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Research Article

On Second-Order Differential Equations with Nonhomogeneous Φ-Laplacian

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Equation with general nonhomogeneous Φ-Laplacian, including classical and singular Φ-Laplacian, is investigated. Necessary and sufficient conditions for the existence of nonoscillatory solutions satisfying certain asymptotic boundary conditions are given and discrepancies between the general and classical Φ are illustrated as well.

1. Introduction

The aim of this paper is to investigate asymptotic properties for second-order nonlinear differential equation

$$\left(a(t)\Phi\left(x'\right)\right)' + b(t)F(x) = 0, \quad (t \geq t_0), \tag{1.1}$$

where

(i) Φ is an increasing odd homeomorphismus, $\Phi : (-\rho, \rho) \to (-\sigma, \sigma)$ such that $\Phi(0) = 0$ and $0 < \rho \leq \infty$, $0 < \sigma \leq \infty$;

(ii) F is a real continuous increasing function on $\mathbb{R}$ such that $F(u)u > 0$ for $u \neq 0$;

(iii) a, b are positive continuous functions for $t \geq t_0$ such that

$$\int_{t_0}^{\infty} b(t)\,dt < \infty, \tag{1.2}$$

and, whenever $\sigma < \infty$,

$$\inf\{\sigma a(t)\} > 0 \quad \text{for any } t \geq t_0. \tag{1.3}$$

For sake of simplicity we will assume also that F is an odd function.

Equation (1.1) is called equation with general Φ-Laplacian because $\operatorname{Dom}\Phi$ and/or $\operatorname{Im}\Phi$ are possibly bounded and it is not required that Φ satisfies the *homogeneity property*

$$\Phi(u)\Phi(v) = \Phi(uv) \quad \text{for any } u, v \in \operatorname{Dom}\Phi. \tag{1.4}$$

Obviously, (1.4) holds for equations with the classical Φ-Laplacian, that is, for

$$\Phi_p(u) = |u|^{p-2}u \quad (p > 1). \tag{1.5}$$

Prototypes of Φ, for which (1.4) does not hold, are the function $\Phi_C : \mathbb{R} \to (-1, 1)$

$$\Phi_C(u) = \frac{u}{\sqrt{1 + |u|^2}}, \tag{1.6}$$

which determines the curvature operator div $(\Phi_C(\nabla u))$, and the function $\Phi_R : (-1, 1) \to \mathbb{R}$

$$\Phi_R(u) = \frac{u}{\sqrt{1 - |u|^2}}, \tag{1.7}$$

which determines the relativity operator $\operatorname{div}(\Phi_R(\nabla u))$. Boundary value problems on compact intervals for equations of type (1.1) are widely studied. The classical Φ-Laplacian is examined in [1, 2], see also references therein; the cases of the curvature or the relativity operator are considered in [3–5]; finally, equations of type (1.1) with nonhomogeneous Φ-Laplacian defined in the whole $\mathbb{R}$ are studied in [6]. As claimed in [6, page 25], the lack of the homogeneity property of Φ causes some difficulties for this study.

Oscillatory and asymptotic properties for (1.1) with classical Φ-Laplacian have attracted attention of many mathematicians in the last two decades; see, for example, [7–17] and references therein. Other papers deal with the qualitative behavior of solutions of systems of the form

$$\begin{aligned} x' &= A(t) f_1(y), \\ y' &= -B(t) f_2(x), \end{aligned} \tag{1.8}$$

see, for example, [18–21]. Since the homogeneity property (1.4) can fail, (1.1) is not equivalent with system (1.8) and so oscillatory and asymptotic properties of (1.1) with general Φ-Laplacian cannot be obtained, in general, from results concerning (1.8).

The aim of this paper is to consider (1.1) with general Φ-Laplacian and to study the existence of all possible types of nonoscillatory solutions of (1.1) and their mutual coexistence. We show that the lack of the homogeneity property of Φ can produce several new phenomena in asymptotic behavior of solutions of (1.1). The discrepancies on the asymptotic properties of solutions of (1.1) with classical and general Φ-Laplacian are presented and illustrated by some examples, as well.

Our main tools for solving the asymptotic boundary value problems are based on topological methods in locally convex spaces and integral inequalities.

We close the introduction by noticing that (1.1) covers a large class of second-order ordinary differential equations which arise in the study of radially symmetric solutions of partial differential equations of the type

$$\operatorname{div}(G(\nabla u)) + B(|x|)F(u) = 0, \quad x \in E, \tag{1.9}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is continuous homeomorphismus, $x = (x_1, \dots x_n) \in \mathbb{R}^n$, $n \geq 2$, $\nabla u = (D_1 u, \dots, D_n u)$, $D_i = \partial/\partial x_i$, $i = 1, \dots, n$, $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, $E = \{x \in R^n : |x| \geq c\}$, $c > 0$. Denote $r = |x|$ and $du/dr = u_r$ the radial derivative of u. If there exists an odd function $g : \mathbb{R} \to \mathbb{R}$ such that

$$G(x) = \frac{x}{r} g(r) \quad x \in E, \tag{1.10}$$

then a direct computation shows that u is a radially symmetric solution of (1.9) if and only if the function $y = y(r) = u(|x|)$ is a solution of the ordinary differential equation

$$\left(r^{n-1} g(y')\right)' + r^{n-1} B(r) F(y) = 0, \quad (r \geq c). \tag{1.11}$$

2. Homogeneity Property of Φ

We start by discussing the homogeneity property (1.4) and the consequences when it fails. To this aim, consider the functional equation

$$X(u)X(v) = X(uv). \tag{2.1}$$

The following holds.

