Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 458015, 16 pages
doi:10.1155/2010/458015

Research Article

Existence of Positive Solutions of a Singular Nonlinear Boundary Value Problem

Ruyun Ma[1] and Jiemei Li[1, 2]

[1] *College of Mathematics and Information Science, Northwest Normal University, Lanzhou 730070, China*
[2] *The School of Mathematics, Physics & Software Engineering, Lanzhou Jiaotong University, Lanzhou 730070, China*

Correspondence should be addressed to Ruyun Ma, ruyun_ma@126.com

Received 21 May 2010; Accepted 11 August 2010

Academic Editor: Vicentiu Radulescu

We are concerned with the existence of positive solutions of singular second-order boundary value problem $u''(t) + f(t, u(t)) = 0$, $t \in (0, 1)$, $u(0) = u(1) = 0$, which is not necessarily linearizable. Here, nonlinearity f is allowed to have singularities at $t = 0, 1$. The proof of our main result is based upon topological degree theory and global bifurcation techniques.

1. Introduction

Existence and multiplicity of solutions of singular problem

$$\begin{gathered} u'' + f(t, u) = 0, \quad t \in (0, 1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where f is allowed to have singularities at $t = 0$ and $t = 1$, have been studied by several authors, see Asakawa [1], Agarwal and O'Regan [2], O'Regan [3], Habets and Zanolin [4], Xu and Ma [5], Yang [6], and the references therein. The main tools in [1–6] are the method of lower and upper solutions, Leray-Schauder continuation theorem, and the fixed point index theory in cones. Recently, Ma [7] studied the existence of nodal solutions of the singular boundary value problem

$$u'' + ra(t)f(u) = 0, \quad t \in (0,1),$$
$$u(0) = u(1) = 0, \tag{1.2}$$

by applying Rabinowitz's global bifurcation theorem, where a is allowed to have singularities at $t = 0, 1$ and f is linearizable at 0 as well as at ∞. It is the purpose of this paper to study the existence of positive solutions of (1.1), which is not necessarily linearizable.

Let X be Banach space defined by

$$X = \left\{ \phi \in L^1_{\text{loc}}(0,1) \mid \int_0^1 t(1-t)\left|\phi(t)\right| dt < \infty \right\}, \tag{1.3}$$

with the norm

$$\|\phi\|_X = \int_0^1 t(1-t)\left|\phi(t)\right| dt. \tag{1.4}$$

Let

$$X_+ = \{\phi \in X \mid \phi(t) \geq 0, a.e.\ t \in (0,1)\},$$
$$X_p = \left\{ \phi \in X_+ \mid \int_0^1 t(1-t)\phi(t)dt > 0 \right\}. \tag{1.5}$$

Definition 1.1. A function $g : (0,1) \times \mathbb{R} \to \mathbb{R}$ is said to be an L^1_{loc}-Carathéodory function if it satisfies the following:

(i) for each $u \in \mathbb{R}$, $g(\cdot, u)$ is measurable;

(ii) for a.e. $t \in (0,1)$, $g(t, \cdot)$ is continuous;

(iii) for any $R > 0$, there exists $h_R \in X_p$, such that

$$\left|g(t,u)\right| \leq h_R(t), \quad \text{a.e. } t \in (0,1),\ |u| \leq R. \tag{1.6}$$

In this paper, we will prove the existence of positive solutions of (1.1) by using the global bifurcation techniques under the following assumptions.

(H1) Let $f : (0,1) \times [0,\infty) \to [0,\infty)$ be an L^1_{loc}-Carathéodory function and there exist functions $a_0(\cdot)$, $a^0(\cdot)$, $c_\infty(\cdot)$, and $c^\infty(\cdot) \in X_p$, such that

$$a_0(t)u - \xi_1(t,u) \leq f(t,u) \leq a^0(t)u + \xi_2(t,u), \tag{1.7}$$

for some L^1_{loc}-Carathéodory functions ξ_1, ξ_2 defined on $(0,1) \times [0,\infty)$ with

$$\xi_1(t,u) = \circ(a_0(t)u), \quad \xi_2(t,u) = \circ\left(a^0(t)u\right), \quad \text{as } u \longrightarrow 0, \tag{1.8}$$

uniformly for a.e. $t \in (0,1)$, and

$$c_\infty(t)u - \zeta_1(t,u) \le f(t,u) \le c^\infty(t)u + \zeta_2(t,u), \tag{1.9}$$

for some L^1_{loc}-Carathéodory functions ζ_1, ζ_2 defined on $(0,1) \times [0,\infty)$ with

$$\zeta_1(t,u) = \circ(c_\infty(t)u), \quad \zeta_2(t,u) = \circ(c^\infty(t)u), \quad \text{as } u \to \infty, \tag{1.10}$$

uniformly for a.e. $t \in (0,1)$.

(H2) $f(t,u) > 0$ for a.e. $t \in (0,1)$ and $u \in (0,\infty)$.

