Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 203248, 11 pages
doi:10.1155/2010/203248

Research Article

A Remark on the Blowup of Solutions to the Laplace Equations with Nonlinear Dynamical Boundary Conditions

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Received 24 April 2010; Revised 19 July 2010; Accepted 7 August 2010

Academic Editor: Zhitao Zhang

We present some sufficient conditions of blowup of the solutions to Laplace equations with semilinear dynamical boundary conditions of hyperbolic type.

1. Introduction

Let Ω be a bounded domain of R^N, $N \geq 1$, with a smooth boundary $\partial\Omega = S = S_1 \cup S_2$, where S_1 and S_2 are closed and disjoint and S_1 possesses positive measure. We consider the following problem:

$$-\Delta u = 0, \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial n} = g(u), \quad \text{on } S_1 \times (0,T), \tag{1.2}$$

$$a\frac{\partial u}{\partial n} + bu = 0, \quad \text{on } S_2 \times (0,T), \tag{1.3}$$

$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad \text{on } S_1, \tag{1.4}$$

where $a \geq 0$, $b \geq 0$, $a + b = 1$, and $k > 0$ are constants, Δ is the Laplace operator with respect to the space variables, and $\partial/\partial n$ is the outer unit normal derivative to boundary S. u_0, u_1 are given initial functions. For convenience, we take $k = 1$ in this paper.

The problem (1.1)–(1.4) can be used as models to describe the motion of a fluid in a container or to describe the displacement of a fluid in a medium without gravity; see [1–5] for more information. In recent years, the problem has attracted a great deal of people. Lions [6] used the theory of maximal monotone operators to solve the existence of solution of the following problem:

$$\Delta u = 0, \quad \text{in } \Omega \times (0, T), \tag{1.5}$$

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial \mathrm{n}} + f(u_t) + |u|^p u = 0, \quad \text{on } S \times (0, T), \tag{1.6}$$

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1, \quad \text{on } S. \tag{1.7}$$

Hintermann [2] used the theory of semigroups in Banach spaces to give the existence and uniqueness of the solution for problem (1.5)–(1.7). Cavalcanti et al. [7–11] studied the existence and asymptotic behavior of solutions evolution problem on manifolds. In this direction, the existence and asymptotic behavior of the related of evolution problem on manifolds has been also considered by Andrade et al. [12, 13], Antunes et al. [14], Araruna et al. [15], and Hu et al. [16]. In addition, Doronin et al. [17] studied a class hyperbolic problem with second-order boundary conditions.

We will consider the blowup of the solution for problem (1.1)–(1.4) with nonlinear boundary source term $g(u)$. Blowup of the solution for problem (1.1)–(1.4) was considered by Kirane [3], when $\partial\Omega = S_1$, by use of Jensen's inequality and Glassey's method [18]. Kirane et al. [19] concerned blowup of the solution for the Laplace equations with a hyperbolic type dynamical boundary inequality by the test function methods. In this paper, we present some sufficient conditions of blowup of the solutions for the problem (1.1)–(1.4) when Ω is a bounded domain and S_2 can be a nonempty set. We use a different approach from those ones used in the prior literature [3, 19].

Another related problem to (1.1)–(1.4) is the following problem:

$$\Delta u = f, \quad \text{in } \Omega \times (0, T), \tag{1.8}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} = g(u), \quad \text{on } S \times (0, T), \tag{1.9}$$

$$u(x, 0) = u_0, \quad \text{on } S. \tag{1.10}$$

Amann and Fila [20], Kirane [3], and Koleva and Vulkov [21] Vulkov [22] considered blowup of the solution of problem (1.8)–(1.10). For more results concerning the related problem (1.8)–(1.10), we refer the reader to [3, 6, 19–31] and their references. In these papers, existence, boundedness, asymptotic behavior, and nonexistence of global solutions for problem (1.8)–(1.10) were studied.

In this paper, the definition of the usually space $H^1(\Omega), H^s(S), L^p(\Omega)$, and $L^p(S)$ can be found in [32] and the norm of $L^2(S)$ is denoted by $\| \bullet \|_S$.

