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Research Article

Multiple Positive Solutions for nth Order Multipoint Boundary Value Problem

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We study the existence of multiple positive solutions for nth-order multipoint boundary value problem. $u^{(n)}(t) + a(t)f(u(t)) = 0$, $t \in (0,1)$, $u^{(j-1)}(0) = 0(j = 1,2,\ldots,n-1)$, $u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i)$, where $n \geq 2$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $\alpha_i > 0$, $i = 1,2,\ldots,m$. We obtained the existence of multiple positive solutions by applying the fixed point theorems of cone expansion and compression of norm type and Leggett-Williams fixed-point theorem. The results obtained in this paper are different from those in the literature.

1. Introduction

The existence of positive solutions for nth-order multipoint boundary problems has been studied by some authors (see [1, 2]). In [1], Pang et al. studied the expression and properties of Green's funtion and obtained the existence of at least one positive solution for nth-order differential equations by applying means of fixed point index theory:

$$\begin{gathered} u^{(n)}(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u^{(j-1)}(0) = 0(j = 1,2,\ldots,n-1), \quad u(1) = \sum_{i=1}^{m}\alpha_i u(\eta_i), \end{gathered} \tag{1.1}$$

where $n \geq 2,\ 0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1,\ \alpha_i > 0,\ i = 1,2,\ldots,m$.

By using the fixed point theorems of cone expansion and compression of norm type, Yang and Wei in [2] also obtained the existence of at least one positive solutions for the BVP (1.1) if $m \geq 2$. This work is motivated by Ma (see [3]). This method is simpler than that of [1]. In addition, Eloe and Ahmad in [4] had solved successfully the existence of positive solutions to the BVP (1.1) if $m = 1$. Hao et al. in [5] had discussed the existence of at least two positive solutions for the BVP (1.1) by applying the Krasonse'skii-Guo fixed point theorem on cone expansion and compression if $m = 1$. However, there are few papers dealing with the existence of multiple positive solutions for nth-order multipoint boundary value problem.

In this paper, we study the existence of at least two positive solutions associated with the BVP (1.1) by applying the fixed point theorems of cone expansion and compression of norm type if $m \geq 2$ and the existence of at least three positive solutions for BVP (1.1) by using Leggett-Williams fixed-point theorem. The results obtained in this paper are different from those in the literature and essentially improve and generalize some well-known results (see [1–8]).

The rest of the paper is organized as follows. In Section 2, we present several lemmas. In Section 3, we give some preliminaries and the fixed point theorems of cone expansion and compression of norm type. The existence of at least two positive solutions for the BVP (1.1) is formulated and proved in Section 4. In Section 5, we give Leggett-Williams fixed-point theorem and obtain the existence of at least three positive solutions for the BVP (1.1).

2. Several Lemmas

Definition 2.1. A function $u(t)$ is said to be a position of the BVP (1.1) if $u(t)$ satisfies the following:

(1) $u(t) \in C[0,1] \cap C^n(0,1)$;

(2) $u(t) > 0$ for $t \in (0,1)$ and satisfies boundary value conditions (1.1);

(3) $u^{(n)}(t) = -a(t)f(u(t))$ hold for $t \in (0,1)$.

Lemma 2.2 (see [1]). *Suppose that*

$$D = \sum_{i=1}^{m} \alpha_i \eta_i^{n-1} \neq 1; \tag{2.1}$$

then for any $y \in C[0,1]$, the problem

$$\begin{gathered} u^{(n)}(t) + y(t) = 0, \quad t \in (0,1), \\ u^{(j-1)}(0) = 0(j = 1,2,\ldots,n-1), \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i) \end{gathered} \tag{2.2}$$

has a unique solution:

$$\begin{aligned} u(t) = & -\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} y(s)\mathrm{d}s + \frac{t^{n-1}}{(n-1)!(1-D)}\int_0^1 (1-s)^{n-1} y(s)\mathrm{d}s \\ & - \frac{t^{n-1}}{(n-1)!(1-D)}\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i} (\eta_i - s)^{n-1} y(s)\mathrm{d}s = \int_0^1 K(t,s)y(s)\mathrm{d}s, \end{aligned} \tag{2.3}$$

where

$$K(t,s) = K_1(t,s) + K_2(t,s),$$

$$K_1(t,s) = \frac{1}{(n-1)!}\begin{cases} t^{n-1}(1-s)^{n-1} - (t-s)^{n-1}, & 0 \le s < t \le 1, \\ t^{n-1}(1-s)^{n-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$K_2(t,s) = \frac{D}{(n-1)!(1-D)} t^{n-1}(1-s)^{n-1} - \frac{1}{(n-1)!(1-D)} \sum_{s \le \eta_i} \alpha_i t^{n-1} (\eta_i - s)^{n-1}.$$

