

Research Article

Positive Solutions of Singular Complementary Lidstone Boundary Value Problems

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We investigate the existence of positive solutions of singular problem $(-1)^m x^{(2m+1)} = f(t, x, \dots, x^{(2m)})$, $x(0) = 0$, $x^{(2i-1)}(0) = x^{(2i-1)}(T) = 0$, $1 \leq i \leq m$. Here, $m \geq 1$ and the Carathéodory function $f(t, x_0, \dots, x_{2m})$ may be singular in all its space variables x_0, \dots, x_{2m} . The results are proved by regularization and sequential techniques. In limit processes, the Vitali convergence theorem is used.

1. Introduction

Let T be a positive constant, $J = [0, T]$ and $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We consider the singular complementary Lidstone boundary value problem

$$(-1)^m x^{(2m+1)}(t) = f(t, x(t), \dots, x^{(2m)}(t)), \quad m \geq 1, \tag{1.1}$$

$$x(0) = 0, \quad x^{(2i-1)}(0) = x^{(2i-1)}(T) = 0, \quad 1 \leq i \leq m, \tag{1.2}$$

where f satisfies the local Carathéodory function on $J \times \mathfrak{D}$ ($f \in \text{Car}(J \times \mathfrak{D})$) with

$$\mathfrak{D} = \begin{cases} \underbrace{\mathbb{R}_+^2 \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \mathbb{R}_+ \times \dots \times \mathbb{R}_+ \times \mathbb{R}_0}_{4k-1} & \text{if } m = 2k - 1, \\ \underbrace{\mathbb{R}_+^2 \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \mathbb{R}_+ \times \dots \times \mathbb{R}_- \times \mathbb{R}_0}_{4k+1} & \text{if } m = 2k. \end{cases} \tag{1.3}$$

The function $f(t, x_0, \dots, x_{2m})$ is positive and may be singular at the value zero of all its space variables x_0, \dots, x_{2m} .

Let $i \in \{0, 1, \dots, 2m\}$. We say that f is *singular at the value zero of its space variable x_i* if for a.e. $t \in J$ and all $x_j, 0 \leq j \leq 2m, j \neq i$ such that $(x_0, \dots, x_i, \dots, x_{2m}) \in \mathfrak{D}$, the relation

$$\lim_{x_i \rightarrow 0} f(t, x_0, \dots, x_i, \dots, x_{2m}) = \infty \quad (1.4)$$

holds.

A function $x \in AC^{2m}(J)$ (i.e., x has absolutely continuous $2m$ th derivative on J) is a *positive solution of problem (1.1), (1.2)* if $x(t) > 0$ for $t \in (0, T]$, x satisfies the boundary conditions (1.2) and (1.1) holds a.e. on J .

The regular complementary Lidstone problem

$$\begin{aligned} (-1)^m x^{(2m+1)}(t) &= h(t, x(t), \dots, x^{(q)}(t)), \quad m \geq 1, q \text{ fixed}, 0 \leq q \leq 2m, \\ x(0) &= \alpha_0, \quad x^{(2i-1)}(0) = \alpha_i, \quad x^{(2i-1)}(1) = \beta_i, \quad 1 \leq i \leq m \end{aligned} \quad (1.5)$$

was discussed in [1]. Here, $h : [0, 1] \times \mathbb{R}^{q+1} \rightarrow \mathbb{R}$ is continuous at least in the interior of the domain of interest. Existence and uniqueness criteria for problem (1.5) are proved by the complementary Lidstone interpolating polynomial of degree $2m$. No contributions exist, as far as we know, concerning the existence of positive solutions of singular complementary Lidstone problems.

We observe that differential equations in complementary Lidstone problems as well as derivatives in boundary conditions are odd orders, in contrast to the Lidstone problem

$$\begin{aligned} (-1)^m x^{(2m)}(t) &= p(t, x(t), \dots, x^{(r)}(t)), \quad m \geq 1, r \text{ fixed}, 0 \leq r \leq 2m-1, \\ x^{(2i)}(0) &= a_i, \quad x^{(2i)}(1) = b_i, \quad 1 \leq i \leq m-1, \end{aligned} \quad (1.6)$$

where the differential equation and derivatives in the boundary conditions are even orders. For $a_i = b_i = 0$ ($1 \leq i \leq m-1$), regular Lidstone problems were discussed in [2–9], while singular ones in [10–15].

The aim of this paper is to give the conditions on the function f in (1.1) which guarantee that the singular problem (1.1), (1.2) has a solution. The existence results are proved by regularization and sequential techniques, and in limit processes, the Vitali convergence theorem [16, 17] is applied.

Throughout the paper, $\|x\|_\infty = \max\{|x(t)| : t \in J\}$ and $\|x\|_{C^n} = \sum_{k=0}^n \|x^{(k)}\|_\infty$, $n \geq 1$ stands for the norm in $C^0(J)$ and $C^n(J)$, respectively. $L^1(J)$ denotes the set of functions (Lebesgue) integrable on J and $\text{meas } \mathcal{M}$ the Lebesgue measure of $\mathcal{M} \subset J$.

We work with the following conditions on the function f in (1.1).