(H3) There exists function $c_1(\cdot) \in X_p$, such that

$$f(t,u) \ge c_1(t)u, \quad \text{a.e. } t \in (0,1), \ u \in [0,\infty). \tag{1.11}$$

Remark 1.2. If $a_0(\cdot)$, $a^0(\cdot)$, $c_\infty(\cdot)$, and $c^\infty(\cdot) \in C([0,1],(0,\infty))$, then (1.8) implies that

$$\xi_1(t,u) = \circ(u), \quad \xi_2(t,u) = \circ(u), \quad \text{as } u \to 0, \tag{1.12}$$

and (1.10) implies that

$$\zeta_1(t,u) = \circ(u), \quad \zeta_2(t,u) = \circ(u), \quad \text{as } u \to \infty. \tag{1.13}$$

The main tool we will use is the following global bifurcation theorem for problem which is not necessarily linearizable.

Theorem A (Rabinowitz, [8]). *Let V be a real reflexive Banach space. Let $F : \mathbb{R} \times V \to V$ be completely continuous, such that $F(\lambda,0) = 0$, for all $\lambda \in \mathbb{R}$. Let $a, b \in \mathbb{R}$ $(a < b)$, such that $u = 0$ is an isolated solution of the following equation:*

$$u - F(\lambda,u) = 0, \quad u \in V, \tag{1.14}$$

for $\lambda = a$ and $\lambda = b$, where $(a,0)$, $(b,0)$ are not bifurcation points of (1.14). *Furthermore, assume that*

$$d(I - F(a,\cdot), B_r(0), 0) \ne d(I - F(b,\cdot), B_r(0), 0), \tag{1.15}$$

where $B_r(0)$ is an isolating neighborhood of the trivial solution. Let

$$\mathcal{S} = \overline{\{(\lambda,u) : (\lambda,u) \text{ is a solution of } (1.14) \text{ with } u \ne 0\}} \cup ([a,b] \times \{0\}), \tag{1.16}$$

then there exists a continuum (i.e., a closed connected set) $\mathcal{C}$ *of* $\mathcal{S}$ *containing* $[a, b] \times \{0\}$, *and either*

(i) $\mathcal{C}$ *is unbounded in* $V \times \mathbb{R}$, *or*

(ii) $\mathcal{C} \cap [(\mathbb{R} \setminus [a, b]) \times \{0\}] \neq \emptyset$.

To state our main results, we need the following.

Lemma 1.3 (see [1, Proposition 4.7]). *Let* $a \in X_p$, *then the eigenvalue problem*

$$u'' + \lambda a(t)u = 0, \quad t \in (0,1),$$
$$u(0) = u(1) = 0 \tag{1.17}$$

has a sequence of eigenvalues as follows:

$$0 < \lambda_1(a) < \lambda_2(a) < \cdots < \lambda_k(a) < \lambda_{k+1}(a) < \cdots, \quad \lim_{k \to \infty} \lambda_k(a) = \infty. \tag{1.18}$$

Moreover, for each $k \in \mathbb{N}$, $\lambda_k(a)$ *is simple and its eigenfunction* $\varphi_k \in C^1[0,1]$ *has exactly* $k-1$ *zeros in* $(0,1)$.

Remark 1.4. Note that $\varphi_k \in C^1[0,1]$ and $\varphi_k(0) = \varphi_k(1) = 0$ for each $k \in \mathbb{N}$. Therefore, there exist constants $M_k > 0$, such that

$$|\varphi_k(t)| \leq M_k t(1-t), \quad t \in [0,1]. \tag{1.19}$$

Our main result is the following.

Theorem 1.5. *Let (H1)–(H3) hold. Assume that either*

$$\lambda_1(c_\infty) < 1 < \lambda_1\left(a^0\right) \tag{1.20}$$

or

$$\lambda_1(a_0) < 1 < \lambda_1(c^\infty), \tag{1.21}$$

then (1.1) *has at least one positive solution.*

Remark 1.6. For other references related to this topic, see [9–14] and the references therein.

2. Preliminary Results

Lemma 2.1 (see [15, Proposition 4.1]). *For any* $h \in X$, *the linear problem*

$$u''(t) + h(t) = 0, \quad t \in (0,1),$$
$$u(0) = u(1) = 0 \tag{2.1}$$

has a unique solution $u \in W^{1,1}(0,1)$ *and* $u' \in AC_{\text{loc}}(0,1)$, *such that*

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Furthermore, if $h \in X_+$, *then*

$$u(t) \ge 0, \quad t \in [0,1]. \tag{2.4}$$

Let $Y = C[0,1]$ be the Banach space with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$, and

$$E = \{u \in C[0,1] \mid u(0) = u(1) = 0\}. \tag{2.5}$$

Let $L : D(L) \subset Y \to X$ be an operator defined by

$$Lu = -u'', \quad u \in D(L), \tag{2.6}$$

where

$$D(L) = \left\{u \in W^{1,1}(0,1) \mid u'' \in X,\, u(0) = u(1) = 0\right\}. \tag{2.7}$$

Then, from Lemma 2.1, $L^{-1} : X \to C[0,1]$ is well defined.