2. Blowup of the Solutions

In this paper, we always assume that the initial data $u_0 \in H^{s+1/2}(S_1)$, $u_1 \in H^s(S_1)$, $s > 1$, and $g \in C$ and that the problem (1.1)–(1.4) possesses a unique local weak solution [2, 3, 6] that is, u is in the class

$$u \in L^\infty\left(0,T;H^{s+1}(\Omega)\right), \qquad u_t \in L^\infty(0,T,H^s(S_1)), \qquad u_{tt} \in L^\infty\left(0,T;L^2(S_1)\right), \tag{2.1}$$

and the boundary conditions are satisfied in the trace sense [2].

Lemma 2.1 (see [33]). *Suppose that $u_t = F(t,u)$, $v_t \geq F(t,v)$, $F \in C$, $t_0 \leq t < +\infty$, $-\infty < u < +\infty$, and $u(t_0) = v(t_0)$. Then, $v(t) \geq u(t)$, $t \geq t_0$.*

Theorem 2.2. *Suppose that $u(x,t)$ is a weak solution of problem* (1.1)–(1.4) *and $g(s)$ satisfies:*

(1) $sg(s) \geq KG(s)$, *where* $K > 2$, $G(s) = \int_0^s g(\rho)d\rho$, $G(s) \geq \beta|s|^{p+1}$, *where* $\beta > 0, p > 1$;

(2) $E_0 = \|u_0\|^2_{S_1} + \|u_1\|^2_{S_1} + (b/a)\|u_0\|_{S_2} - 2\int_{S_1} G(\sigma)d\sigma \leq -2/[(K-2)\beta C_1(p+3)^{-1}]^{2/(p-1)}(1-e^{(1-p)/4})^{4/(p-1)} < 0$

where $C_1 = (mS_1)^{(p+1)/(p-1)}$. Then, the solution of problem (1.1)–(1.4) *blows up in a finite time.*

Proof. Denote

$$E(t) = \|u_t\|^2_{S_1} + \|\nabla u\|^2_{\Omega} + \frac{b}{a}\|u\|_{S_2} - 2\int_{S_1} G(u)\, d\sigma, \tag{2.2}$$

then from (1.1)–(1.4), we have

$$\frac{d}{dt}E(t) = 0, \quad t > 0. \tag{2.3}$$

Hence

$$E(t) = E(0) = E_0. \tag{2.4}$$

Let $H(t) = \|u(t)\|^2_{S_1} + \int_0^t\int_0^\tau \|u(s)\|^2_{S_1} ds\, d\tau$. Using condition (1) of Theorem 2.2, we have

$$\begin{aligned}
\dot{H}(t) &= \frac{d}{dt}H(t) = 2\int_{S_1} uu_t d\sigma + \int_0^t \|u(s)\|^2_{S_1} ds, \\
\ddot{H}(t) &= \frac{d^2}{dt^2}H(t) = 2\int_{S_1} u_t^2 d\sigma + 2\int_{S_1} uu_{tt} d\sigma + \int_{S_1} u^2 d\sigma \\
&= 2\int_{S_1}\left[u_t^2 - u\frac{\partial u}{\partial n} + ug(u) + \frac{1}{2}u^2\right]d\sigma \\
&\geq 2\int_{S_1}\left[u_t^2 - u\frac{\partial u}{\partial n} + KG(u) + \frac{1}{2}u^2\right]d\sigma.
\end{aligned} \tag{2.5}$$

Observing that

$$\int_{S_1} u\frac{\partial u}{\partial n} = \int_\Omega |\nabla u|^2 dx + \frac{b}{a}\int_{S_2} u^2 d\sigma, \tag{2.6}$$

$$K\int_{S_1} G(u)d\sigma = -E_0 + (K-2)\int_{S_1} G(u)d\sigma + \int_{S_1} u_t^2 d\sigma + \frac{b}{a}\int_{S_2} u^2 d\sigma + \int_\Omega |\nabla u|^2 dx, \tag{2.7}$$

we know from (2.5)–(2.7) that

$$\ddot{H}(t) \geq 4\int_{S_1} u_t^2 d\sigma - 2E_0 + \int_{S_1} u^2 d\sigma + 2(K-2)\int_{S_1} G(u)d\sigma \geq -2E_0 + 2(K-2)\beta\int_{S_1} |u|^{p+1} d\sigma. \tag{2.8}$$