(H_1) $f \in \text{Car}(J \times \mathfrak{D})$ and there exists $a \in (0, \infty)$ such that

$$a \leq f(t, x_0, \dots, x_{2m}), \quad (1.7)$$

for a.e. $t \in J$ and each $(x_0, \dots, x_{2m}) \in \mathfrak{D}$.

(H₂) For a.e. $t \in J$ and for all $(x_0, \dots, x_{2m}) \in \mathfrak{D}$, the inequality

$$f(t, x_0, \dots, x_{2m}) \leq h\left(t, \sum_{j=0}^{2m} |x_j|\right) + \sum_{j=0}^{2m} \omega_j(|x_j|) \quad (1.8)$$

is fulfilled, where $h \in \text{Car}(J \times [0, \infty))$ is positive and nondecreasing in the second variable, $\omega_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is nonincreasing, $0 \leq j \leq 2m$,

$$\begin{aligned} \limsup_{v \rightarrow \infty} \frac{1}{v} \int_0^T h(t, Kv) dt < 1, \quad K = \begin{cases} \frac{T^{2m+1} - 1}{T - 1} & \text{if } T \neq 1, \\ 2m + 1 & \text{if } T = 1, \end{cases} \\ \int_0^1 \omega_{2j}(s^2) ds < \infty, \quad \int_0^1 \omega_{2j+1}(s) ds < \infty \quad \text{if } 0 \leq j \leq m-1, \\ \int_0^1 \omega_{2m}(s) ds < \infty. \end{aligned} \quad (1.9)$$

The paper is organized as follows. In Section 2, we construct a sequence of auxiliary regular differential equations associated with (1.1). Section 3 is devoted to the study of auxiliary regular complementary Lidstone problems. We show that the solvability of these problems is reduced to the existence of a fixed point of an operator \mathcal{A} . The existence of a fixed point of \mathcal{A} is proved by a fixed point theorem of cone compression type according to Guo-Krasnosel'skii [18, 19]. The properties of solutions to auxiliary problems are also investigated here. In Section 4, applying the results of Section 3, the existence of a positive solution of the singular problem (1.1), (1.2) is proved.

2. Regularization

Let m be from (1.1). For $n \in \mathbb{N}$, define $\chi_n, \varphi_n, \tau_{n,m} \in C^0(\mathbb{R})$, $\mathbb{R}_n \subset \mathbb{R}$, and $\mathfrak{D}_n \subset \mathbb{R}^{2m+1}$ by the formulas

$$\begin{aligned} \chi_n(u) &= \begin{cases} u & \text{for } u \geq \frac{1}{n}, \\ \frac{1}{n} & \text{for } u < \frac{1}{n}, \end{cases} & \varphi_n(u) &= \begin{cases} -\frac{1}{n} & \text{for } u > -\frac{1}{n}, \\ u & \text{for } u \leq -\frac{1}{n}, \end{cases} \\ \tau_{n,m} &= \begin{cases} \chi_n & \text{if } m = 2k-1, \\ \varphi_n & \text{if } m = 2k, \end{cases} & \mathbb{R}_n &= \left(-\infty, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, \infty\right), \\ \mathfrak{D}_n &= \underbrace{\mathbb{R}^2 \times \mathbb{R}_n \times \mathbb{R} \times \mathbb{R}_n \times \mathbb{R} \times \dots \times \mathbb{R} \times \mathbb{R}_n}_{2m+1}. \end{aligned} \quad (2.1)$$

Let $f \in \text{Car}(J \times \mathfrak{D})$. Chose $n \in \mathbb{N}$ and put

$$\begin{aligned} f_n^*(t, x_0, x_1, x_2, x_3, x_4, \dots, x_{2m-1}, x_{2m}) \\ = f(t, \chi_n(x_0), \chi_n(x_1), x_2, \varphi_n(x_3), x_4, \dots, \tau_{n,m}(x_{2m-1}), x_{2m}) \end{aligned} \quad (2.2)$$

for $(t, x_0, x_1, x_2, x_3, x_4, \dots, x_{2m-1}, x_{2m}) \in J \times \mathfrak{D}_n$. Now, define an auxiliary function f_n by means of the following recurrence formulas:

$$\begin{aligned} f_{n,0}(t, x_0, x_1, \dots, x_{2m}) &= f_n^*(t, x_0, x_1, \dots, x_{2m}) \quad \text{for } (t, x_0, x_1, \dots, x_{2m}) \in J \times \mathfrak{D}_n, \\ f_{n,i}(t, x_0, x_1, \dots, x_{2m}) \\ &= \begin{cases} f_{n,i-1}(t, x_0, x_1, \dots, x_{2m}) & \text{if } |x_{2i}| \geq \frac{1}{n}, \\ \frac{n}{2} \left[f_{n,i-1} \left(t, x_0, \dots, x_{2i-1}, \frac{1}{n}, x_{2i+1}, \dots, x_{2m} \right) \left(x_{2i} + \frac{1}{n} \right) \right. \\ \quad \left. - f_{n,i-1} \left(t, x_0, \dots, x_{2i-1}, -\frac{1}{n}, x_{2i+1}, \dots, x_{2m} \right) \left(x_{2i} - \frac{1}{n} \right) \right] & \text{if } |x_{2i}| < \frac{1}{n}, \end{cases} \end{aligned} \quad (2.3)$$

for $1 \leq i \leq m$, and

$$f_n(t, x_0, x_1, \dots, x_{2m}) = f_{n,m}(t, x_0, x_1, \dots, x_{2m}) \quad \text{for } (t, x_0, x_1, \dots, x_{2m}) \in J \times \mathbb{R}^{2m+1}. \quad (2.4)$$