Proposition 2.1. *Any continuous and increasing solution X of* (2.1) *has the form $X(u) = |u|^\mu u$ for some $\mu > -1$.*

Proof. Denote for $u > 0$

$$Y(u) = \ln X(e^u). \tag{2.2}$$

Then (2.2) transforms (2.1) into the Cauchy functional equation

$$Y(u + v) = Y(u) + Y(v), \tag{2.3}$$

whose continuous solutions are of the form $Y(u) = \lambda u$, $\lambda \in \mathbb{R}$, see, for example, [22]. From here we have $X(u) = u^{\lambda}$ for $u > 0$ and, because X is increasing, it results $\lambda > 0$. Moreover, the continuability of X at 0 gives $X(0) = 0$. If $u < 0$, then from (2.1) we have $X(u) = X(-1)(-u)^{\lambda}$. Because X is increasing, we have $X(-1) < 0$ and from the fact $X^2(-1) = 1$ it follows $X(-1) = -1$. Consequently, $X(u) = |u|^{\lambda} \operatorname{sgn}(u)$, where $\lambda > 0$, for $u \in \mathbb{R}$. □

Let Φ^* be the inverse function to Φ and

$$\Lambda = \bigcap_{t \geq t_0} (0, \sigma a(t)). \tag{2.4}$$

In view of (1.3), Λ is a nonempty interval and, if $\sigma = \infty$, then $\Lambda = (0, \infty)$.

As we will show later, a crucial role for the behavior of nonoscillatory solutions of (1.1) is played by the integral

$$I_{\lambda} = \int_{t_0}^{\infty} \Phi^*\left(\frac{\lambda}{a(t)}\right) dt, \quad \lambda \in \Lambda. \tag{2.5}$$

If Φ satisfies the homogeneity property (1.4), then I_{λ} is either divergent or convergent for any $\lambda > 0$. If (1.4) does not hold, the convergence of (2.5) can depend on the choice of λ, as the following examples illustrate.

Example 2.2. Consider the continuous odd function $\Phi : (-1, 1) \to \mathbb{R}$ given by

$$\Phi(\mathrm{u}) = -(\log u)^{-1} \quad \text{if } 0 < u < 1. \tag{2.6}$$

Thus

$$\Phi^*(w) = e^{-1/w} \quad \text{if } 0 < w < \infty. \tag{2.7}$$

Clearly, $\Lambda = (0, \infty)$. Setting

$$a(t) = \log t \quad \text{on } [2, \infty), \tag{2.8}$$

we have

$$\Phi^*\left(\frac{\lambda}{a(t)}\right) = \frac{1}{t^{1/\lambda}}, \tag{2.9}$$

and so (2.5) converges for $\lambda < 1$ and diverges for $\lambda \geq 1$.

Example 2.3. Consider the continuous odd function $\Phi : \mathbb{R} \to (-\sigma, \sigma)$, $\sigma = 1 + e$, given by

$$\Phi(u) = \begin{cases} -(\log u)^{-1} & \text{if } 0 < u < \dfrac{1}{e}, \\ \dfrac{(1+e)u}{(u+1)} & \text{if } \dfrac{1}{e} \leq u < \infty. \end{cases} \tag{2.10}$$

Thus

$$\Phi^*(w) = \begin{cases} e^{-1/w} & \text{if } 0 < w < 1, \\ \dfrac{w}{(1+e-w)} & \text{if } 1 \le w < 1+e. \end{cases} \tag{2.11}$$

Setting

$$a(t) = \log t \quad \text{on } \left[e^4, \infty\right), \tag{2.12}$$

we have $\Lambda = (0, \overline{\lambda})$, where $\overline{\lambda} = 4(1+e)$. For $\lambda \in \Lambda$ we get (2.9) and thus the same conclusion as in Example 2.2 holds.

Observe that in the above examples the change of the convergence of I_λ depends only on the behavior of Φ^* near zero and thus they can be modified in order to include functions Φ with $\mathrm{Dom}\,\Phi = \mathrm{Im}\,\Phi = \mathbb{R}$.

We close this section by recalling that in the study of oscillatory properties of (1.1) with the classical Φ-Laplacian it is often assumed

$$\int_{t_0}^{\infty} \Phi_p^*\left(\frac{1}{a(t)}\right) dt = \infty. \tag{2.13}$$

In this case the operator $L_1 x \equiv (a(t)\Phi_p(x'))'$ is said to be in the canonical form and it can be reduced by the transformation of the independent variable $s = \int^t 1/\Phi_p^*(a(\tau))d\tau$ to the operator with $a(t) \equiv 1$. In particular, the Sturm-Liouville operator $L_2 x \equiv (a(t)x')'$ in the canonical form can be transformed to the binomial operator $\overline{L}x \equiv d^2x/ds^2$. The lack of the homogeneity property (1.4) makes this approach impossible for a general Φ.

3. Unbounded Solutions

Throughout this paper, by solution of (1.1), we mean a function x which is continuously differentiable together with its quasiderivative $x^{[1]}$

$$x^{[1]}(t) = a(t)\Phi\left(x'(t)\right), \tag{3.1}$$

on some ray $[t_x, \infty)$, where $t_x \ge t_0$, and satisfies (1.1) for $t > t_x$. As usual, a solution x of (1.1), defined on some neighborhood of infinity, is said to be *nonoscillatory* if $x(t) \ne 0$ for any large t, and *oscillatory* otherwise.