Lemma 2.3 (see [1]). *Let $D < 1$; Green's function $K(t,s)$ defined by* (2.4) *satisfies*

$$0 \le K(t,s) \le K(s), \quad \forall t,s \in [0,1],$$
$$\min_{t \in [\eta_1, 1]} K(t,s) \ge \gamma K(s), \quad \forall s \in [0,1], \tag{2.5}$$

where $\gamma = \eta_1^{n-1}$:

$$K(s) = \max_{t\in[0,1]} K_1(t,s) + \max_{t\in[0,1]} K_2(t,s) = \frac{s^{n-1}(1-s)^{n-1}}{(n-1)!}\left[1-(1-s)^{(n-1)/(n-2)}\right]^{2-n} + K_2(1,s). \tag{2.6}$$

We omit the proof Lemma 2.3 here and you can see the detail of Theorem 2.2 in [1].

Lemma 2.4 (see [2]). *Let $D < 1$, $y \in C[0,1]$, and $y \ge 0$; the unique solution $u(t)$ of the BVP* (2.2) *satisfies*

$$\min_{t\in[\eta_1,1]} u(t) \ge \gamma \|u\|, \tag{2.7}$$

where γ is defined by Lemma 2.3, $\|u\| = \max_{t\in[0,1]}|u(t)|$.

3. Preliminaries

In this section, we give some preliminaries for discussing the existence of multiple positive solutions of the BVP (1.1) in the next. In real Banach space $C[0,1]$ in which the norm is defined by

$$\|u\| = \max_{t\in[0,1]} |u(t)|, \tag{3.1}$$

set

$$P = \left\{ u \in C[0,1] \mid u(0) = 0, u(t) > 0 \text{ for } 0 < t \le 1, \min_{t\in[\eta_1,1]} u(t) \ge \gamma \|u\| \right\}. \tag{3.2}$$

Obviously, P is a positive cone in $C[0,1]$, where γ is from Lemma 2.3.

For convenience, we make the following assumptions:

(A_1) $a: [0,1] \to [0,+\infty]$ is continuous and $a(t)$ does not vanish identically, for $t \in [\eta_1, 1]$;

(A_2) $f: [0,+\infty) \to [0,+\infty)$ is continuous;

(A_3) $D = \sum_{i=1}^{m} \alpha_i \eta_i^{n-1} < 1$.

Let

$$(Au)(t) = \int_0^1 K(t,s)a(s)f(u(s))ds, \quad \forall t \in [0,1], \tag{3.3}$$

where $K(t,s)$ is defined by (2.4).

From Lemmas 2.2–2.4, we have the following result.

Lemma 3.1 (see [2]). *Suppose that (A_1)–(A_3) are satisfied, then $A: C[0,1] \to C[0,1]$ is a completely continuous operator, $A(P) \subset P$, and the fixed points of the operator A in P are the positive solutions of the BVP* (1.1).

For convenience, one introduces the following notations. Let

$$\begin{aligned} r &= \frac{1}{(n-1)!(1-D)} \int_0^1 (1-s)^{n-1} a(s) \mathrm{d}s, \\ R &= \frac{\gamma \sum_{i=2}^{m} \alpha_i}{(n-1)!(1-D)} \int_{\eta_1}^{\eta_i} \left[(\eta_i - \eta_i s)^{n-1} - (\eta_i - s)^{n-1} \right] a(s) \mathrm{d}s \quad (m \geq 2). \end{aligned} \tag{3.4}$$

Problem 1. Inspired by the work of the paper [2], whether we can obtain a similar conclusion or not, if

$$\liminf_{u \to 0^+} \frac{f(u)}{u} > R^{-1}, \qquad \liminf_{u \to +\infty} \frac{f(u)}{u} > R^{-1}; \tag{3.5}$$

or

$$\limsup_{u \to 0^+} \frac{f(u)}{u} < r^{-1}, \qquad \limsup_{u \to +\infty} \frac{f(u)}{u} < r^{-1}. \tag{3.6}$$

The aim of the following section is to establish some simple criteria for the existence of multiple positive solutions of the BVP (1.1), which gives a positive answer to the questions stated above. The key tool in our approach is the following fixed point theorem, which is a useful method to prove the existence of positive solutions for differential equations, for example [2–5, 9].