Lemma 2.2. *Let* $a \in X_p$ *and* φ_1 *be the first eigenfunction of* (1.17). *Then for all* $u \in D(L)$, *one has*

$$\int_0^1 u''(t)\varphi_1(t)dt = \int_0^1 u(t)\varphi_1''(t)dt. \tag{2.8}$$

Proof. For any $\delta \in (0,1/2)$, integrating by parts, we have

$$\int_\delta^{1-\delta} u''(t)\varphi_1(t)dt = u'\varphi_1\Big|_\delta^{1-\delta} - u\varphi_1'\Big|_\delta^{1-\delta} + \int_\delta^{1-\delta} u(t)\varphi_1''(t)dt. \tag{2.9}$$

Since $u \in D(L)$ and $\varphi_1 \in C^1[0,1]$, then

$$\lim_{\delta\to 0} u(\delta)\varphi_1'(\delta) = \lim_{\delta\to 0} u(1-\delta)\varphi_1'(1-\delta) = 0. \tag{2.10}$$

Therefore, we only need to prove that

$$\lim_{\delta\to 0} u'(\delta)\varphi_1(\delta) = 0, \qquad \lim_{\delta\to 0} u'(1-\delta)\varphi_1(1-\delta) = 0. \tag{2.11}$$

Let us deal with the first equality, the second one can be treated by the same way. Note that $u \in D(L)$, then

$$\left(tu'(t)\right)' = u' + tu'' \in L^1(0,\delta), \tag{2.12}$$

which implies that $tu'(t) \in AC[0,\delta]$. Then $tu'(t)$ is bounded on $[0,\delta]$. Now, we claim that

$$\lim_{t\to 0} t\left|u'(t)\right| = 0. \tag{2.13}$$

Suppose on the contrary that $\lim_{t\to 0} t|u'(t)| = a > 0$, then for δ small enough, we have

$$t\left|u'(t)\right| \geq \frac{a}{2}, \quad t \in [0,\delta]. \tag{2.14}$$

Therefore,

$$\infty > \int_0^\delta \left|u'(t)\right| dt \geq \int_0^\delta \frac{a}{2t} dt = \infty, \tag{2.15}$$

which is a contradiction. Combining (1.19) with (2.13), we have

$$\left|u'(\delta)\varphi_1(\delta)\right| \leq M_1(1-\delta)\delta\left|u'(\delta)\right| \longrightarrow 0, \quad \delta \to 0. \tag{2.16}$$

This completes the proof. □

Remark 2.3. Under the conditions of Lemma 2.2, for the later convenience, (2.8) is equivalent to

$$\langle Lu, \varphi_1\rangle = \langle u, L\varphi_1\rangle. \tag{2.17}$$

Lemma 2.4 (see [1, Lemma 2.3]). *For every $\rho \in X_+$, the subset K defined by*

$$K = L^{-1}\left(\left\{\phi \in X \mid \left|\phi(t)\right| \leq \rho(t), \text{ a.e. } t \in (0,1)\right\}\right) \tag{2.18}$$

is precompact in $C[0,1]$.

Let $\Sigma \subset \mathbb{R}^+ \times E$ be the closure of the set of positive solutions of the problem

$$Lu = \lambda f(t,u). \tag{2.19}$$

We extend the function f to an L^1_{loc}-Carathéodory function $\overline{f}$ defined on $(0,1)\times\mathbb{R}$ by

$$\overline{f}(t,u)=\begin{cases} f(t,u), & (t,u)\in(0,1)\times[0,\infty),\\ f(t,0), & (t,u)\in(0,1)\times(-\infty,0).\end{cases}\tag{2.20}$$

Then $\overline{f}(t,u)\geq 0$ for $u\in\mathbb{R}$ and a.e. $t\in(0,1)$. For $\lambda\geq 0$, let u be an arbitrary solution of the problem

$$Lu=\lambda\overline{f}(t,u).\tag{2.21}$$

Since $\lambda\overline{f}(t,u(t))\geq 0$ for a.e. $t\in(0,1)$, Lemma 2.2 yields $u(t)\geq 0$ for $t\in[0,1]$. Thus, u is a nonnegative solution of (2.19), and the closure of the set of nontrivial solutions (λ,u) of (2.21) in $\mathbb{R}^+\times E$ is exactly Σ.

Let $g:(0,1)\times\mathbb{R}\to\mathbb{R}$ be an L^1_{loc}-Carathéodory function. Let $\widehat{N}:E\to X$ be the Nemytskii operator associated with the function g as follows:

$$\widehat{N}(u)(t)=g(t,u(t)),\quad u\in E.\tag{2.22}$$

Lemma 2.5. *Let $g(t,u)\geq 0$ on $[0,1]\times\mathbb{R}$. Let $u\in D(L)$ be such that $Lu\geq\lambda\widehat{N}(u)$ in $(0,1)$, $\lambda\geq 0$. Then,*

$$u(t)\geq 0,\quad t\in(0,1).\tag{2.23}$$

Moreover, $u(t)>0, t\in(0,1)$, whenever $u\not\equiv 0$.