Since we assume that F is an odd function, we will restrict our attention only to eventually positive solutions of (1.1) and we denote by $\mathbb{S}$ the set of these solutions.

Let $x \in \mathbb{S}$: we say $x \in \mathbb{M}^{+}$ or $\mathbb{M}^{-}$, if x is eventually increasing or eventually decreasing. If x is eventually positive, then $x^{[1]}$ is decreasing for large t. If $x^{[1]}$ becomes negative for $t \geq T$, because we can suppose also $x(t) > 0$ for $t \geq T$, integrating (1.1) we obtain

$$x^{[1]}(t) \geq x^{[1]}(T) - h\int_{T}^{t} b(s)ds, \tag{3.2}$$

where $h = \sup_{t \geq T} F(x(t))$. Hence, (1.2) gives that $x^{[1]}$ is bounded.

Unbounded solutions of (1.1) are in the class $\mathbb{M}^{+}$ and can be a priori divided into the subclasses:

$$\begin{gathered}\mathbb{M}^{+}_{\infty,\ell} = \left\{x \in \mathbb{M}^{+} : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} x^{[1]}(t) = d_x,\ 0 < d_x < \infty\right\}, \\ \mathbb{M}^{+}_{\infty,0} = \left\{x \in \mathbb{M}^{+} : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} x^{[1]}(t) = 0\right\},\end{gathered} \tag{3.3}$$

while bounded solutions of (1.1) can be a priori divided into the subclasses

$$\begin{gathered}\mathbb{M}^{+}_{\ell,0} = \left\{x \in \mathbb{M}^{+} : \lim_{t\to\infty} x(t) = \ell_x, \lim_{t\to\infty} x^{[1]}(t) = 0,\ 0 < \ell_x < \infty\right\}, \\ \mathbb{M}^{+}_{\ell,\ell} = \left\{x \in \mathbb{M}^{+} : \lim_{t\to\infty} x(t) = \ell_x, \lim_{t\to\infty} x^{[1]}(t) = d_x,\ 0 < \ell_x < \infty,\ 0 < d_x < \infty\right\}, \\ \mathbb{M}^{-}_{\ell,\ell} = \left\{x \in \mathbb{M}^{-} : \lim_{t\to\infty} x(t) = \ell_x, \lim_{t\to\infty} x^{[1]}(t) = d_x,\ 0 < \ell_x < \infty,\ -\infty < d_x < 0\right\}, \\ \mathbb{M}^{-}_{0,\ell} = \left\{x \in \mathbb{M}^{-} : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} x^{[1]}(t) = d_x,\ -\infty < d_x < 0\right\}.\end{gathered} \tag{3.4}$$

In the sequel, we give necessary and sufficient conditions for the existence of unbounded solutions of (1.1). Let I_λ be defined by (2.5) and set

$$\begin{gathered}K_\lambda = \int_{t_0}^{\infty} b(t) F\left(\int_{t_0}^{t} \Phi^{*}\left(\frac{\lambda}{a(s)}\right) ds\right) dt, \\ J_\mu = \int_{t_0}^{\infty} \Phi^{*}\left(\mu \frac{1}{a(t)} \int_{t}^{\infty} b(s)ds\right) dt,\end{gathered} \tag{3.5}$$

where $\lambda \in \Lambda$ and $\mu > 0$. The following holds.

Theorem 3.1. (i_1) *If there exist positive constants* $\overline{\lambda}, L \in \Lambda$ *such that*

$$L < \overline{\lambda}, \quad I_L = \infty, \quad K_{\overline{\lambda}} < \infty, \tag{3.6}$$

then there exist solutions of (1.1) *in* $\mathbb{M}^+_{\infty,\ell}$ *satisfying*

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} x^{[1]}(t) = L. \tag{3.7}$$

(i_2) *Let* $I_\lambda = \infty$ *for any* $\lambda \in \Lambda$. *If for some* $\overline{\lambda} \in \Lambda$ *and* $\mu \in \operatorname{Im} F$

$$K_{\overline{\lambda}} < \infty, \qquad J_\mu = \infty, \tag{3.8}$$

then there exist solutions of (1.1) *in* $\mathbb{M}^+_{\infty,0}$.

Proof. Claim (i_1). Denote

$$A(t;t_0) = \int_{t_0}^{t} \Phi^*\left(\frac{\overline{\lambda}}{a(s)}\right) ds. \tag{3.9}$$

Obviously, $\lim_{t\to\infty} A(t;t_0) = \infty$. Let $\varepsilon > 0$ be such that $L + \varepsilon \le \overline{\lambda}$. Fixed $H > 0$, choose $t_1 > t_0$ large so that

$$\int_{t_1}^{\infty} b(t)F(A(t;t_0))dt < \varepsilon, \quad A(t_1;t_0) \ge H + \varepsilon. \tag{3.10}$$

Denote with $C[t_1,\infty)$ the Fréchet space of all continuous functions on $[t_1,\infty)$ endowed with the topology of uniform convergence on compact subintervals of $[t_1,\infty)$ and consider the set $\Omega \subset C[t_1,\infty)$ given by

$$\Omega = \{u \in C[t_1,\infty) : H + \varepsilon \le u(t) \le H + \varepsilon + A(t;t_1)\}. \tag{3.11}$$

Define in Ω the operator T as follows:

$$T(u)(t) = H + \varepsilon + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\left(L + \int_s^\infty b(\tau)F(u(\tau))d\tau\right)\right) ds. \tag{3.12}$$