Then, under condition (H_1) , $f_n \in \text{Car}(J \times \mathbb{R}^{2m+1})$ and

$$a \leq f_n(t, x_0, x_1, \dots, x_{2m}) \quad \text{for a.e. } t \in J \text{ and all } (x_0, x_1, \dots, x_{2m}) \in \mathbb{R}^{2m+1}. \quad (2.5)$$

Condition (H_2) gives

$$\begin{aligned} f_n(t, x_0, x_1, \dots, x_{2m}) &\leq h \left(t, 2m+1 + \sum_{j=0}^{2m} |x_j| \right) + \sum_{j=0}^{2m} (\omega_j(|x_j|) + \omega_j(1)), \\ &\text{for a.e. } t \in J \text{ and all } (x_0, x_1, \dots, x_{2m}) \in \mathbb{R}_0^{2m+1}, \end{aligned} \quad (2.6)$$

$$\begin{aligned} f_n(t, x_0, x_1, \dots, x_{2m}) &\leq h \left(t, 2m+1 + \sum_{j=0}^{2m} |x_j| \right) + \sum_{j=0}^{2m} \omega_j \left(\frac{1}{n} \right), \\ &\text{for a.e. } t \in J \text{ and all } (x_0, x_1, \dots, x_{2m}) \in \mathbb{R}^{2m+1}. \end{aligned} \quad (2.7)$$

We investigate the regular differential equation

$$(-1)^m x^{(2m+1)}(t) = f_n(t, x(t), \dots, x^{(2m)}(t)). \quad (2.8)$$

If a function $x \in AC^{2m}(J)$ satisfies (2.8) for a.e. $t \in J$, then x is called a solution of (2.8).

3. Auxiliary Regular Problems

Let $j \in \mathbb{N}$ and denote by $G_j(t, s)$ the Green function of the problem

$$x^{(2j)}(t) = 0, \quad x^{(2i)}(0) = x^{(2i)}(T) = 0, \quad 0 \leq i \leq j - 1. \quad (3.1)$$

Then,

$$G_1(t, s) = \begin{cases} \frac{s}{T}(t - T) & \text{for } 0 \leq s \leq t \leq T, \\ \frac{t}{T}(s - T) & \text{for } 0 \leq t \leq s \leq T. \end{cases} \quad (3.2)$$

By [2, 3, 20], the Green function G_j can be expressed as

$$G_j(t, s) = \int_0^T G_1(t, \tau) G_{j-1}(\tau, s) d\tau, \quad j > 1, \quad (3.3)$$

and it is known that (see, e.g., [3, 20])

$$(-1)^j G_j(t, s) > 0 \quad \text{for } (t, s) \in (0, T) \times (0, T), \quad j \geq 1. \quad (3.4)$$

Lemma 3.1 (see [10, Lemmas 2.1 and 2.3]). *For $(t, s) \in J \times J$ and $j \in \mathbb{N}$, the inequalities*

$$(-1)^j G_j(t, s) \leq \frac{T^{2j-3}}{6^{j-1}} s(T - s), \quad (3.5)$$

$$(-1)^j G_j(t, s) \geq \frac{T^{2j-5}}{30^{j-1}} ts(T - t)(T - s) \quad (3.6)$$

hold.

Let $\gamma \in L^1(J)$ and let $u \in AC^{2m-1}(J)$ be a solution of the differential equation

$$(-1)^m u^{(2m)}(t) = \gamma(t), \quad (3.7)$$

satisfying the Lidstone boundary conditions

$$u^{(2i)}(0) = u^{(2i)}(T) = 0, \quad 0 \leq i \leq m - 1. \quad (3.8)$$

It follows from the definition of the Green function G_j that

$$(-1)^j u^{(2j)}(t) = (-1)^{m-j} \int_0^T G_{m-j}(t, s) \gamma(s) ds \quad \text{for } t \in J, \quad 0 \leq j \leq m - 1. \quad (3.9)$$

It is easy to check that $x \in AC^{2m}(J)$ is a solution of problem (2.8), (1.2) if and only if $x(0) = 0$, and its derivative x' is a solution of a problem involving the functional differential equation

$$(-1)^m u^{(2m)}(t) = f_n \left(t, \int_0^t u(s) ds, u(t), \dots, u^{(2m-1)}(t) \right) \quad (3.10)$$

and the Lidstone boundary conditions (3.8). From (3.9) (for $j = 0$), we see that $u \in AC^{2m-1}(J)$ is a solution of problem (3.10), (3.8) exactly if it is a solution of the equation

$$u(t) = (-1)^m \int_0^T G_m(t, s) f_n \left(s, \int_0^s u(\tau) d\tau, u(s), \dots, u^{(2m-1)}(s) \right) ds, \quad (3.11)$$

in the set $C^{2m-1}(J)$. Consequently, x is a solution of problem (2.8), (1.2) if and only if it is a solution of the equation

$$x(t) = (-1)^m \int_0^t \left(\int_0^T G_m(s, \tau) f_n \left(\tau, x(\tau), \dots, x^{(2m)}(\tau) \right) d\tau \right) ds, \quad (3.12)$$

in the set $C^{2m}(J)$. It means that x is a solution of problem (2.8), (1.2) if x is a fixed point of the operator $\mathcal{L} : C^{2m}(J) \rightarrow C^{2m}(J)$ defined as

$$(\mathcal{L}x)(t) = (-1)^m \int_0^t \left(\int_0^T G_m(s, \tau) f_n \left(\tau, x(\tau), \dots, x^{(2m)}(\tau) \right) d\tau \right) ds. \quad (3.13)$$

We prove the existence of a fixed point of \mathcal{L} by the following fixed point result of cone compression type according to Guo-Krasnosel'skii (see, e.g., [18, 19]).