Lemma 3.2 (see [10, 11]). *Suppose that E is a real Banach space and P is cone in E, and let Ω_1, Ω_2 be two bounded open sets in E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let operator $A: P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of two conditions holds:*

(i) $\|Au\| \leq \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Au\| \geq \|u\|$, *for all* $u \in P \cap \partial\Omega_2$;

(ii) $\|Au\| \geq \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Au\| \leq \|u\|$, *for all* $u \in P \cap \partial\Omega_2$.

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

4. The Existence of Two Positive Solutions

Theorem 4.1. *Suppose that the conditions* (A_1)–(A_3) *are satisfied and the following assumptions hold:*

(B_1) $\lim_{u\to 0^+}\inf(f(u)/u) > R^{-1}$;

(B_2) $\lim_{u\to +\infty}\inf(f(u)/u) > R^{-1}$;

(B_3) *There exists a constant* $\rho > 0$ *such that* $f(u) \le r^{-1}\rho,\ u \in [0,\rho]$.

Then the BVP (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \rho < \|u_2\|. \tag{4.1}$$

Proof. At first, it follows from the condition (B_1) that we may choose $\rho_1 \in (0,\rho)$ such that

$$f(u) > R^{-1}u, \quad 0 < u \le \rho_1. \tag{4.2}$$

Set $\Omega_1 = \{u \in C[0,1] : \|u\| < \rho_1\}$, and $u \in P \cap \partial\Omega_1$; from (3.3) and (2.4) and Lemma 2.4, for $0 < t \le 1$, we have

$$\begin{aligned}
Au(1) &= \frac{1}{(n-1)!(1-D)}\left[\int_0^1 D(1-s)^{n-1}a(s)f(u(s))\mathrm{d}s - \sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i - s)^{n-1}a(s)f(u(s))\mathrm{d}s\right]\\
&\ge \frac{\sum_{i=1}^{m}\alpha_i}{(n-1)!(1-D)}\int_0^{\eta_i}\left[(\eta_i-\eta_i s)^{n-1} - (\eta_i - s)^{n-1}\right]a(s)f(u(s))\mathrm{d}s\\
&> \frac{R^{-1}\sum_{i=1}^{m}\alpha_i}{(n-1)!(1-D)}\int_0^{\eta_i}\left[(\eta_i-\eta_i s)^{n-1} - (\eta_i - s)^{n-1}\right]a(s)u(s)\mathrm{d}s\\
&> \frac{R^{-1}\sum_{i=2}^{m}\alpha_i}{(n-1)!(1-D)}\int_{\eta_1}^{\eta_i}\left[(\eta_i-\eta_i s)^{n-1} - (\eta_i - s)^{n-1}\right]a(s)u(s)\mathrm{d}s\\
&> \frac{R^{-1}\gamma\|u\|\sum_{i=2}^{m}\alpha_i}{(n-1)!(1-D)}\int_{\eta_1}^{\eta_i}\left[(\eta_i-\eta_i s)^{n-1} - (\eta_i - s)^{n-1}\right]a(s)\mathrm{d}s\\
&= R^{-1}R\|u\| = \|u\|.
\end{aligned} \tag{4.3}$$

Therefore, we have

$$\|Au\| \ge \|Au(1)\| > \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{4.4}$$

Further, it follows from the condition (B_2) that there exists $\rho_2 > \rho$ such that

$$f(u) > R^{-1}u, \quad u \ge \rho_2. \tag{4.5}$$

Let $\rho^* = \max\{2\rho, \gamma^{-1}\rho_2\}$, set $\Omega_2 = \{u \in C[0,1] : \|u\| < \rho^*\}$, then $u \in P \cap \partial\Omega_2$ and Lemma 2.4 imply

