beta,(\tau\sigma_V)}_{V_i}(\eta)\left[f\left(\sigma^{\eta}_V\right) - f(\sigma_V)\right], \tag{4.1}$$

where $(\tau\sigma_V)$ denotes the configuration in $\Omega_{\mathbb{Z}^2}$ equal to τ outside V and to σ_V inside V, while σ^{η}_V is the configuration in Ω_V equal to η in V_i and to $\sigma_{V\setminus V_i}$ in $V \setminus V_i$. We will refer to the Markov process generated by $L^{\{V_i\},\beta,\tau}$ as the $\{V_i\}$-dynamics. The operator $L^{\{V_i\},\beta,\tau}$ is self-adjoint on $L^2(\Omega, d\mu_V^\tau)$, i.e., the block dynamics is reversible with respect to the Gibbs measure $\mu_V^{\beta,\tau}$. Then

$$\mathrm{gap}_V(\{V_i\}) = \inf_{f\in L^2(\Omega_V, d\mu_V^{\beta,\tau})} \frac{\mathcal{E}(f,f)}{\mathrm{Var}(f)} \tag{4.2}$$

where

$$\begin{aligned}\mathcal{E}(f,f) &= \frac{1}{2}\sum_i \sum_{\sigma_V} \sum_{\eta\in\Omega_{V_i}} \mu_V^{\beta,\tau}(\sigma_V)\mu_{V_i}^{\beta,(\tau\sigma_V)}(\eta)\left[f\left(\sigma_V^\eta\right) - f(\sigma_V)\right]^2, \\ \mathrm{Var}(f) &= \frac{1}{2}\sum_{\sigma,\eta}\mu_V^{\beta,\tau}(\sigma)\mu_V^{\beta,\tau}(\eta)\left[f(\sigma)-f(\eta)\right]^2.\end{aligned} \tag{4.3}$$

Next we introduce some results, which come from [6, 8], we will omit the proofs. Let $\mathfrak{D} = \{V_1, \ldots, V_n\}$ be an arbitrary collection of finite sets and $V = \bigcup_i V_i$. By [6, Proposition 3.4], we have Lemma 4.1.

Lemma 4.1. *For any given boundary condition $\tau \in \Omega$, one has*

$$\mathrm{gap}\left(L_V^{\beta,\tau}\right) \geq \mathrm{gap}\left(L^{\{V_i\},\beta,\tau}\right)\inf_i\inf_{\varphi\in\Omega}\mathrm{gap}\left(L_{V_i}^{\beta,\varphi}\right)\left(\sup_{x\in V}\#\{i : V_i \ni x\}\right)^{-1}. \tag{4.4}$$

The following updates are similar as those of [8, Section 4]. Let $\Lambda(L)$ be a square with sides of $L+1$ and $l = 2[k(\beta)(L\ln L)^{1/2}]$, where $k(\beta)$ is some positive constant, and $[a]$ denotes the integer part of a. Without loss of generality, we can suppose that $N = 2L/l-1$ is an integer. For $i = 1, \ldots, N/2$, we define three kind of rectangles:

$$\begin{aligned}A_i &= \left\{x \in \mathbb{Z}^2 : 0 \leq x_1 \leq L,\ (i-1)\frac{l}{2} \leq x_2 \leq (i+1)\frac{l}{2}\right\}, \\ B_{N/2+i} &= \left\{x \in \mathbb{Z}^2 : 0 \leq x_1 \leq L, L-(i+1)\frac{l}{2} \leq x_2 \leq L-(i-1)\frac{l}{2}\right\}, \\ C_{N+1} &= \left\{x \in \mathbb{Z}^2 : 0 \leq x_1 \leq L,\ -\frac{l}{2}+\frac{l}{2} \leq x_2 \leq \frac{l}{2}+\frac{l}{2}\right\},\end{aligned} \tag{4.5}$$

and let $\{Q\} = \{A_i, B_i, C_{N+1}, i = 1, \ldots, N/2\}$. By the above definition, $\{Q\}$ is the covering of $\Lambda(L)$, and by (3.12), we can construct the $\{Q\}$-dynamics. We will do the updatings in the following order

(a) first, we do the updating of $\{A_i\}$, in the order of $A_1, A_2, \ldots, A_{N/2}$;

(b) second, we do the updating of $\{B_i\}$, in the order of $B_{N/2+1}, B_{N/2+2}, \ldots, B_N$;

(c) at last, we do the updating of C_{N+1}.