Obviously, $T(u)(t) \ge H + \varepsilon$. From (3.10) we have for $s \ge t_1$

$$\int_s^\infty b(\tau)F(u(\tau))d\tau \le \int_s^\infty b(\tau)F(H + \varepsilon + A(\tau;t_1))d\tau \le \int_{t_1}^\infty b(\tau)F(A(\tau;t_0))d\tau < \varepsilon, \tag{3.13}$$

and so, because $L + \varepsilon \le \overline{\lambda}$,

$$T(u)(t) \le H + \varepsilon + \int_{t_1}^{t} \Phi^*\left(\frac{L+\varepsilon}{a(s)}\right) ds \le H + \varepsilon + A(t;t_1), \tag{3.14}$$

that is, T maps Ω into itself. Let us show that $T(\Omega)$ is relatively compact, that is, $T(\Omega)$ consists of functions equibounded and equicontinuous on every compact interval of $[t_1, \infty)$. Because $T(\Omega) \subset \Omega$, the equiboundedness follows. Moreover, in view of the above estimates, for any $u \in \Omega$ we have

$$0 < \frac{d}{dt} T(u)(t) \leq \Phi^*\left(\frac{L+\varepsilon}{a(t)}\right), \tag{3.15}$$

which proves the equicontinuity of the elements of $T(\Omega)$. Now we prove the continuity of T in Ω. Let $\{u_n\}$, $n \in \mathbb{N}$, be a sequence in Ω which uniformly converges on every compact interval of $[t_1, \infty)$ to $\overline{u} \in \Omega$. Because $T(\Omega)$ is relatively compact, the sequence $\{T(u_n)\}$ admits a subsequence $\{T(u_{n_j})\}$ converging, in the topology of $C[t_1, \infty)$, to $\overline{z}_u \in \overline{T(\Omega)}$. Because

$$\int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\left(L + \int_s^\infty b(\tau) F(u_n(\tau)) d\tau\right)\right) ds \leq \int_{t_1}^{t} \Phi^*\left(\frac{L+\varepsilon}{a(s)}\right) ds, \tag{3.16}$$

from the Lebesgue dominated convergence theorem, the sequence $\{T(u_{n_j})(t)\}$ pointwise converges to $T(\overline{u})(t)$. In view of the uniqueness of the limit, $T(\overline{u}) = \overline{z}_u$ is the only cluster point of the compact sequence $\{T(u_n)\}$, that is, the continuity of T in the topology of $C[t_1, \infty)$. Hence, by the Tychonov fixed point theorem there exists a solution x of the integral equation

$$x(t) = H + \varepsilon + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\left(L + \int_s^\infty b(\tau) F(x(\tau)) d\tau\right)\right) ds. \tag{3.17}$$

Clearly, x is a solution of (1.1). Using (3.6) and

$$H + \varepsilon + \int_{t_1}^{t} \Phi^*\left(\frac{L}{a(s)}\right) < x(t) \leq \int_{t_0}^{t} \Phi^*\left(\frac{\overline{\lambda}}{a(s)}\right), \tag{3.18}$$

we get $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} x^{[1]}(t) = L$, and the Claim (i_1) is proved.

Claim (i_2). Let $\{L_n\}_{n\geq 1}$ be a decreasing sequence such that

$$\lim_n L_n = 0, \quad 0 < L_1 < \overline{\lambda}. \tag{3.19}$$

Choose $\varepsilon > 0$ such that $L_1 + \varepsilon < \overline{\lambda}$. Since $\{L_n\}$ is decreasing and $\overline{\lambda} \in \Lambda$, we have $L_n \in \Lambda$ and $L_n + \varepsilon < \overline{\lambda}$. Fixed H such that $H > F^{-1}(\mu)$, using the argument of the claim (i_1), for any $n \geq 1$ there exists $x_n \in \mathbb{M}^+_{\infty,\ell}$ such that $\lim_{t\to\infty} x_n^{[1]}(t) = L_n$. In virtue of the proof of claim (i_1), we have $x_n \in \Omega$ and

$$x_n(t) = H + \varepsilon + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\left(L_n + \int_s^\infty b(\tau) F(x_n(\tau)) d\tau\right)\right) ds. \tag{3.20}$$

Since $T(\Omega)$ is compact (in the topology of $C[t_1,\infty)$), there exists a subsequence of $\{x_n\}$ converging to x in any compact interval of $C[t_1,\infty)$. For sake of simplicity, let $\{x_n\}$ be such a sequence, that is, $\lim_n x_n = x$. Because

$$\int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\left(L_n + \int_s^\infty b(\tau)F(x_n(\tau))d\tau\right)\right)ds \le \int_{t_1}^{t} \Phi^*\left(\frac{\overline{\lambda}}{a(s)}\right)ds, \tag{3.21}$$

using the Lebesgue dominated convergence theorem, it results for $t \ge t_1$

$$x(t) = H + \varepsilon + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\left(\int_s^\infty b(\tau)F(x(\tau))d\tau\right)\right)ds. \tag{3.22}$$

Hence x is a solution of (1.1) and $\lim_{t\to\infty} x^{[1]}(t) = 0$. Because $x(t) \ge H + \varepsilon$ and F, Φ are increasing, we have $F(x(t)) \ge F(H+\varepsilon) > \mu$ and

$$x(t) \ge H + \varepsilon + \int_{t_1}^{t} \Phi^*\left(\frac{\mu}{a(s)}\left(\int_s^\infty b(\tau)d\tau\right)\right)ds. \tag{3.23}$$

