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Research Article

Mixed Monotone Iterative Technique for Impulsive Periodic Boundary Value Problems in Banach Spaces

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This paper deals with the existence of L-quasi-solutions for impulsive periodic boundary value problems in an ordered Banach space E. Under a new concept of upper and lower solutions, a new monotone iterative technique on periodic boundary value problems of impulsive differential equations has been established. Our result improves and extends some relevant results in abstract differential equations.

1. Introduction

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, chemical, biological, and engineering background and realistic mathematical model, and hence has been emerging as an important area of investigation in the last few decades; see [1]. Correspondingly, applications of the theory of impulsive differential equations to different areas were considered by many authors, and some basic results on impulsive differential equations have been obtained; see [2–5]. But many of them are about impulsive initial value problem; see [2, 3] and the references therein. The research on impulsive periodic boundary value problems is seldom; see [4, 5].

In this paper, we use a monotone iterative technique in the presence of coupled lower and upper L-quasisolutions to discuss the existence of solutions to the impulsive periodic boundary value problem (IPBVP) in an ordered Banach space E

$$\begin{gathered} u'(t) = f(t, u(t), u(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = u(\omega), \end{gathered} \tag{1.1}$$

where $f \in C(J \times E \times E, E)$, $J = [0, \omega]$, $\omega > 0$; $0 < t_1 < t_2 < \cdots < t_m < \omega$; $I_k \in C(E \times E, E)$ is an impulsive function, $k = 1, 2, \ldots, m$. $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, that is, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

The monotone iterative technique in the presence of lower and upper solutions is an important method for seeking solutions of differential equations in abstract spaces. Early on, Lakshmikantham and Leela [4] built a monotone iterative method for the periodic boundary value problem of first-order differential equation in $\mathbb{R}$

$$\begin{gathered} u'(t) = f(t, u(t)), \quad t \in J, \\ u(0) = u(\omega), \end{gathered} \tag{1.2}$$

and they proved that, if PBVP(1.2) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and nonlinear term f satisfies the monoton condition

$$f(t, x_2) - f(t, x_1) \geq -M(x_2 - x_1), \quad \forall t \in J,\ v_0(t) \leq x_1 \leq x_2 \leq w_0(t), \tag{1.3}$$

with a positive constant M, then PBVP(1.2) has minimal and maximal solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively. Later, He and Yu [5] developed the problem to impulsive differential equation

$$\begin{gathered} u'(t) = f(t, u(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = u(\omega), \end{gathered} \tag{1.4}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R})$, $J = [0, \omega]$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $0 < t_1 < t_2 < \cdots < t_m < \omega$.

But all of these results are in real spaces $\mathbb{R}$. We not only consider problems in Banach spaces, but also expand the nonlinear term to the case of $f(t, u, u)$. If $f(t, u) = f_1(t, u) + f_2(t, u)$, $f_1(t, u)$ is nondecreasing in u and $f_2(t, u)$ is nonincreasing in u, then the monotonity condition (1.3) is not satisfied, and the results in [4, 5] are not right, in this case, we studied the IPBVP(1.1). As far as we know, no work has been done for the existence of solutions for IPBVP(1.1) in Banach spaces.

In order to apply the monotone iterative technique to the initial value problem without impulse

$$\begin{gathered} u'(t) = f(t, u(t), u(t)), \quad t \in [0, a], \\ u(0) = x_0, \end{gathered} \tag{1.5}$$

Lakshmikantham et al. [6] and Guo and Lakshmikantham [7] obtained the existence of coupled quasisolutions of problem (1.5) by mixed monotone sequence of coupled quasiupper and lower solutions under the concept of quasiupper and lower solutions. In this paper, we improve and extend the above-mentioned results, and obtain the existence of the coupled minimal and maximal L-quasisolutions and the solutions between the coupled minimal and maximal L-quasisolutions of the problem (1.1) through the mixed monotone iterative about the coupled L-quasisolutions. If $L \equiv 0$, the coupled upper and lower L-quasisolutions are equivalent to coupled upper and lower quasisolutions of the IPBVP(1.1). If $L \equiv 0$, $f(t,u,u) = f(t,u)$ and $I_k(u,u) = I_k(u)$, the coupled upper and lower L-quasisolutions are equivalent to upper and lower solutions of IPBVP(1.4).