The reason why we do the updatings is that we want to enforce the (+) spins and (−) spins to agree after the updatings. Next we give a result, which comes from [6, Theorem 6.4]. First, let $Q_{L,M}$ be a rectangle with sides of L, M and $L \geq M$, then

$$\inf_{\tau} \text{ gap } \{Q_{L,M}, \beta, \tau\} \geq \frac{1}{|Q_{L,M}|} c_m \exp\{-4\beta(2M+1)\}, \tag{4.6}$$

where the constant c_m has been defined in Section 1.

By the arguments of Lemmas 3.2 and 4.1, we will study the spectral gaps for the boundary conditions τ_2, τ_3, and τ_4 in the next section.

5. The Estimates of the Spectral Gaps for the Boundary Conditions τ_2, τ_3, and τ_4

We consider the Gibbs probability measure and the corresponding spectral gaps of the Ising model with mixed boundary conditions τ_2, τ_3, τ_4. At inverse temperature $\beta > \beta_c$, a lower bound on the spectral gap for the two-dimensional stochastic Ising model has been given for the boundary conditions τ_2, τ_3, τ_4, which is of order $-(L \ln L)^{1/2}$ in the exponent. Lemma 3.2 and the results of Section 4 are applied to analyze and estimate the spectral gap in this section. Next we give the following Theorem 5.1.

Theorem 5.1. *Let $\beta > \beta_c$, and let τ_i, $i = 2, 3, 4$ be defined in* (2.2), (2.3) *and* (2.4) *respectively, then for some* $C > 0$ *and for any integer L, we have*

$$\text{gap } (\Lambda(L), \beta, \tau_i) \geq \exp\left\{-C\beta(L \ln L)^{1/2}\right\}. \tag{5.1}$$

Proof of Theorem 5.1. First we consider the case $\tau = \tau_3$. Afterwards we consider the cases τ_2, τ_4. Let

$$\Lambda(L) = \left\{x \in \mathbb{Z}^2 : 0 \leq x_1 \leq L,\ 0 \leq x_2 \leq L\right\}. \tag{5.2}$$

Now the definitions of (4.5) will be modified, and we will show the proof in two steps. In order to simplify the proof, for $\tau = \tau_3$, we give another condition, $l_2 - N/2 = N/2 - l_1$, where l_1, l_2 are defined in (2.3). We redefine C_{N+1} of (4.5) to be

$$C_{N+1} = \left\{x \in \mathbb{Z}^2 : 0 \leq x_1 \leq L,\ -\frac{C_3(L \ln L)^{1/2}}{2} + \frac{L}{2} \leq x_2 \leq \frac{C_3(L \ln L)^{1/2}}{2} + \frac{L}{2}\right\}. \tag{5.3}$$

Step 1. In this part, we give the estimate for a special sequence of updatings. Let us use the following convention

$$V_i = \begin{cases} A_i, & 1 \le i \le \dfrac{N}{2}, \\ B_i, & \dfrac{N}{2} + 1 \le i \le N, \\ C_i, & i = N + 1. \end{cases} \tag{5.4}$$

Let $\Lambda(L)$ be a finite square of side $L+1$, let $S_{N+1} = \{t_1, \dots, t_N, t_{N+1}\}$ be a fixed ordered sequence with $t_1 = 0$, and let $\sigma_{t_i}^{\{Q\},\tau}$ be the configuration of $\{Q\}$-dynamics (see Section 4) at time t_i starting from the initial configuration σ, and the ith updating occurs in the box V_i. For $m = 1, \dots, N/2$ and $l = 2[k(\beta)(L \ln L)^{1/2}]$, let

$$\begin{gathered} R_m^A = \left\{ x \in \bigcup_{j \le m} A_j : x_2 \le (m+1)\frac{l}{2} - \left[\frac{l}{4}\right] \right\}, \\ R_{N/2+m}^B = \left\{ x \in \bigcup_{j \le m} B_{N/2+j} : x_2 \ge L - (m+1)\frac{l}{2} + \left[\frac{l}{4}\right] \right\} \cup R_{N/2}^A, \\ R_{N+1}^C = C_{N+1} \cup R_N^B = \Lambda(L). \end{gathered} \tag{5.5}$$