Since $J_\mu = \infty$, the solution x is unbounded and the proof is complete. □

Remark 3.2. If $I_\lambda = \infty$ for any $\lambda \in \Lambda$ and there exists $\overline{\lambda} \in \Lambda$ such that $K_{\overline{\lambda}} < \infty$, then (1.1) has solutions in $\mathbb{M}^+_{\infty,\ell}$ satisfying $\lim_{t\to\infty} x^{[1]}(t) = L$ for any $L < \overline{\lambda}$. In some particular situations, the existence of solutions $x \in \mathbb{M}^+_{\infty,\ell}$, satisfying the limit case $\lim_{t\to\infty} x^{[1]}(t) = \overline{\lambda}$, is examined in [23]. See also Example 3.4 below.

The following result is the partial converse of Theorem 3.1.

Theorem 3.3. *If there exists a nonoscillatory unbounded solution x of* (1.1) *such that $\lim_{t\to\infty} x^{[1]}(t) = L \ge 0$, then $I_\lambda = \infty$ for any $\lambda \in \Lambda$, $\lambda > L$, and*

$$\int_{t_0}^{\infty} b(t)F\left(\mu\int_{t_0}^{t} \Phi^*\left(\frac{L}{a(s)}\right)ds\right)dt < \infty \tag{3.24}$$

for any μ, $0 < \mu < 1$.

Proof. Let $\varepsilon > 0$ be such that $L < \lambda - \varepsilon$. Without loss of generality, we can suppose $x(t) > 0$, $0 < x^{[1]}(t) < L + \varepsilon$ for $t \ge T$. Thus

$$x(t) < x(T) + \int_T^t \Phi^*\left(\frac{L+\varepsilon}{a(s)}\right)ds < x(T) + \int_T^t \Phi^*\left(\frac{\lambda}{a(s)}\right)ds. \tag{3.25}$$

Since x is unbounded, the first assertion follows.

Clearly, (3.24) holds for $L = 0$. Now, assume $L > 0$. Since $x^{[1]}$ is decreasing for $t \geq T$, we have

$$x(t) \geq \int_T^t \Phi^*\left(\frac{L}{a(s)}\right) ds. \tag{3.26}$$

Integrating (1.1), we obtain

$$x^{[1]}(t) \geq L + \int_t^\infty b(s) F\left(\int_T^s \Phi^*\left(\frac{L}{a(\tau)}\right) d\tau\right) ds. \tag{3.27}$$

If $I_L < \infty$, from (1.2) inequality (3.24) follows. If $I_L = \infty$, using l'Hopital rule, we have

$$\lim_{t\to\infty} \frac{\int_T^t \Phi^*\left(La^{-1}(s)\right) ds}{\int_{t_0}^t \Phi^*\left(La^{-1}(s)\right) ds} = \lim_{t\to\infty} \frac{\Phi^*\left(La^{-1}(t)\right)}{\Phi^*\left(La^{-1}(t)\right)} = 1. \tag{3.28}$$

Thus we have for large s, say $s \geq T$,

$$\int_T^s \Phi^*\left(\frac{L}{a(\tau)}\right) d\tau \geq \mu \int_{t_0}^s \Phi^*\left(\frac{L}{a(\tau)}\right) d\tau. \tag{3.29}$$

From here, inequality (3.27) yields

$$x^{[1]}(t) \geq \int_t^\infty b(s) F\left(\mu \int_{t_0}^s \Phi^*\left(\frac{L}{a(\tau)}\right) d\tau\right) ds. \tag{3.30}$$

Since $x^{[1]}$ is bounded, (3.24) again follows. □

The following example illustrates Theorem 3.1 and a possible discrepancy between equations with nonhomogeneous Φ-Laplacian and ones with Φ_p. More precisely, if $x, y \in \mathbb{M}^+_{\infty,\ell}$, then it may happen

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = 0, \tag{3.31}$$

while, when Φ is the classical Φ-Laplacian, in view of the l'Hopital rule and the homogeneity property (1.4), the limit in (3.31) is finite and different from zero, that is, solutions in the class $\mathbb{M}^+_{\infty,\ell}$ have the same growth at infinity when $\Phi = \Phi_p$.

Example 3.4. Consider the equation

$$\left(\frac{t+1}{t}\Phi_C(x')\right)' + \sqrt[3]{\frac{9}{4}} \frac{1}{2t^2\sqrt{t^2+t}} F(x) = 0, \quad t \geq 1, \tag{3.32}$$

where Φ_C is given by (1.6) and $F(u) = |u|^{2/3}\operatorname{sgn} u$. We have $\Lambda = (0, 1]$ and for $\lambda \in \Lambda$

$$\Phi_C^*\left(\frac{\lambda t}{t+1}\right) = \frac{\lambda t}{\sqrt{(1-\lambda^2)t^2 + 2t + 1}}. \tag{3.33}$$

Obviously, $I_\lambda = \infty$ for any $\lambda \in (0,1]$. One can verify that $y(t) = (2/3)t^{3/2}$ is a solution of (3.32) in the class $\mathbb{M}^+_{\infty,\ell}$ and $\lim_{t\to\infty} y^{[1]}(t) = 1$. Because $K_\lambda < \infty$ for $\lambda \in (0,1]$, from Theorem 3.1, (3.32) has solutions x in $\mathbb{M}^+_{\infty,\ell}$ satisfying $\lim_{t\to\infty} x^{[1]}(t) = L$ for any $L \in (0,1)$. Since $\lim_{t\to\infty} a(t) = 1$, we get $\lim_{t\to\infty} x'(t) = \Phi^*(L) > 0$ and so (3.31) holds.