It follows from (2.8) that

$$\dot{H}(t) \geq -2E_0 t + 2(K-2)\beta\int_0^t\int_{S_1} |u|^{p+1} d\sigma\, ds + \dot{H}(0), \tag{2.9}$$

$$H(t) \geq -E_0 t^2 + 2(K-2)\beta\int_0^t\int_0^\tau\int_{S_1} |u(s)|^{p+1} d\sigma\, ds\, d\tau + t\dot{H}(0) + H(0), \tag{2.10}$$

where $H(0) = \|u_0\|_{S_1}^2$, $\dot{H}(0) = 2\int_{S_1} u_0 u_1 d\sigma$. From (2.8) and (2.10), we have

$$\begin{aligned}\ddot{H}(t)+H(t) \geq 2(K-2)\beta\left[\int_{S_1} |u|^{p+1} d\sigma + \int_0^t\int_0^\tau\int_{S_1} |u(s)|^{p+1} d\sigma\, ds\, d\tau\right] \\ + t\dot{H}(0) - E_0 t^2 + H(0) - 2E_0.\end{aligned} \tag{2.11}$$

Using the inversion of the Hölder inequality, we obtain

$$\int_{S_1} |u|^{p+1} d\sigma \geq \left(\int_{S_1} |u|^2 d\sigma\right)^{(p+1)/2} (mS_1)^{(1-p)/2}, \tag{2.12}$$

$$\int_0^t\int_0^\tau\int_{S_1} |u(s)|^{p+1} d\sigma\, ds\, d\tau \geq \left(\int_0^t\int_0^\tau\int_{S_1} |u(s)|^2 d\sigma\, ds\, d\tau\right)^{(p+1)/2} \left(\frac{1}{2}t^2 mS_1\right)^{(p-1)/2}. \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.11), we have

$$\begin{aligned}
&\ddot{H}(t) + H(t) \\
&\geq 2(K-2)\beta(mS_1)^{(p+1)/(p-1)} \\
&\quad \times \left[\left(\int_{S_1} |u|^2 d\sigma \right)^{(p+1)/2} \left(\frac{1}{2}t^2 \right)^{(p+1)/(p-1)} \left(\int_0^t \int_0^\tau \int_{S_1} |u(s)|^{p+1} d\sigma\, ds\, d\tau \right)^{2/(p+1)} \right] \\
&\quad + t\dot{H}(0) - E_0 t^2 + H(0) - 2E_0 \\
&\geq 2(K-2)\beta(mS_1)^{(p+1)/(p-1)} \left[\left(\int_{S_1} |u|^2 d\sigma \right)^{(p+1)/2} + \left(\int_0^t \int_0^\tau \int_{S_1} |u(s)|^{p+1} d\sigma\, ds\, d\tau \right)^{(p+1)/2} \right] \\
&\quad + t\dot{H}(0) - E_0 t^2 + H(0) - 2E_0, \quad t \geq 1.
\end{aligned} \tag{2.14}$$

Noticing that

$$(a+b)^n \leq 2^{n-1}(a^n + b^n), \quad a > 0,\ b > 0,\ n > 1, \tag{2.15}$$

we have

$$\ddot{H}(t) + H(t) \geq 2^{(3-p)/2}(K-2)\beta(mS_1)^{(p+1)/(p-1)} H^{(p+1)/2}(t) + t\dot{H}(0) \ - E_0 t^2 + H(0) - 2E_0. \tag{2.16}$$

We see from (2.9) and (2.10) that $\dot{H}(t) \to +\infty, H(t) \to +\infty$ as $t \to +\infty$. Therefore, there is a $t_0 \geq 1$ such that

$$\dot{H}(t) > 0, \quad H(t) > 0, \quad t \geq t_0. \tag{2.17}$$

Multiplying both sides of (2.16) by $2\dot{H}(t)$ and using (2.9), we get

$$\frac{d}{dt}\left[\dot{H}^2(t) + H^2(t)\right] \geq \frac{1}{p+3} 2^{(5-p)/2}(K-2)\beta(mS_1)^{(p+1)/(p-1)} \frac{d}{dt} H^{(p+3)/2}(t) + I(t), \quad t \geq t_0, \tag{2.18}$$

where

$$I(t) = \left(-4E_0 t + 2\dot{H}(0)\right)\left(-E_0 t^2 + \dot{H}(0)t + H(0) - 2E_0\right). \tag{2.19}$$