Lemma 3.2. *Let X be a Banach space, and let $P \subset X$ be a cone in X . Let Ω_1, Ω_2 be bounded open balls of X centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $\mathcal{F} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow P$ is completely continuous operator such that*

$$\|\mathcal{F}x\| \geq \|x\| \quad \text{for } x \in P \cap \partial\Omega_1, \quad \|\mathcal{F}x\| \leq \|x\| \quad \text{for } x \in P \cap \partial\Omega_2 \quad (3.14)$$

holds. Then, \mathcal{F} has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We are now in the position to prove that problem (2.8), (1.2) has a solution.

Lemma 3.3. *Let (H_1) and (H_2) hold. Then, problem (2.8), (1.2) has a solution.*

Proof. Let the operator $\mathcal{L} : C^{2m}(J) \rightarrow C^{2m}(J)$ be given in (3.13), and let

$$P = \left\{ x \in C^{2m}(J) : x(t) \geq 0 \text{ for } t \in J \right\}. \quad (3.15)$$

Then, P is a cone in $C^{2m}(J)$ and since $(-1)^m G_m(t, s) > 0$ for $(t, s) \in (0, T) \times (0, T)$ by (3.4) and f_n satisfies (2.5), we see that $\mathcal{L} : C^{2m}(J) \rightarrow P$. The fact that \mathcal{L} is a completely continuous

operator follows from $f_n \in \text{Car}(J \times \mathbb{R}^{2m+1})$, from Lebesgue dominated convergence theorem, and from the Arzelà-Ascoli theorem.

Choose $x \in P$ and put $y(t) = (\mathcal{A}x)(t)$ for $t \in J$. Then, (cf. (2.5))

$$(-1)^m y^{(2m+1)}(t) = f_n(t, x(t), \dots, x^{(2m)}(t)) \geq a > 0 \quad \text{for a.e. } t \in J. \quad (3.16)$$

Since $y(0) = 0$ and $y^{(2i-1)}(0) = y^{(2i-1)}(T) = 0$ for $1 \leq i \leq m$, the equality $y^{(j)}(\xi_j) = 0$ holds with some $\xi_j \in J$ for $0 \leq j \leq 2m$. We now use the equality $y^{(2m)}(\xi_{2m}) = 0$ and have

$$\left| y^{(2m)}(t) \right| = \left| \int_{\xi_{2m}}^t y^{(2m+1)}(s) ds \right| \geq a|t - \xi_{2m}| \quad \text{for } t \in J. \quad (3.17)$$

Hence, $\|y^{(2m)}\|_\infty \geq aT/2$, and so

$$\|\mathcal{A}x\|_{C^{2m}} > \frac{aT}{2}. \quad (3.18)$$

Next, we deduce from the relation

$$\left| y^{(2m)}(t) \right| = \left| \int_{\xi_{2m}}^t f_n(s, x(s), \dots, x^{(2m)}(s)) ds \right| \leq \int_0^T f_n(s, x(s), \dots, x^{(2m)}(s)) ds \quad (3.19)$$

and from (2.7) that

$$\left| y^{(2m)}(t) \right| \leq \int_0^T h(s, 2m+1 + \|x\|_{C^{2m}}) ds + T \sum_{j=0}^{2m} \omega_j \left(\frac{1}{n} \right) \quad \text{for } t \in J. \quad (3.20)$$

Therefore,

$$\left\| y^{(2m)} \right\|_\infty \leq \int_0^T h(s, 2m+1 + \|x\|_{C^{2m}}) ds + V, \quad (3.21)$$

where $V = T \sum_{j=0}^{2m} \omega_j(1/n)$. Since $y^{(j)}(\xi_j) = 0$ for $0 \leq j \leq 2m$, we have

$$\left\| y^{(j)} \right\|_\infty \leq T^{2m-j} \left\| y^{(2m)} \right\|_\infty, \quad 0 \leq j \leq 2m. \quad (3.22)$$

The last inequality together with (3.21) gives

$$\|y\|_{C^{2m}} \leq K \left\| y^{(2m)} \right\|_\infty \leq K \left(\int_0^T h(s, 2m+1 + \|x\|_{C^{2m}}) ds + V \right), \quad (3.23)$$

where K is from (H_2) . Since $x \in P$ is arbitrary, relations (3.18) and (3.21) imply that for all