4. Bounded Solutions

Here we study the existence of bounded solutions of (1.1). The following holds.

Theorem 4.1. (i_1) *If there exists a solution x of* (1.1) *in the class $\mathbb{M}^+_{\ell,0}$ such that $\lim_{t\to\infty} x(t) = L$, then $J_\mu < \infty$ for any μ, $0 < \mu < F(L)$.*

(i_2) *If there exists a positive constant $\overline{\mu} \in \operatorname{Im} F$ such that $J_{\overline{\mu}} < \infty$, then* (1.1) *has solutions satisfying*

$$\lim_{t\to\infty} x(t) = L, \qquad \lim_{t\to\infty} x^{[1]}(t) = 0, \tag{4.1}$$

where $L = F^{-1}(\overline{\mu})$.

Proof. Claim (i_1). Let L_ε such that $L > L_\varepsilon > F^{-1}(\mu)$. We can suppose, without loss of generality, $x(t) > L_\varepsilon$, $x^{[1]}(t) > 0$ for any $t \ge T \ge t_0$. We have for $t \ge T$

$$x'(t) = \Phi^*\left(\frac{1}{a(t)}\int_t^\infty b(\tau)F(x(\tau))d\tau\right) \tag{4.2}$$

or

$$L - x(t) \ge \int_t^\infty \Phi^*\left(\frac{1}{a(s)}\int_s^\infty b(\tau)F(L_\varepsilon)d\tau\right) \ge \int_t^\infty \Phi^*\left(\frac{\mu}{a(s)}\int_s^\infty b(\tau)d\tau\right) \tag{4.3}$$

which gives the assertion.

Claim (i_2). The assertion follows by applying the Tychonov fixed point theorem to the operator T given by

$$T(u)(t) = L - \int_t^\infty \Phi^*\left(\frac{1}{a(t)}\int_t^\infty b(\tau)F(u(\tau))d\tau\right)ds \tag{4.4}$$

in the set $\Omega \subset C[t_1, \infty)$

$$\Omega = \left\{u \in C[t_1,\infty) : \frac{1}{2}L \le u(t) \le L\right\}, \tag{4.5}$$

where

$$\int_{t_1}^{\infty} \Phi^*\left(\frac{\overline{\mu}}{a(t)}\int_t^{\infty} b(\tau)d\tau\right)dt \le \frac{L}{2}. \tag{4.6}$$

The argument is similar to the one given in the proof of Theorem 3.1(i$_1$), with minor changes. □

From Theorem 4.1, we have the following.

Corollary 4.2. *It holds $\mathbb{M}^+_{\ell,0} \neq \emptyset$ if and only if there exists $\overline{\mu} \in \operatorname{Im} F$, $\overline{\mu} > 0$ such that $J_{\overline{\mu}} < \infty$.*

If there exists $\overline{\lambda} \in \Lambda$ such that $I_{\overline{\lambda}} < \infty$, then, in view of Theorem 3.3, solutions in the class $\mathbb{M}^+_{\infty,0}$ do not exist. Now we show that, in this case, (1.1) has bounded solutions both in $\mathbb{M}^-$ and $\mathbb{M}^+_{\ell,\ell}$.

Theorem 4.3. (i$_1$) *If there exists $\overline{\lambda} \in \Lambda$ such that $I_{\overline{\lambda}} < \infty$, then*

$$\mathbb{M}^-_{\ell,\ell} \neq \emptyset, \quad \mathbb{M}^-_{0,\ell} \neq \emptyset, \quad \mathbb{M}^+_{\ell,\ell} \neq \emptyset, \quad \mathbb{M}^+_{\ell,0} \neq \emptyset. \tag{4.7}$$

(i$_2$) *Conversely, if any subclass $\mathbb{M}^-_{\ell,\ell}, \mathbb{M}^-_{0,\ell}, \mathbb{M}^+_{\ell,\ell}$ is nonempty, then there exists $\overline{\lambda} \in \Lambda$ such that $I_{\overline{\lambda}} < \infty$.*

Proof. Without loss of generality, we assume $\overline{\lambda}$ small so that $\overline{\lambda} \in \operatorname{Im} F$.

Claim (i$_1$). Choose λ_1, $0 < \lambda_1 \le \overline{\lambda}$ such that $\lambda_1 \int_t^{\infty} b(s)ds \le \overline{\lambda}$. Thus

$$\frac{\lambda_1}{a(t)}\int_t^{\infty} b(s)ds \le \frac{\overline{\lambda}}{a(t)}, \tag{4.8}$$

and so the class $\mathbb{M}^+_{\ell,0}$ is nonempty in virtue of Theorem 4.1.