From (2.18) we have

$$\frac{d}{dt}\left[\dot{H}^2(t) + H^2(t) - C_2 H^{(p+3)/2}(t)\right] \geq I(t), \quad t \geq t_0, \tag{2.20}$$

where $C_2 = (1/(p+3))2^{(5-p)/2}(K-2)\beta(mS_1)^{(p+1)/(p-1)}$. Integrating (2.20) over (t, t_0), we arrive at

$$\dot{H}^2(t) + H^2(t) - C_2 H^{(p+3)/2}(t) \geq \int_{t_0}^{t} I(\tau)d\tau + \dot{H}^2(t_0) + H^2(t_0) - C_2 H^{(p+3)/2}(t_0), \quad t \geq t_0. \tag{2.21}$$

Observe that when $t \to +\infty$, the right-hand side of (2.21) approaches to positive infinity since $I(t) > 0$ for sufficiently large t; hence, there is a $t_1 \geq t_0$ such that the right side of (2.21) is larger than or equal to zero when $t \geq t_1$. We thus have

$$\dot{H}^2(t) + H^2(t) \geq C_2 H^{(p+3)/2}(t), \quad t \geq t_1. \tag{2.22}$$

Extracting the square root of both sides of (2.22) and noticing that $\dot{H}(t)H(t) \geq 0$, we obtain

$$\dot{H}(t) + H(t) \geq C_3 H^{(p+3)/4}(t) \geq C_3 t^{(1-p)/2} H^{(p+3)/4}(t), \quad t \geq t_1, \tag{2.23}$$

since $1-p < 0, t > t_1 > t_0 > 1$, where $C_3 = \sqrt{C_2}$.

Consider the following initial value problem of the Bernoulli equation:

$$\dot{Z} + Z = C_3 t^{(1-p)/2} Z^{(p+3)/4}, \quad t \geq t_1, \quad Z(t_1) = H(t_1). \tag{2.24}$$

Solving the problem (2.24), we obtain the solution

$$\begin{aligned} Z(t) &= e^{-(t-t_1)} \left[H^{(1-p)/4}(t_1) - \frac{p-1}{4} \int_{t_1}^{t} C_3 \tau^{(1-p)/2} e^{((1-p)/4)(\tau - t_1)} d\tau \right]^{4/(1-p)} \\ &= e^{-(t-t_1)} H(t_1) J^{4/(1-p)}(t), \quad t \geq t_1, \end{aligned} \tag{2.25}$$

where $J(t) = (1-(p-1)/4)H^{(p-1)/4}(t_1)C_3 \int_{t_1}^{t} \tau^{(1-p)/2} e^{((1-p)/4)(\tau-t_1)} d\tau$. Obviously, $J(t_1) = 1 > 0$, and for $t > t_1 + 1$

$$\begin{aligned} \delta(t) &= \frac{p-1}{4} H^{(p-1)/4}(t_1) C_3 \int_{t_1}^{t} \tau^{(1-p)/2} e^{((1-p)/4)(\tau-t_1)} d\tau \\ &\geq \frac{p-1}{4} H^{(p-1)/4}(t_1) C_3 \int_{t_1}^{t_1+1} \tau^{(1-p)/2} e^{((1-p)/4)(\tau-t_1)} d\tau \\ &\geq \frac{p-1}{4} H^{(p-1)/4}(t_1) C_3 (t_1+1)^{(1-p)/2} \int_{t_1}^{t_1+1} e^{((1-p)/4)(\tau-t_1)} d\tau \\ &= H^{(p-1)/4}(t_1) C_3 (t_1+1)^{(1-p)/2} \left(1 - e^{(1-p)/4}\right). \end{aligned} \tag{2.26}$$

From (2.10), we see that

$$H^{(p-1)/4}(t)(t+1)^{(1-p)/2} \geq \left[\frac{-E_0t^2+\dot{H}(0)t+H(0)}{t^2+2t+1}\right]^{(p-1)/4} \longrightarrow (-E_0)^{(p-1)/4} \tag{2.27}$$

as $t \longrightarrow +\infty$. Take t_1 sufficiently large such that $H^{(p+1)/4}(t_1)(t_1+1)^{(1-p)/2} \geq 1/2(-E_0)^{(p-1)/4}$. It follows from (2.26) and the condition of Theorem 2.2 that

$$\delta(t) \geq \frac{1}{2}(-E_0)^{(p-1)/4}C_3\left(1-e^{(1-p)/4}\right) \geq 1, \quad t \geq t_1+1. \tag{2.28}$$