$x \in P$, inequalities (3.18) and

$$\|\mathcal{L}x\|_{C^{2m}} \leq K \left(\int_0^T h(s, 2m+1 + \|x\|_{C^{2m}}) ds + V \right) \quad (3.24)$$

hold. By (H_2) , there exists $C > 0$ such that

$$\frac{1}{v} \left(\int_0^T h(s, 2m+1 + Kv) ds + V \right) \leq 1 \quad \forall v \geq \frac{C}{K}, \quad (3.25)$$

and therefore,

$$K \left(\int_0^T h(s, 2m+1 + v) ds + V \right) \leq v \quad \forall v \geq C. \quad (3.26)$$

Let

$$\Omega_1 = \left\{ x \in C^{2m}(J) : \|x\|_{C^{2m}} < \frac{aT}{2} \right\}, \quad \Omega_2 = \left\{ x \in C^{2m}(J) : \|x\|_{C^{2m}} < C \right\}. \quad (3.27)$$

Then, it follows from (3.18), (3.24), and (3.26) that

$$\|\mathcal{L}x\|_{C^{2m}} \geq \|x\|_{C^{2m}} \quad \text{for } x \in P \cap \partial\Omega_1, \quad \|\mathcal{L}x\|_{C^{2m}} \leq \|x\|_{C^{2m}} \quad \text{for } x \in P \cap \partial\Omega_2. \quad (3.28)$$

The conclusion now follows from Lemma 3.2 (for $X = C^{2m}(J)$ and $\mathcal{F} = \mathcal{L}$). \square

The properties of solutions to problem (2.8), (1.2) are collected in the following lemma.

Lemma 3.4. *Let (H_1) and (H_2) be satisfied. Let x_n be a solution of problem (2.8), (1.2). Then, for all $n \in \mathbb{N}$, the following assertions hold:*

- (i) $(-1)^j x_n^{(2j+1)}(t) > 0$ for $t \in (0, T)$, $0 \leq j \leq m-1$, and $(-1)^m x_n^{(2m+1)}(t) \geq a$ for a.e. $t \in J$,
- (ii) x_n is increasing on J , and for $0 \leq j \leq m-1$, $(-1)^j x_n^{(2j+2)}$ is decreasing on J , and there is a unique $\xi_{j,n} \in (0, T)$ such that $x_n^{(2j+2)}(\xi_{j,n}) = 0$,
- (iii) there exists a positive constant A such that

$$\begin{aligned} |x_n^{(2m)}(t)| &\geq A|t - \xi_{m-1,n}|, \\ |x_n^{(2j+2)}(t)| &\geq A(t - \xi_{j,n})^2 \quad \text{if } 0 \leq j \leq m-2, \\ |x_n^{(2j+1)}(t)| &\geq At(T-t) \quad \text{if } 0 \leq j \leq m-1, \\ x_n(t) &\geq At^2, \end{aligned} \quad (3.29)$$

for $t \in J$,

- (iv) the sequence $\{x_n\}$ is bounded in $C^{2m}(J)$.

Proof. Let us choose an arbitrary $n \in \mathbb{N}$. By (2.5),

$$(-1)^m x_n^{(2m+1)}(t) = f_n(t, x_n(t), \dots, x_n^{(2m)}(t)) \geq a \quad \text{for a.e. } t \in J, \quad (3.30)$$

and it follows from the definition of the Green function G_j that the equality

$$(-1)^j x_n^{(2j+1)}(t) = (-1)^{m-j} \int_0^T G_{m-j}(t, s) f_n(s, x_n(s), \dots, x_n^{(2m)}(s)) ds \quad (3.31)$$

holds for $t \in J$ and $0 \leq j \leq m-1$. Now, using (1.2), (3.4), (3.30), and (3.31), we see that assertion (i) is true. Hence, $(-1)^j x_n^{(2j+2)}$ is decreasing on J for $0 \leq j \leq m-1$ and x_n is increasing on this interval. Due to $x_n^{(2i-1)}(0) = x_n^{(2i-1)}(T) = 0$ for $1 \leq i \leq m$, there exists a unique $\xi_{j,n} \in (0, T)$ such that $x_n^{(2j+2)}(\xi_{j,n}) = 0$ for $0 \leq j \leq m-1$. Consequently, assertion (ii) holds.

Next, in view of (2.5), (3.6), and (3.31),

$$\begin{aligned} \left| x_n^{(2j+1)}(t) \right| &\geq \frac{T^{2(m-j)-5} a}{30^{m-j-1}} t(T-t) \int_0^T s(T-s) ds \\ &= \frac{T^{2(m-j)-2} a}{6 \cdot 30^{m-j-1}} t(T-t) \quad \text{for } t \in J, 0 \leq j \leq m-1. \end{aligned} \quad (3.32)$$

Since

$$x_n^{(2j+2)}(t) = \int_{\xi_{j,n}}^t x_n^{(2j+3)}(s) ds \quad (3.33)$$

and, by [13, Lemma 6.2],

$$\left| \int_{\xi_{j,n}}^t s(T-s) ds \right| \geq \frac{T}{6} (t - \xi_{j,n})^2, \quad (3.34)$$

we have

$$\left| x_n^{(2j+2)}(t) \right| \geq \frac{T^{2(m-j)-3} a}{36 \cdot 30^{m-j-2}} (t - \xi_{j,n})^2 \quad \text{for } t \in J, 0 \leq j \leq m-2. \quad (3.35)$$