2. Preliminaries

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{x \in E \mid x \geq 0\}$ is normal with normal constant N. Let $J = [0,\omega]$, $\omega > 0$ is a constant; $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = \omega$; $J_k = [t_{k-1}, t_k]$, $k = 1,2,\ldots,m+1$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$. Let $\mathrm{PC}(J,E) = \{u : J \to E \mid u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1,2,\ldots,m\}$. Evidently, $\mathrm{PC}(J,E)$ is a Banach space with the norm $\|u\|_{\mathrm{PC}} = \sup_{t\in J}\|u(t)\|$. An abstract function $u \in \mathrm{PC}(J,E) \cap C^1(J',E)$ is called a solution of IPBVP(1.1) if $u(t)$ satisfies all the equalities of (1.1).

Let $\mathrm{PC}^1(J,E) = \{u \in \mathrm{PC}(J,E) \cap C^1(J',E) \mid u'(t_k^+)$ and $u'(t_k^-)$ exist, $k = 1,2,\ldots,m\}$. For $u \in \mathrm{PC}^1(J,E)$, it is easy to see that the left derivative $u'_-(t_k)$ of $u(t)$ at $t = t_k$ exists and $u'_-(t_k) = u'(t_k^-)$, and set $u'(t_k) = u'(t_k^-)$, then $u' \in \mathrm{PC}(J,E)$. If $u \in \mathrm{PC}(J,E) \cap C^1(J',E)$ is a solution of IPBVP(1.1), by the continuity of $f, u \in \mathrm{PC}^1(J,E)$.

Let $C(J,E)$ denote the Banach space of all continuous E-value functions on interval J with the norm $\|u\|_C = \max_{t\in J}\|u(t)\|$. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [8]. For any $B \subset C(J,E)$ and $t \in J$, set $B(t) = \{u(t) \mid u \in B\} \subset E$. If B is bounded in $C(J,E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \leq \alpha(B)$.

Now, we first give the following lemmas in order to prove our main results.

Lemma 2.1 (see [9]). *Let $B = \{u_n\} \subset \mathrm{PC}(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\left(\left\{\int_J u_n(t)dt \mid n \in \mathbb{N}\right\}\right) \leq 2\int_J \alpha(B(t))dt. \tag{2.1}$$

Lemma 2.2 (see [10]). *Let $D \subset E$ be bounded. Then exist a countable set $D_0 \subset D$, such that $\alpha(D) \leq 2\alpha(D_0)$.*

Lemma 2.3 (see [11]). *Let $B \subset C(J,E)$ be equicontinuous. Then $\alpha(B(t))$ is continuous on J, and*

$$\alpha(B) = \max_{t\in J} \alpha(B(t)) = \alpha(B(J)). \tag{2.2}$$

Lemma 2.4 (see [8]). *Let X be a Banach space and Ω is a bounded convex closed set in $X, Q : \Omega \to \Omega$ be condensing, then Q has a fixed point in Ω.*

To prove our main results, for any $h \in \mathrm{PC}(J, E)$, we consider the periodic boundary value problem (PBVP) of linear impulsive differential equation in E

$$\begin{gathered} u'(t) + Mu(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \ldots, m, \\ u(0) = u(\omega), \end{gathered} \tag{2.3}$$

where $M \geq 0$, $y_k \in E$, $k = 1, 2, \ldots, m$.

Lemma 2.5. *For any $h \in \mathrm{PC}(J, E)$, $x \in E$ and $y_k \in E$, $k = 1, 2, \ldots, m$, the linear PBVP*(2.3) *has a unique solution $u \in PC^1(J, E)$ given by*

$$u(t) = e^{-Mt} y_h + \int_0^t e^{-M(t-s)} h(s)\,ds + \sum_{t_k < t} e^{-M(t-t_k)} y_k, \tag{2.4}$$

where $y_h = (1/(1 - e^{-M\omega}))(\int_0^\omega e^{-M(\omega-s)} h(s)\,ds + \sum_{k=1}^m e^{-M(\omega-t_k)} y_k)$.

Proof. For any $h \in \mathrm{PC}(J, E)$, $x \in E$ and $y_k \in E$, $k = 1, 2, \ldots, m$, the linear initial value problem

$$\begin{gathered} u'(t) + Mu(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \ldots, m, \\ u(0) = x \end{gathered} \tag{2.5}$$

has a unique solution $u \in \mathrm{PC}^1(J, E)$ given by

$$u(t) = e^{-Mt} x + \int_0^t e^{-M(t-s)} h(s)\,ds + \sum_{t_k < t} e^{-M(t-t_k)} y_k, \tag{2.6}$$

where $M \geq 0$ is a constant [3].