Let $N:E\to X$ be the Nemytskii operator associated with the function $\overline{f}$ as follows:

$$N(u)(t)=\overline{f}(t,u),\quad u\in E.\tag{2.24}$$

Then (2.21), with $\lambda\geq 0$, is equivalent to the operator equation

$$u=\lambda L^{-1}N(u),\quad u\in E,\tag{2.25}$$

that is,

$$u(t)=\lambda\int_0^1 G(t,s)N(u(s))ds,\quad u\in E.\tag{2.26}$$

Lemma 2.6. *Let (H1) and (H2) hold. Then the operator $L^{-1}N:C[0,1]\to C[0,1]$ is completely continuous.*

Proof. From (1.10) in (H1), there exists $R > 0$, such that, for a.e. $t \in (0,1)$ and $|u| > R$,

$$|\zeta_1(t,u)| \leq \frac{1}{2}c_\infty(t)u, \qquad |\zeta_2(t,u)| \leq \frac{1}{2}c^\infty(t)u. \tag{2.27}$$

Since $\overline{f}$ is an L^1_{loc}-Carathéodory function, then there exists $h_R \in X_p$, such that, for a.e. $t \in (0,1)$ and $|u| \leq R$, $|\overline{f}(t,u)| \leq h_R(t)$. Therefore, for a.e. $t \in (0,1)$ and $u \in \mathbb{R}$, we have

$$\left|\overline{f}(t,u)\right| \leq \frac{3}{2}c^\infty(t)u + h_R(t). \tag{2.28}$$

For convenience, let $T = L^{-1}N$. We first show that $T : C[0,1] \to C[0,1]$ is continuous. Suppose that $u_m \to u$ in $C[0,1]$ as $m \to \infty$. Clearly, $\overline{f}(t,u_m) \to \overline{f}(t,u)$ as $m \to \infty$ for a.e. $t \in (0,1)$ and there exists $M > 0$ such that $\|u_m\| \leq M$ for every $m \in \mathbb{N}$. It is easy to see that

$$\begin{gathered}|Tu_m(t) - Tu(t)| \leq \int_0^1 s(1-s)\left|\overline{f}(s,u_m(s)) - \overline{f}(s,u(s))\right|ds, \\ \left|\overline{f}(s,u_m(s)) - \overline{f}(s,u(s))\right| \leq 3c^\infty(s)M + 2h_R(s), \quad \text{a.e. } s \in (0,1).\end{gathered} \tag{2.29}$$

By the Lebesgue dominated convergence theorem, we have that $Tu_m \to Tu$ in $C[0,1]$ as $m \to \infty$. Thus, $L^{-1}N$ is continuous.

Let D be a bounded set in $C[0,1]$. Lemma 2.4 together with (2.28) shows that $T(D)$ is precompact in $C[0,1]$. Therefore, T is completely continuous. □

In the following, we will apply the Leray-Schauder degree theory mainly to the mapping $\Phi_\lambda : E \to E$,

$$\Phi_\lambda(u) = u - \lambda L^{-1}N(u). \tag{2.30}$$

For $R > 0$, let $B_R = \{u \in E : \|u\| < R\}$, let $\deg(\Phi_\lambda, B_R, 0)$ denote the degree of Φ_λ on B_R with respect to 0.

Lemma 2.7. *Let $\Lambda \subset \mathbb{R}^+$ be a compact interval with $[\lambda_1(a^0), \lambda_1(a_0)] \cap \Lambda = \emptyset$, then there exists a number $\delta_1 > 0$ with the property*

$$\Phi_\lambda(u) \neq 0, \quad \forall u \in Y : 0 < \|u\| \leq \delta_1,\ \forall \lambda \in \Lambda. \tag{2.31}$$

Proof. Suppose to the contrary that there exist sequences $\{\mu_n\} \subset \Lambda$ and $\{u_n\}$ in $Y : \mu_n \to \mu^* \in \Lambda$, $u_n \to 0$ in Y, such that $\Phi_{\mu_n}(u_n) = 0$ for all $n \in \mathbb{N}$, then, $u_n \geq 0$ in $[0,1]$.

Set $v_n = u_n/\|u_n\|$. Then $Lv_n = \mu_n\|u_n\|^{-1}N(u_n) = \mu_n\|u_n\|^{-1}f(t,u_n)$ and $\|v_n\| = 1$. Now, from condition (H1), we have the following:

$$a_0(t)u_n - \xi_1(t,u_n) \leq f(t,u_n) \leq a^0(t)u_n + \xi_2(t,u_n), \tag{2.32}$$

and accordingly

$$\mu_n\left(a_0(t)v_n - \frac{\xi_1(t,u_n)}{\|u_n\|}\right) \leq \mu_n\frac{f(t,u_n)}{\|u_n\|} \leq \mu_n\left(a^0(t)v_n + \frac{\xi_2(t,u_n)}{\|u_n\|}\right). \tag{2.33}$$