Furthermore,

$$\left| x_n^{(2m)}(t) \right| = \left| \int_{\xi_{m-1,n}}^t f_n(s, x_n(s), \dots, x_n^{(2m)}(s)) ds \right| \geq a |t - \xi_{m-1,n}|, \quad t \in J, \quad (3.36)$$

and (cf. (3.32) for $j = 0$)

$$\begin{aligned} x_n(t) &= \int_0^t x'_n(s) ds \geq \frac{T^{2m-2}a}{6 \cdot 30^{m-1}} \int_0^t s(T-s) ds \\ &= \frac{T^{2m-2}a}{36 \cdot 30^{m-1}} t^2(3T-2t) \geq \frac{T^{2m-1}a}{36 \cdot 30^{m-1}} t^2 \quad \text{for } t \in J, \end{aligned} \quad (3.37)$$

since $x'_n > 0$ on $(0, T)$ by assertion (ii). Let

$$A = a \cdot \min \left\{ 1, A_1, A_2, \frac{T^{2m-1}}{36 \cdot 30^{m-1}} \right\}, \quad (3.38)$$

where

$$\begin{aligned} A_1 &= \min \left\{ \frac{T^{2(m-j)-2}}{6 \cdot 30^{m-j-1}} : 0 \leq j \leq m-1 \right\}, \\ A_2 &= \min \left\{ \frac{T^{2(m-j)-3}}{36 \cdot 30^{m-j-2}} : 0 \leq j \leq m-2 \right\}. \end{aligned} \quad (3.39)$$

Then estimate (3.29) follows from relations (3.32)–(3.37).

It remains to prove the boundedness of the sequence $\{x_n\}$ in $C^{2m}(J)$. We use estimate (3.29), the properties of ω_j given in (H_2) , and the inequality

$$t(T-t) \geq \begin{cases} \frac{T}{2}t & \text{for } 0 < t \leq \frac{T}{2}, \\ \frac{T}{2}(T-t) & \text{for } \frac{T}{2} < t < T \end{cases} \quad (3.40)$$

and have

$$\begin{aligned} \int_0^T \omega_{2m}(|x_n^{(2m)}(s)|) ds &\leq \int_0^T \omega_{2m}(A|s - \xi_{m-1,n}|) ds \\ &= \frac{1}{A} \left(\int_0^{A\xi_{m-1,n}} \omega_{2m}(s) ds + \int_0^{A(T-\xi_{m-1,n})} \omega_{2m}(s) ds \right) \\ &< \frac{2}{A} \int_0^{AT} \omega_{2m}(s) ds, \\ \int_0^T \omega_{2j+2}(|x_n^{(2j+2)}(s)|) ds &\leq \int_0^T \omega_{2j+2}(A(s - \xi_{j,n})^2) ds \\ &= \frac{1}{\sqrt{A}} \int_{-\sqrt{A}\xi_{j,n}}^{\sqrt{A}(T-\xi_{j,n})} \omega_{2j+2}(s^2) ds \end{aligned}$$

$$\begin{aligned}
&< \frac{2}{\sqrt{A}} \int_0^{\sqrt{AT}} \omega_{2j+2}(s^2) ds \quad \text{for } 0 \leq j \leq m-2, \\
\int_0^T \omega_{2j+1}(|x_n^{(2j+1)}(s)|) ds &\leq \int_0^T \omega_{2j+1}(As(T-s)) ds \\
&< \int_0^{T/2} \omega_{2j+1}\left(\frac{ATs}{2}\right) ds + \int_{T/2}^T \omega_{2j+1}\left(\frac{AT(T-s)}{2}\right) ds \\
&< \frac{4}{AT} \int_0^{AT^2/4} \omega_{2j+1}(s) ds \quad \text{for } 0 \leq j \leq m-1, \\
\int_0^T \omega_0(|x_n(s)|) ds &\leq \int_0^T \omega_0(As^2) ds = \frac{1}{\sqrt{A}} \int_0^{\sqrt{AT}} \omega_0(s^2) ds.
\end{aligned} \tag{3.41}$$

In particular,

$$\begin{aligned}
\int_0^T \omega_{2m}(|x_n^{(2m)}(s)|) ds &< \frac{2}{A} \int_0^{AT} \omega_{2m}(s) ds, \\
\int_0^T \omega_{2j+2}(|x_n^{(2j+2)}(s)|) ds &< \frac{2}{\sqrt{A}} \int_0^{\sqrt{AT}} \omega_{2j+2}(s^2) ds \quad \text{for } 0 \leq j \leq m-2, \\
\int_0^T \omega_{2j+1}(|x_n^{(2j+1)}(s)|) ds &< \frac{4}{AT} \int_0^{AT^2/4} \omega_{2j+1}(s) ds \quad \text{for } 0 \leq j \leq m-1, \\
\int_0^T \omega_0(|x_n(s)|) ds &\leq \frac{1}{\sqrt{A}} \int_0^{\sqrt{AT}} \omega_0(s^2) ds,
\end{aligned} \tag{3.42}$$

for all $n \in \mathbb{N}$. Now, from the above estimates, from (2.6) and from $x_n^{(2m)}(\xi_{m-1,n}) = 0$ for some $\xi_{m-1,n} \in (0, T)$, which is proved in (ii), we get