Therefore,

$$J(t) = 1-\delta(t) \leq 0, \quad t \geq t_1+1. \tag{2.29}$$

By virtue of the continuity of $J(t)$ and the theorem of the intermediate values, there is a constant $t_1 < \widetilde{T} \leq t_1+1$ such that $J(\widetilde{T}) = 0$. Hence, $Z(t) \longrightarrow +\infty$ as $t \longrightarrow \widetilde{T}^-$. It follows from Lemma 2.1 that $H(t) \geq Z(t)$, $t \geq t_1$. Thus, $H(t) \longrightarrow +\infty$ as $t \longrightarrow \widetilde{T}^-$. The theorem is proved. □

Theorem 2.3. *Suppose that $g(s)$ is a convex function, $g(0)=0, g(s) \geq ls^p$, where a is a real number $p>1$, and $u(x,t)$ is a weak solution of problem* (1.1)–(1.4)

$$\int_{S_1} u_0(\sigma)\psi_1(\sigma)d\sigma = \alpha \geq \left(\frac{\lambda_1}{l}^{1/(p-1)}\right) > 0, \qquad \int_{S_1} u_1(\sigma)\psi_1(\sigma)d\sigma = \beta > 0, \tag{2.30}$$

where ψ_1 is the normalized eigenfunction (i.e., $\psi_1 \geq 0, \int_{S_1}\psi_1(\sigma)d\sigma = 1$) corresponding the smallest eigenvalue $\lambda_1 > 0$ of the following Steklov spectral problem [23]:

$$\Delta\psi = 0, \quad in\ \Omega, \tag{2.31}$$

$$\frac{\partial\psi}{\partial n} = \lambda\psi, \quad on\ S_1, \tag{2.32}$$

$$a\frac{\partial\psi}{\partial n} + b\psi = 0, \quad on\ S_2, \tag{2.33}$$

where $\Omega, S_1, S_2, k, a, b$ are defined as in Section 1. Then, the solution of problem (1.1)–(1.4) *blows up in a finite time.*

Proof. Let

$$y(t) = \int_{S_1} u(\sigma,t)\psi_1(\sigma)d\sigma. \tag{2.34}$$

Then, $y(0) = y_0 = \alpha > 0$, $y_t(0) = y_1 = \beta > 0$. It follows from (1.1)–(1.4) that $y(t)$ satisfies

$$y_{tt} = -\int_{S_1} \frac{\partial u}{\partial n}\psi_1 d\sigma + \int_{S_1} g(u)\psi_1 d\sigma. \tag{2.35}$$

Using Green's formula, we have

$$\begin{aligned}
0 &= \int_{\Omega} \Delta u \psi_1 dx = \int_{S} \frac{\partial u}{\partial n}\psi_1 d\sigma - \int_{\Omega} \nabla u \cdot \nabla \psi_1 dx \\
&= \int_{S} \frac{\partial u}{\partial n}\psi_1 d\sigma - \int_{S} u\frac{\partial \psi_1}{\partial n} d\sigma + \int_{\Omega} u \Delta \psi_1 dx \\
&= \left(\int_{S_1} \frac{\partial u}{\partial n}\psi_1 d\sigma - \int_{S_1} u\frac{\partial \psi_1}{\partial n} d\sigma\right) + \left(\int_{S_2} \frac{\partial u}{\partial n}\psi_1 d\sigma - \int_{S_2} u\frac{\partial \psi_1}{\partial n} d\sigma\right) + \int_{\Omega} u \Delta \psi_1 dx \\
&= B_1 + B_2,
\end{aligned} \tag{2.36}$$

where we have used (2.31) and the fact that ψ_1 is the eigenfunction of the problem (1.1)–(1.4), B_1 and B_2 are denoted as the expressions in the first and the second parenthesis, respectively. From (2.32), we have

$$B_1 = \int_{S_1} \frac{\partial u}{\partial n}\psi_1 d\sigma - \lambda_1 \int_{S_1} u\psi_1 d\sigma. \tag{2.37}$$

If $a = 0$, it is clear that $B_2 = 0$ otherwise, by (1.3) and (2.33),

$$B_2 = \int_{S_2} \left(-\frac{b}{a}u\right)\psi_1 d\sigma - \int_{S_2} u\left(-\frac{b}{a}\psi_1\right) d\sigma = 0. \tag{2.38}$$