$$\min_{\eta_1 \le t \le 1} u(t) \ge \gamma\|u\| \ge \rho_2, \tag{4.6}$$

and by the condition (B_2), (2.4), (3.3), and Lemma 2.4, we have

$$\begin{aligned}
Au(1) &= \frac{1}{(n-1)!(1-D)}\left[\int_0^1 D(1-s)^{n-1}a(s)f(u(s))\mathrm{d}s - \sum_{i=1}^{m}\alpha_i\int_0^{\eta_i}(\eta_i - s)^{n-1}a(s)f(u(s))\mathrm{d}s\right] \\
&\ge \frac{\sum_{i=1}^{m}\alpha_i}{(n-1)!(1-D)}\int_0^{\eta_i}\left[(\eta_i - \eta_i s)^{n-1} - (\eta_i - s)^{n-1}\right]a(s)f(u(s))\mathrm{d}s \\
&> \frac{R^{-1}\sum_{i=1}^{m}\alpha_i}{(n-1)!(1-D)}\int_0^{\eta_i}\left[(\eta_i - \eta_i s)^{n-1} - (\eta_i - s)^{n-1}\right]a(s)u(s)\mathrm{d}s \\
&> \frac{R^{-1}\sum_{i=2}^{m}\alpha_i}{(n-1)!(1-D)}\int_{\eta_1}^{\eta_i}\left[(\eta_i - \eta_i s)^{n-1} - (\eta_i - s)^{n-1}\right]a(s)u(s)\mathrm{d}s \\
&> \frac{R^{-1}\gamma\|u\|\sum_{i=2}^{m-2}\alpha_i}{(n-1)!(1-D)}\int_{\eta_1}^{\eta_i}\left[(\eta_i - \eta_i s)^{n-1} - (\eta_i - s)^{n-1}\right]a(s)\mathrm{d}s \\
&= R^{-1}R\|u\| = \|u\|.
\end{aligned} \tag{4.7}$$

Therefore, we have

$$\|Au\| \ge \|Au(1)\| > \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{4.8}$$

Finally, let $\Omega_3 = \{u \in C[0,1] : \|u\| < \rho\}$ and $u \in P \cap \partial\Omega_3$. By (2.3), (3.3), and the condition (B_3), we have

$$\begin{aligned}
Au(t) &\le \frac{t^{n-1}}{(n-1)!(1-D)}\int_0^1 (1-s)^{n-1}a(s)f(u(s))\mathrm{d}s \\
&\le \frac{r^{-1}\rho}{(n-1)!(1-D)}\int_0^1 (1-s)^{n-1}a(s)\mathrm{d}s = r^{-1}r\rho = \|u\|,
\end{aligned} \tag{4.9}$$

which implies

$$\|Au\| \le \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{4.10}$$

Thus from (4.4)–(4.10) and Lemmas 3.1 and 3.2, A has a fixed point u_1 in $P \cap (\overline{\Omega}_3 \setminus \Omega_1)$ and a fixed u_2 in $P \cap (\overline{\Omega}_2 \setminus \Omega_3)$. Both are positive solutions of BVP (1.1) and satisfy

$$0 < \|u_1\| < \rho < \|u_2\|. \tag{4.11}$$

The proof is complete. □

Corollary 4.2. *Suppose that the conditions* (A_1)–(A_3) *are satisfied and the following assumptions hold:*

(B_1') $\lim_{u\to 0^+}\inf(f(u)/u) = +\infty$;

(B_2') $\lim_{u\to +\infty}\inf(f(u)/u) = +\infty$;

(B_3') *there exists a constant* $\rho' > 0$ *such that* $f(u) \le r^{-1}\rho',\ u \in [0,\rho']$.

Then the BVP (1.1) *has at least two positive solutions* u_1*and* u_2 *such that*

$$0 < \|u_1\| < \rho' < \|u_2\|. \tag{4.12}$$

Proof. From the conditions (B_i') $(i = 1,2)$, there exist sufficiently big positive constants $M_i(i = 1,2)$ such that

$$\lim_{u\to 0^+}\sup\frac{f(u)}{u} > M_2, \qquad \lim_{u\to +\infty}\sup\frac{f(u)}{u} > M_1 \tag{4.13}$$

by the condition (B_3'); so all the conditions of Theorem 4.1 are satisfied; by an application of Theorem 4.1, the BVP (1.1) has two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < \rho' < \|u_2\|. \tag{4.14}$$

□

Theorem 4.3. *Suppose that the conditions* (A_1)–(A_3) *are satisfied and the following assumptions hold:*

(C_1) $\lim_{u\to 0^+}\sup(f(u)/u) < r^{-1}$;

(C_2) $\lim_{u\to +\infty}\sup(f(u)/u) < r^{-1}$;

(C_3) *there exists a constant* $l > 0$ *such that* $f(u) \ge R^{-1}l,\ u \in [\gamma l, l]$.