Let φ^0 and φ_0 denote the nonnegative eigenfunctions corresponding to $\lambda_1(a^0)$ and $\lambda_1(a_0)$, respectively, then we have from the first inequality in (2.33) that

$$\left\langle \mu_n\left(a_0(t)v_n - \frac{\xi_1(t,u_n)}{\|u_n\|}\right), \varphi_0\right\rangle \leq \left\langle \mu_n\frac{f(t,u_n)}{\|u_n\|}, \varphi_0\right\rangle = \langle Lv_n, \varphi_0\rangle. \tag{2.34}$$

From Lemma 2.2, we have that

$$\langle Lv_n, \varphi_0\rangle = \langle v_n, L\varphi_0\rangle = \lambda_1(a_0)\langle v_n, a_0(t)\varphi_0\rangle. \tag{2.35}$$

Since $u_n \to 0$ in E, from (1.12), we have that

$$\frac{\xi_1(t,u_n)}{\|u_n\|} \longrightarrow 0, \quad \text{as } \|u_n\| \longrightarrow 0. \tag{2.36}$$

By the fact that $\|v_n\| = 1$, we conclude that $v_n \rightharpoonup v$ in E. Thus,

$$\langle v_n, a_0(t)\varphi_0\rangle \longrightarrow \langle v, a_0(t)\varphi_0\rangle. \tag{2.37}$$

Combining this and (2.35) and letting $n \to \infty$ in (2.34), it follows that

$$\langle \mu^* a_0(t)v, \varphi_0\rangle \leq \lambda_1(a_0)\langle a_0(t)\varphi_0, v\rangle, \tag{2.38}$$

and consequently

$$\mu^* \leq \lambda_1(a_0). \tag{2.39}$$

Similarly, we deduce from second inequality in (2.33) that

$$\lambda_1\left(a^0\right) \leq \mu^*. \tag{2.40}$$

Thus, $\lambda_1(a^0) \leq \mu^* \leq \lambda_1(a_0)$. This contradicts $\mu^* \in \Lambda$. □

Corollary 2.8. *For $\lambda \in (0, \lambda_1(a^0))$ and $\delta \in (0, \delta_1)$, $\deg(\Phi_\lambda, B_\delta, 0) = 1$.*

Proof. Lemma 2.7, applied to the interval $\Lambda = [0, \lambda]$, guarantees the existence of $\delta_1 > 0$, such that for $\delta \in (0, \delta_1)$,

$$u - \tau\lambda L^{-1}N(u) \neq 0, \quad u \in E : 0 < \|u\| \leq \delta, \ \tau \in [0,1]. \tag{2.41}$$

This together with Lemma 2.6 implies that for any $\delta \in (0, \delta_1)$,

$$\deg(\Phi_\lambda, B_\delta, 0) = \deg(I, B_\delta, 0) = 1, \tag{2.42}$$

which ends the proof. □

Lemma 2.9. *Suppose $\lambda > \lambda_1(a_0)$, then there exists $\delta_2 > 0$ such that for all $u \in E$ with $0 < \|u\| \leq \delta_2$, for all $\tau \geq 0$,*

$$\Phi_\lambda(u) \neq \tau\varphi_0, \tag{2.43}$$

where φ_0 is the nonnegative eigenfunction corresponding to $\lambda_1(a_0)$.

Proof. Suppose on the contrary that there exist $\tau_n \geq 0$ and a sequence $\{u_n\}$ with $\|u_n\| > 0$ and $u_n \to 0$ in E such that $\Phi_\lambda(u_n) = \tau_n\varphi_0$ for all $n \in \mathbb{N}$. As

$$Lu_n = \lambda N(u_n) + \tau_n\lambda_1(a_0)a_0(t)\varphi_0 \tag{2.44}$$

and $\tau_n\lambda_1(a_0)a_0(t)\varphi_0 \geq 0$ in $(0,1)$, it concludes from Lemma 2.2 that

$$u_n(t) \geq 0, \quad t \in [0,1]. \tag{2.45}$$

Notice that $u_n \in D(L)$ has a unique decomposition

$$u_n = w_n + s_n\varphi_0, \tag{2.46}$$

where $s_n \in \mathbb{R}$ and $\langle w_n, a_0(t)\varphi_0\rangle = 0$. Since $u_n \geq 0$ on $[0,1]$ and $\|u_n\| > 0$, we have from (2.46) that $s_n > 0$.