Therefore, (2.36) implies that $B_1 = 0$, that is,

$$\int_{S_1} \frac{\partial u}{\partial n}\psi_1 d\sigma = \lambda_1 \int_{S_1} u\psi_1 d\sigma = \lambda_1 y(t). \tag{2.39}$$

Now, (2.35) takes the form

$$y_{tt} = -\lambda_1 y + \int_{S_1} g(u)\psi_1 d\sigma. \tag{2.40}$$

From Jensen's inequality and the condition $g(s) \geq ls^p$, we have

$$\int_{S_1} g(u)\psi_1 d\sigma \geq g\left(\int_{S_1} u\psi_1 d\sigma\right) \geq l y^p. \tag{2.41}$$

Substituting the above inequality into (2.40), we get

$$y_{tt} + \lambda_1 y \geq l y^p, \quad t > 0. \tag{2.42}$$

Since $y(0) = \alpha > 0$, $y_t(0) = \beta > 0$, from the continuity of $y(t)$, it follows that there is a right neighborhood $(0, \delta)$ of the point $t = 0$, in which $\dot{y}(t) > 0$, and hence $y(t) > y_0 > 0$. If there exists a point t_0 such that $\dot{y}(t) > 0(t \in [0, t_0))$, but $\dot{y}(t_0) = 0$, then $y(t)$ is monotonically increasing on $[0, t_0]$. It follows from (2.42) that on $(0, t_0]$

$$y_{tt} \geq y\left(l y^{p-1} - \lambda_1\right) \geq y_0\left(l y_0^{p-1} - \lambda_1\right) \geq 0, \tag{2.43}$$

and thus $y_t(t)$ is monotonically increasing on $[0, t_0]$. This contradicts $\dot{y}(t_0) = 0$. Therefore, $\dot{y}(t) > 0$ and hence $y(t) > y_0$ as $t > 0$.

Multiplying both sides of (2.42) by $2y_t$ and integrating the product over $[0, t]$, we get

$$y_t^2 \geq \frac{2l}{p+1}\left(y^{p+1} - y_0^{p+1}\right) - \lambda_1\left(y^2 - y_0^2\right) + y_1^2 = B(y). \tag{2.44}$$

Since $B(y_0) = y_1^2 > 0$ and

$$B'(y) = 2l y^p - 2\lambda_1 y > 2y_0\left(l y_0^{p-1} - \lambda_1\right) \geq 0, \tag{2.45}$$

then $B(y) > B(y_0) > 0, ct > 0$. Extracting the square root of both sides of (2.44), we have

$$y_t \geq \left[\frac{2l}{p+1}\left(y^{p+1} - y_0^{p+1}\right) - \lambda_1\left(y^2 - y_0^2\right) + y_1^2\right]^{-1/2}, \quad t > 0. \tag{2.46}$$

Equation (2.46) means that the interval $[0, \widetilde{T}]$ of the existence of $y(t)$ is finite this, that is,

$$\overline{T} \leq \int_{y_0}^{+\infty} \left[\frac{2l}{p+1}\left(y^{p+1} - \alpha^{p+1}\right) - \lambda_1\left(y^2 - \alpha^2\right) + \beta^2\right]^{1/2} ds < +\infty, \tag{2.47}$$

and $y(t) \to +\infty$ as $t \to \widetilde{T}^-$. The theorem is proved. □

Remark 2.4. The results of the above theorem hold when one considers (1.1)–(1.4) with more general elliptic operator, like

$$Lu \equiv -\operatorname{div}(k(x)\nabla u) + c(x)u, \quad 0 < k_0 \leq k(x) \leq k_1, \ c(x) \geq 0, \quad \text{in } \Omega \times (0, T), \tag{2.48}$$

and the corresponding boundary conditions

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + k(x)\frac{\partial u}{\partial n} &= g(u), \quad \text{on } S_1 \times (0,T), \\
k(x)\frac{\partial u}{\partial n} + bu = 0, \quad b(x) &\geq 0, \quad \text{on } S_2 \times (0,T).
\end{aligned} \tag{2.49}$$

Acknowledgments

The authors are very grateful to the referee's suggestions and comments. The authors are supported by National Natural Science Foundation of China and Foundation of Henan University of Technology.

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