For $i = 1, \dots, N+1$, let

$$R_i \in \left\{ R_1^A, \dots, R_N^A, R_{\frac{N}{2}}^B, \dots, R_N^B, R_{N+1}^C \right\}, \tag{5.6}$$

for example, $R_{N+1} = R_{N+1}^C$. Next, we define the events

$$\begin{gathered} F_i(x) = \left\{ (+)_{t_i}^{\{Q\},\tau}(x) \ne (-)_{t_i}^{\{Q\},\tau}(x) \right\}, \quad i = 1, \dots, N+1, \\ F_i = \bigcup_{\{x \in R_i\}} F_i(x), \quad i = 1, \dots, N+1. \end{gathered} \tag{5.7}$$

In particular, we have

$$F_{N+1} = \bigcup_{x \in \Lambda(L)} F_{N+1}(x). \tag{5.8}$$

Let $q_i = P(F_i),\ i = 1, 2, \dots, N+1$, then we have for every $n \le N$

$$q_{i+1} \le q_i + P\left(F_{i+1} \cap F_i^c\right) \le \sum_{n=1}^{N} P\left(F_{n+1} \cap F_n^c\right) + P(F_1). \tag{5.9}$$

Hence by induction, we have

$$q_{N+1} \leq \sum_{n=1}^{N/2-1} P(F_{n+1} \cap F_n^c) + \sum_{n=N/2}^{N-1} P(F_{n+1} \cap F_n^c) \\ + P(F_1) + P\left(F_{N+1} \cap F_N^c\right). \tag{5.10}$$

Next we want to show that

$$q_{N+1} \leq N(L+1)^2 \exp\{-m(\ln L)\}, \tag{5.11}$$

where $m = m(\beta, \varepsilon)$ has been defined in Lemma 3.2. First, we consider the first term of (5.10), $\sum_{n=1}^{N/2-1} P(F_{n+1} \cap F_n^c)$.

$$P(F_{n+1} \cap F_n^c) \leq \sum_{\substack{x \in R_{n+1} \cap A_{n+1}, \\ \sigma \in \Omega_{\Lambda(L)}}} \mu_{\Lambda(L)}^{\beta,\tau}(\sigma) \\ \times P\left(F_{n+1}(x) \cap \left[\bigcap_{y \in R_n} \left\{ (+)_{t_n}^{\{Q\},\tau}(y) = (-)_{t_n}^{\{Q\},\tau}(y) = \sigma_{t_n}^{\{Q\},\tau}(y) \right\} \right] \right) \tag{5.12}$$

where $n \in \{1, \ldots, N/2 - 1\}$. Then the summand in the right-hand side of after mentioned inequality can be estimated from above by

$$\mu_{\Lambda(L)}^{\beta,\tau}(\sigma) E\left[\mu_{A_{n+1}}^{\beta,\sigma_{t_n}^{\{Q\},\tau},+,(+)_{t_n}^{\{Q\},\tau},+}(\eta(x) = 1) - \mu_{A_{n+1}}^{\beta,\sigma_{t_n}^{\{Q\},\tau},+,(-)_{t_n}^{\{Q\},\tau},+}(\eta(x) = 1) \right], \tag{5.13}$$

where E is the expectation over the random configuration $\sigma_{t_n}^{\{Q\},\tau}$. Since the dynamics is reversible with respect to $\mu_{\Lambda(L)}^{\beta,\tau}(\sigma)$, and by the DLR property,