Let us show that $\mathbb{M}^-_{\ell,\ell} \neq \emptyset$. Choose t_1 large so that

$$F\left(\overline{\lambda}\right)\int_{t_1}^{\infty} b(s)ds < \frac{\overline{\lambda}}{2}, \qquad \int_{t_1}^{\infty} \Phi^*\left(\frac{\overline{\lambda}}{a(t)}\right)dt < \frac{\overline{\lambda}}{4}. \tag{4.9}$$

In the Fréchet space $C[t_1, \infty)$ consider the set Ω given by

$$\Omega = \left\{u \in C[t_1, \infty) : \frac{3}{4}\overline{\lambda} \le u(t) \le \overline{\lambda}\right\} \tag{4.10}$$

and define in Ω the operator

$$T(u)(t) = \overline{\lambda} + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\left(-\frac{\overline{\lambda}}{2} - \int_{t_1}^{s} b(\xi)F(u(\xi))d\xi\right)\right)ds. \tag{4.11}$$

Clearly $T(u)(t) \leq \overline{\lambda}$. Moreover

$$\int_{t_1}^{s} b(\xi)F(u(\xi))d\xi \leq F\left(\overline{\lambda}\right)\int_{t_1}^{\infty} b(\xi)d\xi < \frac{\overline{\lambda}}{2}. \tag{4.12}$$

Thus

$$T(u)(t) \geq \overline{\lambda} - \int_{t_1}^{t} \Phi^*\left(\frac{\overline{\lambda}}{a(s)}\right)ds > \overline{\lambda} - \frac{\overline{\lambda}}{4} = \frac{3}{4}\overline{\lambda}. \tag{4.13}$$

Hence $T(\Omega) \subset \Omega$. Using an argument similar to the one given in the proof of Theorem 3.1-(i_1), with minor changes, we get that $T(\Omega)$ is relatively compact. Applying the Tychonov fixed point theorem we obtain that $\mathbb{M}^-_{\ell,\ell} \neq \emptyset$.

In a similar way, we can prove that $\mathbb{M}^+_{\ell,\ell} \neq \emptyset$. Choose $t_1 \geq t_0$ satisfying (4.9) and consider in the set Ω given by (4.10) the operator

$$T_1(u)(t) = \frac{3}{4}\overline{\lambda} + \int_{t_1}^{t} \Phi^*\left(\frac{1}{a(s)}\left(\overline{\lambda} - \int_{t_1}^{s} b(\xi)F(u(\xi))d\xi\right)\right)ds. \tag{4.14}$$

We have

$$\int_{t_1}^{s} b(\xi)F(u(\xi))d\xi \leq F\left(\overline{\lambda}\right)\int_{t_1}^{\infty} b(\xi)d\xi < \frac{\overline{\lambda}}{2}, \tag{4.15}$$

and so $T_1(u)(t) \geq 3\overline{\lambda}/4$. Moreover

$$T_1(u)(t) \leq \frac{3}{4}\overline{\lambda} + \int_{t_1}^{t} \Phi^*\left(\frac{\overline{\lambda}}{a(s)}\right)ds \leq \overline{\lambda}. \tag{4.16}$$

Hence $T_1(\Omega) \subset \Omega$ and applying the Tychonov fixed point theorem we obtain the existence of a solution of (1.1) in the class $\mathbb{M}^+_{\ell,\ell}$.

To show that $\mathbb{M}^-_{0,\ell} \neq \emptyset$, it is sufficient to choose the same $t_1 \geq t_0$ satisfying (4.9) and consider, into the set Ω given by

$$\Omega = \left\{u \in C[t_1, \infty) : 0 \leq u(t) \leq \overline{\lambda}\right\}, \tag{4.17}$$

the operator

$$T_2(u)(t) = \int_{t}^{\infty} \Phi^*\left(\frac{1}{a(s)}\left(\frac{\overline{\lambda}}{2} + \int_{t_1}^{s} b(\xi)F(u(\xi))d\xi\right)\right)ds. \tag{4.18}$$

Clearly $T_2(u)(t) > 0$. Moreover

$$\frac{\overline{\lambda}}{2} + \int_{t_1}^{s} b(\xi)F(u(\xi))d\xi \leq \frac{\overline{\lambda}}{2} + F\left(\overline{\lambda}\right)\int_{t_1}^{\infty} b(\xi)d\xi \leq \overline{\lambda}, \tag{4.19}$$

and so

$$T_2(u)(t) \leq \int_{t}^{\infty} \Phi^*\left(\frac{\overline{\lambda}}{a(s)}\right)ds \leq \frac{\overline{\lambda}}{4} < \overline{\lambda}. \tag{4.20}$$

Hence $T(\Omega) \subset \Omega$. From the Tychonov fixed point theorem we get $\mathbb{M}^-_{0,\ell} \neq \emptyset$.

Claim (i_2). By contradiction, assume $I_\lambda = \infty$ for any $\lambda \in \Lambda$. Let $x \in \mathbb{M}^-_{\ell,\ell} \cup \mathbb{M}^-_{0,\ell}$ and assume $x(t) > 0$, $x^{[1]}(t) < 0$ for $t \geq T \geq t_0$. Since $x^{[1]}$ is decreasing for $t \geq T$, we have $x^{[1]}(t) < x^{[1]}(T)$ or

$$x'(t) < \Phi^*\left(\frac{x^{[1]}(T)}{a(t)}\right). \tag{4.21}$$

Integrating this inequality, we get a contradiction with the positiveness of x.