Then the BVP (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < l < \|u_2\|. \tag{4.15}$$

Proof. It follows from the condition (C_1) that we may choose $\rho_3 \in (0,l)$ such that

$$f(u) < r^{-1}u, \qquad 0 < u \le \rho_3. \tag{4.16}$$

Set $\Omega_4 = \{u \in C[0,1] : \|u\| < \rho_3\}$, and $u \in P \cap \partial\Omega_4$; from (3.2) and (2.4), for $0 < t \le 1$, we have

$$\begin{aligned} Au(t) &\le \frac{t^{n-1}}{(n-1)!(1-D)}\int_0^1 (1-s)^{n-1}a(s)f(u(s))\mathrm{d}s \\ &< \frac{r^{-1}\|u\|}{(n-1)!(1-D)}\int_0^1 (1-s)^{n-1}a(s)\mathrm{d}s = r^{-1}r\|u\| = \|u\|. \end{aligned} \tag{4.17}$$

Therefore, we have

$$\|Au\| < \|u\|, \quad u \in P \cap \partial\Omega_4. \tag{4.18}$$

It follows from the condition (C_2) that there exists $\rho_4 > l$ such that $f(u) < r^{-1}u$ for $u \geq \rho_4$, and we consider two cases.

Case i. Suppose that f is unbounded; there exists $l^* > \rho_4$ such that $f(u) \leq f(l^*)$ for $0 < u \leq l^*$. Then for $u \in P$ and $\|u\| = l^*$, we have

$$\begin{aligned} Au(t) &\leq \frac{t^{n-1}}{(n-1)!(1-D)} \int_0^1 (1-s)^{n-1} a(s) f(u(s)) \mathrm{d}s \\ &\leq \frac{t^{n-1}}{(n-1)!(1-D)} \int_0^1 (1-s)^{n-1} a(s) f(l^*) \mathrm{d}s \\ &< \frac{r^{-1}l^*}{(n-1)!(1-D)} \int_0^1 (1-s)^{n-1} a(s) \mathrm{d}s = r^{-1}rl^* = l^* = \|u\|. \end{aligned} \tag{4.19}$$

Case ii. If f is bounded, that is, $f(u) \leq N$ for all $u \in [0, +\infty)$, taking $l^* \geq \max\{2l, Nr\}$, for $u \in P$ and $\|u\| = l^*$, we have

$$\begin{aligned} Au(t) &\leq \frac{t^{n-1}}{(n-1)!(1-D)} \int_0^1 (1-s)^{n-1} a(s) f(u(s)) \mathrm{d}s \\ &\leq \frac{N}{(n-1)!(1-D)} \int_0^1 (1-s)^{n-1} a(s) \mathrm{d}s \leq Nr \leq l^* = \|u\|. \end{aligned} \tag{4.20}$$

Hence, in either case, we always may set $\Omega_5 = \{u \in \mathrm{C}[0,1] : \|u\| < l^*\}$ such that

$$\|Au\| \leq \|u\|, \quad u \in P \cap \partial\Omega_5. \tag{4.21}$$

Finally, set $\Omega_6 = \{u \in C[0,1] : \|u\| < l\}$; then $u \in P \cap \partial\Omega_6$ and Lemma 2.4 imply

$$\min_{t \in [\eta_1, 1]} u(t) \geq \gamma \|u\| = \gamma l, \tag{4.22}$$

and by the condition (C_3), (2.4), and (3.3), we have

$$\begin{aligned}
Au(1) &= \frac{1}{(n-1)!(1-D)}\left[\int_0^1 D(1-s)^{n-1}a(s)f(u(s))\mathrm{d}s - \sum_{i=1}^{m}\alpha_i\int_0^{\eta_i}(\eta_i-s)^{n-1}a(s)f(u(s))\mathrm{d}s\right] \\
&\geq \frac{\sum_{i=1}^{m}\alpha_i}{(n-1)!(1-D)}\int_0^{\eta_i}\left[(\eta_i-\eta_i s)^{n-1}-(\eta_i-s)^{n-1}\right]a(s)f(u(s))\mathrm{d}s \\
&\geq \frac{R^{-1}l\sum_{i=2}^{m}\alpha_i}{(n-1)!(1-D)}\int_{\eta_1}^{\eta_i}\left[(\eta_i-\eta_i s)^{n-1}-(\eta_i-s)^{n-1}\right]a(s)\mathrm{d}s \\
&\geq \frac{R^{-1}l\gamma\sum_{i=2}^{m}\alpha_i}{(n-1)!(1-D)}\int_{\eta_1}^{\eta_i}\left[(\eta_i-\eta_i s)^{n-1}-(\eta_i-s)^{n-1}\right]a(s)\mathrm{d}s \\
&= R^{-1}lR = \|u\|.
\end{aligned} \tag{4.23}$$