$$\sum_{\sigma \in \Omega_{\Lambda(L)}} \mu_{\Lambda(L)}^{\beta,\tau}(\sigma) E \mu_{A_{n+1}}^{\beta,\sigma_{t_n}^{\{Q\},\tau},+,(+)_{t_n}^{\{Q\},\tau},+}(\eta(x) = 1) \\ \leq \sum_{\sigma \in \Omega_{\Lambda(L)}} \mu_{\Lambda(L)}^{\beta,\tau}(\sigma) \mu_{A_{n+1}}^{\beta,\sigma,+,+,+}(\eta(x) = 1) \\ \leq \sum_{\sigma \in \Omega_{R_{n+1} \cup A_{n+1}}} \mu_{R_{n+1} \cup A_{n+1}}^{\beta,+}(\sigma) \mu_{A_{n+1}}^{\beta,\sigma,+,+,+}(\eta(x) = 1) \\ = \mu_{R_{n+1} \cup A_{n+1}}^{\beta,+}(\eta(x) = 1). \tag{5.14}$$

Similarly we obtain the following:

$$\sum_{\sigma \in \Omega_{\Lambda(L)}} \mu_{\Lambda(L)}^{\beta,\tau}(\sigma) E \mu_{A_{n+1}}^{\beta,\sigma_{t_n}^{\{Q\},\tau},+,(-)_{t_n}^{\{Q\},\tau},+}(\eta(x)=1) \geq \mu_{R_{n+1}\cup A_{n+1}}^{\beta,[0]}(\eta(x)=1). \tag{5.15}$$

By Lemma 3.2, we have

$$\sum_{x \in R_{n+1}\cap A_{n+1}} \left[\mu_{R_{n+1}\cup A_{n+1}}^{\beta,+}(\eta(x)=1) - \mu_{R_{n+1}\cup A_{n+1}}^{\beta,[0]}(\eta(x)=1)\right] \leq (L+1)^2 \exp\{-m(\ln L)\}. \tag{5.16}$$

Thus, for $1 \leq n \leq N/2-1$, we have

$$\begin{gathered} P(F_{n+1}\cap F_n^c) \leq (L+1)^2 \exp\{-m(\ln L)\}, \\ \sum_{n=1}^{N/2-1} P(F_{n+1}\cap F_n^c) \leq \left(\frac{N}{2}-1\right)(L+1)^2 \exp\{-m(\ln L)\}. \end{gathered} \tag{5.17}$$

We can use the same method to estimate $\sum_{n=N/2}^{N-1} P(F_{n+1}\cap F_n^c)$, but in this case, the vertical boundary conditions of B_{N+i} (for $i = 1,\ldots,N/2$, see (4.5)) becomes minus boundary conditions instead of plus boundary conditions. For this case of minus boundary condition, we can get similar results as in the argument above. So we have

$$\sum_{n=N/2}^{N-1} P(F_{n+1}\cap F_n^c) \leq \frac{N}{2}(L+1)^2 \exp\{-m(\ln L)\}. \tag{5.18}$$

Similarly we can estimate $P(F_1)$ in (5.10). Note that, by the definition of $\{Q\}$-dynamics, we have $P(F_{N+1}\cap F_N^c)=0$. Thus, we finally obtain (5.11)

$$q_{N+1} \leq N(L+1)^2 \exp\{-m(\ln L)\}. \tag{5.19}$$

Step 2. In this part, we will use the results of the first step to finish the proof of Theorem 5.1.

Given a sequence $S_{N+1} = \{t_1,\ldots,t_{N+1}\}$ of updatings we say that S_{N+1} is a good sequence if and only if S_{N+1} is ordered and the event F_{N+1}^c occurs at the end of the sequence. Because of (5.11) we know that the probability that an ordered sequence of updatings S_{N+1} is also a good sequence is larger than

$$1-(N+1)(L+1)^2 \exp\{-m(\ln L)\} > \frac{1}{2} \tag{5.20}$$

for L large enough. By [7, Lemma 3.1], for any N large enough (independent of t)

$$P\big(\text{there exists no ordered sequence in } [0,t]\big) \leq \exp\left\{-\frac{tN^{-N}}{2}\right\}. \tag{5.21}$$

Let $T = \exp\{(L \ln L)^{1/2}\}$ and L be large enough, then

$$P(\text{there exists a good sequence in } [0,T]) \geq \frac{1}{3}. \tag{5.22}$$