Choose $\sigma > 0$, such that

$$\sigma < \frac{\lambda - \lambda_1(a_0)}{\lambda}. \tag{2.47}$$

By (H1), there exists $r_1 > 0$, such that

$$|\xi_1(t,u)| \leq \sigma a_0(t)u, \quad \text{a.e. } t \in (0,1), \ u \in [0, r_1]. \tag{2.48}$$

Therefore, for a.e. $t \in (0,1)$, $u \in [0, r_1]$,

$$f(t,u) \geq a_0(t)u - \xi_1(t,u) \geq (1-\sigma)a_0(t)u. \tag{2.49}$$

Since $\|u_n\| \to 0$, there exists $N^* > 0$, such that

$$0 \leq u_n \leq r_1, \quad \forall n \geq N^*, \tag{2.50}$$

and consequently

$$f(t,u_n) \geq (1-\sigma)a_0(t)u_n, \quad \forall n \geq N^*. \tag{2.51}$$

Applying (2.51), it follows that

$$\begin{aligned} s_n\lambda_1(a_0)\langle\varphi_0, a_0(t)\varphi_0\rangle &= \langle u_n, L\varphi_0\rangle = \langle Lu_n, \varphi_0\rangle \\ &= \lambda\langle N(u_n), \varphi_0\rangle + \tau_n\lambda_1(a_0)\langle a_0(t)\varphi_0, \varphi_0\rangle \\ &\geq \lambda\langle N(u_n), \varphi_0\rangle \geq \lambda\langle (1-\sigma)a_0(t)u_n, \varphi_0\rangle \\ &= \lambda(1-\sigma)\langle a_0(t)\varphi_0, u_n\rangle \\ &= \lambda(1-\sigma)s_n\langle a_0(t)\varphi_0, \varphi_0\rangle. \end{aligned} \tag{2.52}$$

Thus,

$$\lambda_1(a_0) \geq \lambda(1-\sigma). \tag{2.53}$$

This contradicts (2.47). □

Corollary 2.10. *For $\lambda > \lambda_1(a_0)$ and $\delta \in (0, \delta_2)$,* $\deg(\Phi_\lambda, B_\delta, 0) = 0$.

Proof. Let $0 < \delta \leq \delta_2$, where δ_2 is the number asserted in Lemma 2.9. As Φ_λ is bounded in $\overline{B}_\delta$, there exists $c > 0$ such that $\Phi_\lambda(u) \neq c\varphi_0$, for all $u \in \overline{B}_\delta$. By Lemma 2.9, one has

$$\Phi_\lambda(u) \neq \tau c\varphi_0, \quad u \in \partial B_\delta, \ \tau \in [0,1]. \tag{2.54}$$

This together with Lemma 2.6 implies that

$$\deg(\Phi_\lambda, B_\delta, 0) = \deg(\Phi_\lambda - c\varphi_0, B_\delta, 0) = 0. \tag{2.55}$$

□

Now, using Theorem A, we may prove the following.

Proposition 2.11. $[\lambda_1(a^0), \lambda_1(a_0)]$ *is a bifurcation interval from the trivial solution for* (2.30)*. There exists an unbounded component* $\mathcal{C}$ *of positive solutions of* (2.30) *which meets* $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$*. Moreover,*

$$\mathcal{C} \cap \left[\left(\mathbb{R} \setminus \left[\lambda_1\left(a^0\right), \lambda_1(a_0)\right]\right) \times \{0\}\right] = \emptyset. \tag{2.56}$$

Proof. For fixed $n \in \mathbb{N}$ with $\lambda_1(a^0) - (1/n) > 0$, let us take that $a_n = \lambda_1(a^0) - (1/n)$, $b_n = \lambda_1(a_0) + (1/n)$ and $\widehat{\delta} = \min\{\delta_1, \quad \delta_2\}$. It is easy to check that, for $0 < \delta < \widehat{\delta}$, all of the conditions of Theorem A are satisfied. So there exists a connected component $\mathcal{C}_n$ of solutions of (2.30) containing $[a_n, \ b_n] \times \{0\}$, and either

(i) $\mathcal{C}_n$ is unbounded, or

(ii) $\mathcal{C}_n \cap [(\mathbb{R} \setminus [a_n, b_n]) \times \{0\}] \neq \emptyset$.

By Lemma 2.7, the case (ii) can not occur. Thus, $\mathcal{C}_n$ is unbounded bifurcated from $[a_n, b_n] \times \{0\}$ in $\mathbb{R} \times E$. Furthermore, we have from Lemma 2.7 that for any closed interval $I \subset [a_n, b_n] \setminus [\lambda_1(a^0), \lambda_1(a_0)]$, if $u \in \{y \in E \mid (\lambda, y) \in \Sigma, \lambda \in I\}$, then $\|u\| \to 0$ in E is impossible. So $\mathcal{C}_n$ must be bifurcated from $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ in $\mathbb{R} \times E$. □

3. Proof of the Main Results

Proof of Theorem 1.5. It is clear that any solution of (2.30) of the form $(1, u)$ yields solutions u of (1.1). We will show that $\mathcal{C}$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. To do this, it is enough to show that $\mathcal{C}$ joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(c^\infty), \lambda_1(c_\infty)] \times \{\infty\}$. Let $(\eta_n, y_n) \in \mathcal{C}$ satisfy

$$\eta_n + \|y_n\| \longrightarrow \infty. \tag{3.1}$$

We note that $\eta_n > 0$ for all $n \in \mathbb{N}$ since $(0, 0)$ is the only solution of (2.30) for $\lambda = 0$ and $\mathcal{C} \cap (\{0\} \times E) = \emptyset$.