Hence, we have

$$\|Au\| \geq \|u\|, \quad u \in P\cap\partial\Omega_6. \tag{4.24}$$

From (4.18)–(4.24) and Lemmas 3.1 and 3.2, A has a fixed point u_1 in $P\cap(\overline{\Omega}_6\setminus\Omega_4)$ and a fixed u_2 in $P\cap(\overline{\Omega}_5\setminus\Omega_6)$. Both are positive solutions of the BVP(1.1) and satisfy

$$0 < \|u_1\| < l < \|u_2\|. \tag{4.25}$$

The proof is complete. □

Corollary 4.4. *Suppose that the conditions* (A_1)–(A_3) *are satisfied and the following assumptions hold:*

(C_1') $\lim_{u\to 0^+}\sup(f(u)/u) = 0$;

(C_2') $\lim_{u\to +\infty}\sup(f(u)/u) = 0$;

(C_3') *there exists a constant* $\rho'' > 0$ *such that* $f(u) \geq R^{-1}\rho''$, $u \in [\gamma\rho'', \rho'']$.

Then BVP (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \rho'' < \|u_2\|. \tag{4.26}$$

The proof of Corollary 4.4 is similar to that of Corollary 4.2; so we omit it.

5. The Existence of Three Positive Solutions

Let E be a real Banach space with cone P. A map $\beta : P \to [0, +\infty)$ is said to be a nonnegative continuous concave functional on P if β is continuous and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y) \tag{5.1}$$

for all $x, y \in P$ and $t \in [0,1]$. Let a, b be two numbers such that $0 < a < b$ and let β be a nonnegative continuous concave functional on P. We define the following convex sets:

$$\begin{gathered} P_a = \{x \in P : \|x\| < a\}, \qquad \partial P_a = \{x \in P : \|x\| = a\}, \qquad \overline{P}_a = \{x \in P : \|x\| \leq a\}, \\ P(\beta, a, b) = \{x \in P : a \leq \beta(x), \|x\| \leq b\}. \end{gathered} \tag{5.2}$$

Lemma 5.1 (see [12]). *Let $A : \overline{P}_c \to \overline{P}_c$ be completely continuous and let β be a nonnegative continuous concave functional on P such that $\beta(x) \leq \|x\|$ for $x \in \overline{P}_c$. Suppose that there exist $0 < d < a < b \leq c$ such that*

(i) $\{x \in P(\beta, a, b) : \beta(x) > a\} \neq \emptyset$ *and* $\beta(Ax) > a$ *for* $x \in P(\beta, a, b)$,

(ii) $\|Ax\| < d$ *for* $\|x\| \leq d$,

(iii) $\beta(Ax) > a$ *for* $x \in P(\beta, a, c)$ *with* $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2, x_3 in $\overline{P}_c$ such that

$$\|x_1\| < d, a < \beta(x_2), \text{ and } \|x_3\| > d \quad \text{with } \beta(x_3) < a. \tag{5.3}$$

Now, we establish the existence conditions of three positive solutions for the BVP (1.1).

Theorem 5.2. *Suppose that (A_1)–(A_3) hold and there exist numbers a and d with $0 < d < a$ such that the following conditions are satisfied:*

(D_1) $\lim_{u\to\infty}(f(u)/u) < (1/G)$,

(D_2) $f(u) < d/G,\ u \in [0, d]$,

(D_3) $f(u) > a/F,\ u \in [a, a/\gamma]$,

where

$$F = \min_{t\in[\eta_1,1]} \int_{\eta_1}^{1} K(t,s)a(s)ds, \qquad G = \max_{t\in[0,1]} \int_0^1 K(t,s)a(s)ds, \tag{5.4}$$

Then the boundary value problem (1.1) *has at least three positive solutions.*

Proof. Let P be defined by (3.2) and let A be defined by (3.3). For $u \in P$, let

$$\beta(u) = \min_{t\in[\eta_1,1]} u(t). \tag{5.5}$$

Then it is easy to check that β is a nonnegative continuous concave functional on P with $\beta(u) \leq \|u\|$ for $u \in P$ and $A : P \to P$ is completely continuous.