We conclude by observing that, if there exists a good sequence in $[0,t]$, then, by monotonicity, at the end of the sequence, the configurations $(+)_t^{\{Q\},\tau}$ and $(-)_t^{\{Q\},\tau}$ will be identical. Therefore we can estimate

$$P\left((+)_t^{\{Q\},\tau} \neq (-)_t^{\{Q\},\tau}\right) \leq \left(\frac{2}{3}\right)^{[t/T]} \tag{5.23}$$

which immediately implies that

$$\operatorname{gap}(\{Q\},\tau) \geq T^{-1}\log\left(\frac{3}{2}\right) = \exp\left\{-(L \ln L)^{1/2}\right\}\log\left(\frac{3}{2}\right). \tag{5.24}$$

By Lemma 4.1, we want to estimate the term "$(\sup_{x\in V} \#\{i : V_i \ni x\})^{-1}$", by the construction of covering defined in (a)–(c) of Section 4, we have $(\sup_{x\in V}\#\{i : V_i \ni x\}) \leq 2$, so by Lemma 4.1, (4.6), (5.22) and (5.24), we have

$$\begin{aligned}\operatorname{gap}(\Lambda(L),\beta,\tau) &\geq \frac{1}{2}\inf_i \inf_\varphi \operatorname{gap}\left(L_{V_i}^{\beta,\varphi}\right) \operatorname{gap}(\{Q\},\tau)\\ &\geq \frac{1}{2}(L+1)^{-2}c_m \exp\left\{-4\beta 2k(\beta)(L \ln L)^{1/2}\right\}\exp\left\{-(L \ln L)^{1/2}\right\}\log\left(\frac{3}{2}\right)\\ &\geq \exp\left\{-C\beta(L \ln L)^{1/2}\right\}\end{aligned} \tag{5.25}$$

for some $C > 0$.

For the case that $\tau = \tau_2$, $\tau = \tau_4$, we follow the similar arguments as above. Specifically, for the case that $\tau = \tau_2$, we replace the free boundary condition with $\delta+$ or $\delta-$ boundary conditions, where δ is a small positive constant. Then we use almost the same arguments as in the above proof, we can prove Theorem 5.1 for $\tau = \tau_2$. For the case that $\tau = \tau_4$, by (2.4) and Theorem 3.1, we can get

$$\operatorname{gap}(\Lambda(L),\beta,\tau_4) \geq \exp\left\{-8\beta C(L \ln L)^{1/2}\right\} \operatorname{gap}(\Lambda(L),\beta,\tau') \tag{5.26}$$

where τ' denotes the boundary conditions that on the bottom side of $\Lambda(L)$ is the plus boundary condition and on the other three sides of $\Lambda(L)$ are open boundary conditions. For $\operatorname{gap}(\Lambda(L),\beta,\tau')$, by using the arguments of the present paper, we can prove Theorem 5.1 for

$\tau = \tau_4$. Note that in this case, (4.5) should be changed to be

$$A_i = \left\{ x \in Z^2 : 0 \le x_1 \le L,\ (i-1)\frac{l}{2} \le x_2 \le (i+1)\frac{l}{2} \right\} \tag{5.27}$$

where $l = 2[(L \ln L)^{1/2}]$ and $N = 2L/l - 1$, for $i = 1, \ldots, N$.

Combining the above proofs for boundary conditions τ_2, τ_3, τ_4, these complete the proof of Theorem 5.1. □

6. Conclusion

In the present paper, we estimate the Gibbs measures and the spectral gaps of Ising model with four classes of mixed boundary conditions in a finite square of side $L+1$, in the absence of an external field and at the inverse temperature $\beta > \beta_c$. The results show to which extent boundary conditions can affect the speed at which the stochastic Ising model relaxes to the equilibrium.

Acknowledgments

The author is supported in part by National Natural Science Foundation of China Grant no. 70771006, BJTU Foundation no. 2006XM044, and grant-in-Aid for JSPS fellows no. 00026. The author would like to thank Yasunari Higuchi for his kind support on this research work, and thank the support of the Department of Mathematics, Kobe University of Japan.

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