Case 1. consider the following:

$$\lambda_1(c_\infty) < 1 < \lambda_1\left(a^0\right). \tag{3.2}$$

In this case, we show that the interval

$$\left(\lambda_1(c_\infty), \lambda_1\left(a^0\right)\right) \subseteq \{\lambda \in \mathbb{R} \mid (\lambda, u) \in \mathcal{C}\}. \tag{3.3}$$

We divide the proof into two steps.

Step 1. We show that $\{\eta_n\}$ is bounded.

Since $(\eta_n, y_n) \in \mathcal{C}$, $Ly_n = \eta_n f(t, y_n)$. From (H3), we have

$$Ly_n \geq \eta_n c_1(t) y_n. \tag{3.4}$$

Let $\overline{\varphi}$ denote the nonnegative eigenfunction corresponding to $\lambda_1(c_1)$.
From (3.4), we have

$$\langle Ly_n, \overline{\varphi}\rangle \geq \eta_n \langle c_1(t)y_n, \overline{\varphi}\rangle. \tag{3.5}$$

By Lemma 2.2, we have

$$\lambda_1(c_1)\langle y_n, c_1(t)\overline{\varphi}\rangle = \langle y_n, L\overline{\varphi}\rangle \geq \eta_n \langle c_1(t)\overline{\varphi}, y_n\rangle. \tag{3.6}$$

Thus,

$$\eta_n \leq \lambda_1(c_1). \tag{3.7}$$

Step 2. We show that $\mathcal{C}$ joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(c^\infty), \lambda_1(c_\infty)] \times \{\infty\}$.

From (3.1) and (3.7), we have that $\|y_n\| \to \infty$. Notice that (2.30) is equivalent to the integral equation

$$y_n(t) = \eta_n \int_0^1 G(t,s) f(s, y_n(s)) ds, \tag{3.8}$$

which implies that

$$\begin{aligned} \eta_n \int_0^1 G(t,s)\left[c^\infty(s)y_n(s) + \zeta_2(s, y_n(s))\right]ds &\geq y_n(t) \\ &\geq \eta_n \int_0^1 G(t,s)\left[c_\infty(s)y_n(s) - \zeta_1(s, y_n(s))\right]ds. \end{aligned} \tag{3.9}$$

We divide the both sides of (3.9) by $\|y_n\|$ and set $v_n = y_n/\|y_n\|$. Since v_n is bounded in E, there exist a subsequence of $\{v_n\}$ and $v^* \in E$ with $v^* \geq 0$ and $v^* \not\equiv 0$ on $(0,1)$, such that

$$\eta_n \longrightarrow \eta^*, \quad v_n \stackrel{\omega}{\rightharpoonup} v^* \quad \text{in } E, \tag{3.10}$$

relabeling if necessary. Thus, (3.9) yields that

$$\eta^* \int_0^1 G(t,s)c^\infty(s)v^*(s)ds \geq v^*(t) \geq \eta^* \int_0^1 G(t,s)c_\infty(s)v^*(s)ds. \tag{3.11}$$

Let φ^∞ and φ_∞ denote the nonnegative eigenfunctions corresponding to $\lambda_1(c^\infty)$ and $\lambda_1(c_\infty)$, respectively, then it follows from the second inequality in (3.11) that

$$\begin{aligned}
\lambda_1(c_\infty)\langle c_\infty\varphi_\infty, v^*\rangle &= \langle L\varphi_\infty, v^*\rangle = \langle -\varphi_\infty'', v^*\rangle = -\int_0^1 \varphi_\infty''(t)v^*(t)dt \\
&\geq -\int_0^1 \varphi_\infty''(t)\eta^*\int_0^1 G(t,s)c_\infty(s)v^*(s)dsdt \\
&= -\eta^*\int_0^1 c_\infty(s)v^*(s)\int_0^1 G(t,s)\varphi_\infty''(t)dtds \\
&= \eta^*\int_0^1 c_\infty(s)v^*(s)\varphi_\infty(s)ds \\
&= \eta^*\langle c_\infty\varphi_\infty, v^*\rangle,
\end{aligned} \tag{3.12}$$

and consequently

$$\eta^* \leq \lambda_1(c_\infty). \tag{3.13}$$

Similarly, we deduce from the first inequality in (3.11) that

$$\lambda_1(c^\infty) \leq \eta^*. \tag{3.14}$$

Thus,

$$\lambda_1(c^\infty) \leq \eta^* \leq \lambda_1(c_\infty). \tag{3.15}$$

So $\mathcal{C}$ joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(c^\infty), \lambda_1(c_\infty)] \times \{\infty\}$.

Case 2. $\lambda_1(a_0) < 1 < \lambda_1(c^\infty)$.