First, we prove that if (D_1) holds, then there exists a number $c > a/\gamma$ and $A : \overline{P}_c \to \overline{P}_c$. To do this, by (D_1), there exist $M > 0$ and $\lambda < 1/G$ such that

$$f(u) < \lambda u, \quad \text{for } u > M. \tag{5.6}$$

Set

$$\delta = \max_{u\in[0,M]} f(u); \tag{5.7}$$

it follows that $f(u) < \lambda u + \delta$ for all $u \in [0, +\infty)$. Take

$$c > \max\left\{ \frac{\delta G}{1-\lambda G}, \frac{a}{\gamma} \right\}. \tag{5.8}$$

If $u \in \overline{P}_c$, then

$$(Au)(t) \leq \max_{t\in[0,1]} \int_0^1 K(t,s)a(s)f(u(s))ds < \max_{t\in[0,1]} \int_0^1 K(t,s)a(s)ds(\lambda\|u\| + \delta) < (\lambda c + \delta)G < c, \tag{5.9}$$

that is,

$$\|Au\| < c. \tag{5.10}$$

Hence (5.10) show that if (D_1) holds, then there exists a number $c > a/\gamma$ such that A maps $\overline{P}_c$ into P_c.

Now we show that $\{u \in P(\beta, a, a/\gamma) : \beta(u) > a\} \neq \emptyset$ and $\beta(Au) > a$ for all $u \in P(\beta, a, a/\gamma)$. In fact, take $x(t) \equiv (a + (a/\gamma))/2 > a$, so $x \in \{u \in P(\beta, a, a/\gamma) : \beta(u) > a\}$. Moreover, for $u \in P(\beta, a, a/\gamma)$, then $\beta(u) > a$, and we have

$$\frac{a}{\gamma} \geq \|u\| \geq \beta(u) > a. \tag{5.11}$$

Therfore, by (D_3) we obtain

$$\beta(Au) = \min_{t\in[\eta_1,1]} \int_0^1 K(t,s)a(s)f(u(s))ds > \frac{a}{F} \min_{t\in[\eta_1,1]} \int_{\eta_1}^1 K(t,s)a(s)ds = a. \tag{5.12}$$

Next, we assert that $\|Au\| < d$ for $\|u\| \leq d$. In fact, if $u \in \overline{P}_d$, by (D_2) we have

$$\|Au\| < \frac{d}{G}\left(\max_{t\in[0,1]} \int_0^1 K(t,s)a(s)ds \right) = d. \tag{5.13}$$

Hence, $A : \overline{P}_d \to P_d$ for $u \in \overline{P}_d$.

Finally, we assert that if $u \in P(\beta, a, c)$ and $\|Au\| > a/\gamma$, then $\beta(Au) > a$. To see this, if $u \in P(\beta, a, c)$ and $\|Au\| > a/\gamma$,then we have from Lemma 2.3 that

$$\begin{aligned}\beta(Au) &= \min_{t\in[\eta_1,1]} \int_0^1 K(t,s)a(s)f(u(s))ds \\ &\geq \int_0^1 \min_{t\in[\eta_1,1]} K(t,s)a(s)f(u(s))ds \geq \gamma \int_0^1 K(s)a(s)f(u(s))ds \\ &\geq \gamma \int_0^1 \max_{t\in[0,1]} K(t,s)a(s)f(u(s))ds \geq \gamma \max_{t\in[0,1]} \int_0^1 K(t,s)a(s)f(u(s))ds = \gamma\|Au\|.\end{aligned} \tag{5.14}$$

So we have

$$\beta(Au) \geq \gamma\|Au\| > \gamma \cdot \frac{a}{\gamma} = a. \tag{5.15}$$

To sum up (5.10)~(5.15), all the conditions of Lemma 5.1 are satisfied by taking $b = a/\gamma$. Hence, A has at least three fixed points; that is, BVP (1.1) has at least three positive solutions u_1, u_2, and u_3 such that

$$\|u_1\| < d, a < \beta(u_2), \text{ and } \|u_3\| > d \quad \text{with } \beta(u_3) < a. \tag{5.16}$$

The proof is complete. □

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