$$\begin{aligned}
|x_n^{(2m)}(t)| &= \left| \int_{\xi_{m-1,n}}^t f_n(s, x_n(s), \dots, x_n^{(2m)}(s)) ds \right| \\
&\leq \int_0^T f_n(s, x_n(s), \dots, x_n^{(2m)}(s)) ds \\
&\leq \int_0^T h\left(s, 2m+1 + \sum_{j=0}^{2m} |x_n^{(j)}(s)|\right) ds + \sum_{j=0}^{2m} \int_0^T (\omega_j(|x_n^{(j)}(s)|) + \omega_j(1)) ds \\
&< \int_0^T h\left(s, 2m+1 + \sum_{j=0}^{2m} \|x_n^{(j)}\|_\infty\right) ds + \Lambda,
\end{aligned} \tag{3.43}$$

where

$$\begin{aligned} \Lambda = & \frac{2}{A} \int_0^{AT} \omega_{2m}(s) ds + \frac{2}{\sqrt{A}} \sum_{j=0}^{m-2} \int_0^{\sqrt{AT}} \omega_{2j+2}(s^2) ds \\ & + \frac{4}{AT} \sum_{j=0}^{m-1} \int_0^{AT^2/4} \omega_{2j+1}(s) ds + \frac{1}{\sqrt{A}} \int_0^{\sqrt{AT}} \omega_0(s^2) ds + T \sum_{j=0}^{2m} \omega_j(1). \end{aligned} \quad (3.44)$$

Notice that $\Lambda < \infty$ by (H_2) . Consequently,

$$\|x_n^{(2m)}\|_\infty < \int_0^T h\left(s, 2m+1 + \sum_{j=0}^{2m} \|x_n^{(j)}\|_\infty\right) ds + \Lambda \quad \text{for } n \in \mathbb{N}. \quad (3.45)$$

Since $\|x_n^{(j)}\|_\infty \leq T^{2m-j} \|x_n^{(2m)}\|_\infty$ for $0 \leq j \leq 2m$, which follows from the fact that $x_n^{(j)}$ vanishes in J by (1.2) and assertion (ii), inequality (3.45) yields

$$\|x_n^{(2m)}\|_\infty < \int_0^T h\left(s, 2m+1 + K \|x_n^{(2m)}\|_\infty\right) ds + \Lambda \quad \text{for } n \in \mathbb{N}, \quad (3.46)$$

where K is from (H_2) . Due to the condition

$$\limsup_{v \rightarrow \infty} \frac{1}{v} \int_0^T h(t, Kv) dt < 1 \quad (3.47)$$

in (H_2) , there exists a positive constant S such that for all $v \geq S$ the inequality

$$\int_0^T h(t, 2m+1 + Kv) dt + \Lambda \leq v \quad (3.48)$$

is fulfilled. The last inequality together with estimate (3.46) gives $\|x_n^{(2m)}\|_\infty < S$ for $n \in \mathbb{N}$. Consequently, $\|x_n^{(j)}\|_\infty < T^{2m-j} S$ for $0 \leq j \leq 2m$, $n \in \mathbb{N}$, and assertion (iv) follows. \square

The following result gives the important property of $\{f_n(t, x_n(t), \dots, x_n^{(2m)}(t))\}$ for applying the Vitali convergent theorem in the proof of Theorem 4.1.

Lemma 3.5. *Let (H_1) and (H_2) hold. Let x_n be a solution of problem (2.8), (1.2). Then, the sequence*

$$\left\{ f_n\left(t, x_n(t), \dots, x_n^{(2m)}(t)\right) \right\} \subset L^1(J) \quad (3.49)$$

is uniformly integrable on J , that is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{M} \subset J$ and $\text{meas } \mathcal{M} < \delta$, then

$$\int_{\mathcal{M}} f_n\left(t, x_n(t), \dots, x_n^{(2m)}(t)\right) dt < \varepsilon \quad \text{for } n \in \mathbb{N}. \quad (3.50)$$

Proof. By Lemma 3.4 (iv), there exists $S > 0$ such that for $n \in \mathbb{N}$, the inequality $\|x_n\|_{C^{2m}} < S$ holds. Now, we conclude from (2.5) and (2.6), from the properties of h and ω_j given in (H_2) , and finally from (3.29) that for $j \in J$ and $n \in \mathbb{N}$, the estimate

$$\begin{aligned} a &\leq f_n(t, x_n(t), \dots, x_n^{(2m)}(t)) \\ &\leq h(t, 2m + 1 + S) + \omega_0(At^2) + \sum_{j=0}^{m-1} \omega_{2j+1}(At(T-t)) \\ &\quad + \sum_{j=0}^{m-2} \omega_{2j+2}(A(t - \xi_{j,n})^2) + \omega_{2m}(A|t - \xi_{m-1,n}|) + \sum_{j=0}^m \omega_j(1) \end{aligned} \tag{3.51}$$

is fulfilled, where A is a positive constant. Since the functions $h(t, 2m + 1 + S)$, $\omega_0(At^2)$, and $\omega_{2j+1}(At(T-t))$ ($0 \leq j \leq m - 1$) belong to the set $L^1(J)$ by assumption (H_2) , in order to prove that $\{f_n(t, x_n(t), \dots, x_n^{(2m)}(t))\}$ is uniformly integrable on J , it suffices to show that the sequences

$$\{\omega_{2m}(A|t - \xi_{m-1,n}|\}), \quad \{\omega_{2j+2}(A(t - \xi_{j,n})^2)\}, \quad 0 \leq j \leq m - 2 \tag{3.52}$$

are uniformly integrable on J . Due to $\int_0^1 \omega_{2m}(s)ds < \infty$ and $\int_0^1 \omega_{2j}(s^2)ds < \infty$ for $1 \leq j \leq m - 1$ by (H_2) , this fact follows from [13, Criterion 11.10 (with $b = A$ and $r = 1, 2$)]. \square

4. The Main Result

The following theorem is the existence result for the singular problem (1.1), (1.2).