Now let $x \in \mathbb{M}^+_{\ell,\ell}$ and assume $x(t) > 0$, $x^{[1]}(t) > 0$ for $t \geq T \geq t_0$ and $\lim_{t\to\infty} x^{[1]}(t) = \ell_x > 0$. Since $x^{[1]}$ is decreasing for $t \geq T$, it results $x^{[1]}(t) > \ell_x$, or

$$x(t) \geq x(t) + \int_{T}^{t} \Phi^*\left(\frac{\ell_x}{a(s)}\right)ds, \tag{4.22}$$

which contradicts the boundedness of x. □

It is known that, for equations with classical Φ-Laplacian in the canonical form, the set $\mathbb{S}$ is given by $\mathbb{M}^+_{\ell,0} \cup \mathbb{M}^+_{\infty,0} \cup \mathbb{M}^+_{\infty,\ell}$ (see [13, 21]). Theorem 4.3(i_2) shows that this remains valid for equations with the general Φ-Laplacian, when $I_\lambda = \infty$ for any $\lambda \in \Lambda$.

Remark 4.4. The converse of Theorem 4.3(i_1) does not hold for the class $\mathbb{M}^+_{\ell,0}$. Indeed, in view of Corollary 4.2, it may happen $\mathbb{M}^+_{\ell,0} \neq \emptyset$ also when $I_\lambda = \infty$ for any $\lambda \in \Lambda$, as the following example shows.

Example 4.5. Consider the equation

$$\left(t\Phi_R(x')\right)' + t^{-2}F(x) = 0, \quad t \geq 1, \tag{4.23}$$

where Φ_R is defined in (1.7). Since $\sigma = \infty$, it results $\Lambda = (0, \infty)$. Moreover we have for any $\lambda > 0$

$$I_\lambda = \int_{1}^{\infty} \frac{\lambda}{\sqrt{t^2 + \lambda^2}} dt = \infty \tag{4.24}$$

and $J_\mu < \infty$ for any $\mu > 0$. Hence, in virtue of Corollary 4.2, (4.23) has solutions in the class $\mathbb{M}^+_{\ell,0}$.

From Theorems 3.1, 4.1, and 4.3, the following coexistence results hold.

Corollary 4.6. *Let $I_\lambda = \infty$ for any $\lambda \in \Lambda$. Assume that there exist $\overline{\lambda}, \mu, \nu$ such that $0 < \mu < \nu$, $\mu \in \operatorname{Im} F$, $\overline{\lambda} \in \Lambda$, and*

$$J_\mu < \infty, \quad J_\nu = \infty, \quad K_{\overline{\lambda}} < \infty. \tag{4.25}$$

Then

$$\mathbb{M}^+_{\ell,0} \neq \emptyset, \quad \mathbb{M}^+_{\infty,0} \neq \emptyset, \quad \mathbb{M}^+_{\infty,\ell} \neq \emptyset. \tag{4.26}$$

Corollary 4.7. *Let $\lambda_1, \lambda_2 \in \Lambda$ such that $I_{\lambda_1} < \infty$, $I_{\lambda_2} = \infty$. If there exists $\overline{\lambda} \in \Lambda$, $\overline{\lambda} > \lambda_2$ such that $K_{\overline{\lambda}} < \infty$, then*

$$\mathbb{M}^-_{\ell,\ell} \neq \emptyset, \quad \mathbb{M}^-_{0,\ell} \neq \emptyset, \quad \mathbb{M}^+_{\ell,0} \neq \emptyset, \quad \mathbb{M}^+_{\ell,\ell} \neq \emptyset, \quad \mathbb{M}^+_{\infty,\ell} \neq \emptyset. \tag{4.27}$$

Due to the homogeneity property (1.4), the coexistence described in Corollaries 4.6 and 4.7 is impossible for (1.1) with classical Φ-Laplacian. Hence Corollaries 4.6 and 4.7 illustrate a discrepancy between equations with classical or general Φ-Laplacian. The following examples illustrate that this kind of coexistence can occur for (1.1).

Example 4.8. Consider the equation

$$\left(\frac{t-1}{t}\Phi(x')\right)' - \frac{d}{dt}\left(\frac{t-1}{t\log t}\right)\sqrt{\log|x|}\operatorname{sgn} x = 0 \quad (t \geq e), \tag{4.28}$$

where Φ is as in Example 2.2. Hence $\Lambda = (0, \infty)$. Since $\lim_{t\to\infty} a(t) = 1$, we have $I_\lambda = \infty$ for any $\lambda > 0$. Because

$$\frac{1}{a(t)}\int_t^\infty b(s)ds = \frac{1}{\log t}, \tag{4.29}$$

reasoning as in Example 2.2, we get $J_{\lambda_1} < \infty$ and $J_{\lambda_2} = \infty$ for $0 < \lambda_1 < 1$ and $\lambda_2 > 1$. Moreover,

$$K_1 \leq \int_e^\infty -\frac{d}{dt}\left(\frac{t-1}{t\log t}\right)(\log t)^{1/2}dt \leq \int_e^\infty \frac{1}{t\,\log^{3/2} t}\,dt < \infty. \tag{4.30}$$

Hence, from Corollary 4.6, we obtain the existence of solutions of (4.28) in the classes $\mathbb{M}^+_{\infty,\ell}$, $\mathbb{M}^+_{\ell,0}$, and $\mathbb{M}^+_{\infty,0}$.

Example 4.9. Consider the equation

$$\left(\log t\Phi\left(x'\right)\right)' + t^{-3}x^2 \operatorname{sgn} x = 0 \quad (t \geq 2), \tag{4.31}$$

where Φ is as in Example 2.2. Reasoning as in Example 2.2, we have $I_{1/2} < \infty$, $I_1 = \infty$. Since

$$K_2 \leq 4\int_2^{\infty} t^{-2} dt, \tag{4.32}$$

we can apply Corollary 4.7 to (4.31), and obtain existence of solutions as stipulated in (4.27).

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