In this case, if $(\eta_n, y_n) \in \mathcal{C}$ is such that

$$\begin{gathered}
\lim_{n\to\infty}(\eta_n + \|y_n\|) = \infty, \\
\lim_{n\to\infty}\eta_n = \infty,
\end{gathered} \tag{3.16}$$

then

$$(\lambda_1(a_0), \lambda_1(c^\infty)) \subseteq \{\lambda \in (0,\infty) \mid (\lambda, u) \in \mathcal{C}\}, \tag{3.17}$$

and moreover,

$$(\{1\} \times E) \cap \mathcal{C} \neq \emptyset. \tag{3.18}$$

Assume that $\{\eta_n\}$ is bounded, applying a similar argument to that used in Step 2 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$\eta_n \longrightarrow \eta^* \in [\lambda_1(c^\infty), \lambda_1(c_\infty)], \quad \|y_n\| \longrightarrow \infty, \quad \text{as} \quad n \longrightarrow \infty. \tag{3.19}$$

Again C joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(c^\infty), \lambda_1(c_\infty)] \times \{\infty\}$ and the result follows.

□

Remark 3.1. Lomtatidze [13, Theorem 1.1] proved the existence of solutions of singular two-point boundary value problems as follows:

$$\begin{gathered} u''(t) = g(t,u), \\ u(a) = 0, \qquad u(b) = 0, \end{gathered} \tag{3.20}$$

under the following assumptions:

(A1)

$$\begin{gathered} g(t,x) \le h_1(t)x, \quad 0 < x < \delta, \\ g(t,x) \ge h_2(t)x, \quad x > \frac{1}{\delta}, \end{gathered} \tag{3.21}$$

where $h_i : (a,b) \rightarrow R (i = 1,2)$ satisfies the following condition:

$$\int_a^b (t-a)(b-t)|h_i(t)|dt < +\infty \quad (i = 1,2), \tag{3.22}$$

(A2) For $i = 1,2$, let v_i be the solution of singular IVPs

$$v''(t) = h_i(t)v, \qquad v(a) = 0, \qquad v'(a) = 1, \tag{3.23}$$

satisfying v_1 has at least one zero in $(a,b]$ and v_2 has no zeros in $(a,b]$.

It is worth remarking that (A1)-(A2) imply Condition (1.21) in Theorem 1.5. However, Condition (1.21) is easier to be verified than (A1)-(A2) since $\lambda_1(c^\infty)$ and $\lambda_1(a_0)$ are easily estimated by Rayleigh's Quotient.

The language of eigenvalue of singular linear eigenvalue problem did not occur until Asakawa [1] in 2001. The first part of Theorem 1.5 is new.

Acknowledgments

The authors are very grateful to the anonymous referees for their valuable suggestions. This work was supported by the NSFC 11061030, the Fundamental Research Funds for the Gansu Universities.

References

[1] H. Asakawa, "Nonresonant singular two-point boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, no. 6, pp. 791–809, 2001.

[2] R. P. Agarwal and D. O'Regan, *Singular Differential and Integral Equations with Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[3] D. O'Regan, *Theory of Singular Boundary Value Problems*, World Scientific, River Edge, NJ, USA, 1994.

[4] P. Habets and F. Zanolin, "Upper and lower solutions for a generalized Emden-Fowler equation," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 3, pp. 684–700, 1994.

[5] X. Xu and J. Ma, "A note on singular nonlinear boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 293, no. 1, pp. 108–124, 2004.

[6] X. Yang, "Positive solutions for nonlinear singular boundary value problems," *Applied Mathematics and Computation*, vol. 130, no. 2-3, pp. 225–234, 2002.

[7] R. Ma, "Nodal solutions for singular nonlinear eigenvalue problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 6, pp. 1417–1427, 2007.

[8] P. H. Rabinowitz, "Some aspects of nonlinear eigenvalue problems," *The Rocky Mountain Journal of Mathematics*, vol. 3, pp. 161–202, 1973.

[9] R. P. Agarwal and D. O'Regan, *An Introduction to Ordinary Differential Equations*, Universitext, Springer, New York, NY, USA, 2008.

[10] R. P. Agarwal and D. O'Regan, *Ordinary and Partial Differential Equations*, Universitext, Springer, New York, NY, USA, 2009.

[11] M. Ghergu and V. D. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, vol. 37 of *Oxford Lecture Series in Mathematics and Its Applications*, Oxford University Press, Oxford, UK, 2008.

[12] A. Kristály, V. Rădulescu, and C. Varga, *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and Its Applications, no. 136, Cambridge University Press, Cambridge, UK, 2010.

[13] A. G. Lomtatidze, "Positive solutions of boundary value problems for second-order ordinary differential equations with singularities," *Differentsial'nye Uravneniya*, vol. 23, no. 10, pp. 1685–1692, 1987.

[14] I. T. Kiguradze and B. L. Shekhter, "Singular boundary-value problems for ordinary second-order differential equations," *Journal of Soviet Mathematics*, vol. 43, no. 2, pp. 2340–2417, 1988, translation from: Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh., vol. 30, pp. 105–201, 1987.

[15] C. D. Coster and P. Habets, *Two-Point Boundary Value Problems: Lower and Upper Solutions*, vol. 205 of *Mathematics in Science and Engineering*, 2006.