Theorem 4.1. *Let (H_1) and (H_2) hold. Then, problem (1.1), (1.2) has a positive solution $x \in AC^{2m}(J)$ and*

$$x(t) > 0 \quad \text{for } t \in (0, T], \quad (-1)^j x^{(2j+1)}(t) > 0 \quad \text{for } t \in (0, T), \quad 0 \leq j \leq m - 1. \tag{4.1}$$

Proof. Lemma 3.3 guarantees that problem (2.8), (1.2) has a solution x_n . Consider the sequence $\{x_n\}$. By Lemma 3.4, $\{x_n\}$ is bounded in $C^{2m}(J)$,

$$(-1)^j x_n^{(2j+1)}(t) > 0 \quad \text{for } t \in (0, T), \quad 0 \leq j \leq m - 1, \tag{4.2}$$

and x_n fulfils estimate (3.29), where A is a positive constant and $\xi_{j,n} \in (0, T)$. Furthermore, the sequence $\{f_n(t, x_n(t), \dots, x_n^{(2m)}(t))\}$ is uniformly integrable on J by Lemma 3.5, and therefore, we deduce from the equality $(-1)^m x_n^{(2m+1)}(t) = f_n(t, x_n(t), \dots, x_n^{(2m)}(t))$ for a.e. $t \in J$ that $\{x_n^{(2m)}\}$ is equicontinuous on J . Now, by the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem, we may assume without loss of generality that $\{x_n\}$ is convergent in $C^{2m}(J)$ and $\{\xi_{j,n}\}$ is convergent in \mathbb{R} for $0 \leq j \leq m - 1$. Let $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} \xi_{j,n} = \xi_j$

($0 \leq j \leq m-1$). Then $x \in C^{2m}(J)$ and x satisfies the boundary conditions (1.2). Letting $n \rightarrow \infty$ in (3.29) and (4.2), we get (for $t \in J$)

$$\begin{aligned} |x^{(2m)}(t)| &\geq A|t - \xi_{m-1}|, & |x^{(2j+2)}(t)| &\geq A(t - \xi_j)^2 \quad \text{if } 0 \leq j \leq m-2 \\ (-1)^j x^{(2j+1)}(t) &\geq At(T-t) \quad \text{if } 0 \leq j \leq m-1, & x(t) &\geq At^2. \end{aligned} \quad (4.3)$$

Keeping in mind the definition of f_n , we conclude from (4.3) that

$$\lim_{n \rightarrow \infty} f_n(t, x_n(t), \dots, x_n^{(2m)}(t)) = f(t, x(t), \dots, x^{(2m)}(t)) \quad \text{for a.e. } t \in J. \quad (4.4)$$

Then, by the Vitali theorem, $f(t, x(t), \dots, x^{(2m)}(t)) \in L^1(J)$ and

$$\lim_{n \rightarrow \infty} \int_0^t f_n(s, x_n(s), \dots, x_n^{(2m)}(s)) ds = \int_0^t f(s, x(s), \dots, x^{(2m)}(s)) ds \quad \text{for } t \in J. \quad (4.5)$$

Letting $n \rightarrow \infty$ in the equality

$$x_n^{(2m)}(t) = x_n^{(2m)}(0) + \int_0^t f_n(s, x_n(s), \dots, x_n^{(2m)}(s)) ds, \quad (4.6)$$

we get

$$x^{(2m)}(t) = x^{(2m)}(0) + \int_0^t f(s, x(s), \dots, x^{(2m)}(s)) ds \quad \text{for } t \in J. \quad (4.7)$$

As a result, $x \in AC^{2m}(J)$ and x is a solution of (1.1). Consequently, x is a positive solution of problem (1.1), (1.2) and inequality (4.1) follows from (4.3). \square

Example 4.2. Consider problem (1.1), (1.2) with

$$f(t, x_0, \dots, x_{2m}) = p(t) + \sum_{k=0}^{2m} \left(a_k(t) |x_k|^{\alpha_k} + \frac{b_k(t)}{|x_k|^{\beta_k}} \right) \quad (4.8)$$

on $J \times \mathfrak{D}$, where $p, a_k \in L^1(J)$, $b_k \in L^\infty(J)$ (that is, b_k is essentially bounded and measurable on J) are nonnegative, $p(t) \geq a > 0$ for a.e. $t \in J$. If $\alpha_k \in [0, 1)$ for $0 \leq k \leq 2m$ and $\beta_{2k} \in [0, 1/2)$, $\beta_{2m}, \beta_{2k+1} \in [0, 1)$ for $0 \leq k \leq m-1$, then, by Theorem 4.1, the problem has a positive solution $x \in AC^{2m}(J)$ satisfying inequality (4.1).

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