If u is a solution of the linear initial value problem (2.5) satisfies $u(\omega) = x$, namely

$$x = e^{-M\omega} x + \int_0^\omega e^{-M(\omega-s)} h(s)\,ds + \sum_{k=1}^m e^{-M(\omega-t_k)} y_k, \tag{2.7}$$

then it is the solution of the linear PBVP(2.3). From (2.7), we have

$$x = \frac{1}{1 - e^{-M\omega}} \left(\int_0^\omega e^{-M(\omega-s)} h(s)\,ds + \sum_{k=1}^m e^{-M(\omega-t_k)} y_k \right) \triangleq y_h. \tag{2.8}$$

So, (2.4) is satisfied.

Inversely, we can verify directly that the function $u \in \mathrm{PC}(J, E)$ defined by (2.4) is a solution of the linear PBVP(2.3). Therefore, the conclusion of Lemma 2.5 holds. □

Definition 2.6. Let $L \geq 0$ be a constant. If functions $v_0, w_0 \in \mathrm{PC}^1(J, E)$ satisfy

$$\begin{gathered} v_0'(t) \leq f(t, v_0(t), w_0(t)) + L(v_0(t) - w_0(t)), \quad t \in J', \\ \Delta v_0|_{t=t_k} \leq I_k(v_0(t_k), w_0(t_k)), \quad k = 1, 2, \ldots, m, \\ v_0(0) \leq v_0(\omega), \end{gathered} \tag{2.9}$$

$$\begin{gathered} w_0'(t) \geq f(t, w_0(t), v_0(t)) + L(w_0(t) - v_0(t)), \quad t \in J', \\ \Delta w_0|_{t=t_k} \geq I_k(w_0(t_k), v_0(t_k)), \quad k = 1, 2, \ldots, m, \\ w_0(0) \geq w_0(\omega), \end{gathered} \tag{2.10}$$

we call v_0, w_0 coupled lower and upper L-quasisolutions of the IPBVP(1.1). Only choose "=" in (2.9) and (2.10), we call (v_0, w_0) coupled L-quasisolution pair of the IPBVP(1.1). Furthermore, if $v_0 = w_0 := u_0$, we call u_0 a solution of the IPBVP(1.1).

Now, we define an operator $Q : \mathrm{PC}(J, E) \to \mathrm{PC}(J, E)$ as following:

$$\begin{aligned} Q(u, v)(t) = e^{-Mt} S(u, v) &+ \int_0^t e^{-M(t-s)} \left[f(s, u(s), v(s)) + (M + L)u(s) - Lv(s) \right] ds \\ &+ \sum_{t_k < t} e^{-M(t-t_k)} I_k(u(t_k), v(t_k)), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} S(u, v) = \frac{1}{1 - e^{-M\omega}} \Bigg(&\int_0^\omega e^{-M(\omega-s)} \left[f(s, u(s), v(s)) + (M + L)u(s) - Lv(s) \right] ds \\ &+ \sum_{k=1}^m e^{-M(\omega-t_k)} I_k(u(t_k), v(t_k)) \Bigg). \end{aligned} \tag{2.12}$$

Evidently, $\mathrm{PC}(J, E)$ is also an ordered Banach space with the partial order "$\leq$" reduced by the positive cone $K_{\mathrm{PC}} = \{u \in \mathrm{PC}(J, E) \mid u(t) \geq 0,\ t \in J\}$. K_{PC} is also normal with the same normal constant N. For $v, w \in \mathrm{PC}(J, E)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in \mathrm{PC}(J, E) \mid v \leq u \leq w\}$ in $\mathrm{PC}(J, E)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in E \mid v(t) \leq u(t) \leq w(t),\ t \in J\}$ in E.

3. Main Results

Theorem 3.1. *Let E be an ordered Banach space, whose positive* cone P *is normal,* $f \in C(J\times E\times E, E)$ *and* $I_k \in C(E \times E, E)$*,* $k = 1, 2, \dots, m$*. Assume that the IPBVP*(1.1) *has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0 \leq w_0$. Suppose that the following conditions are satisfied:*

($H1$) *There exist constants* $M > 0$ *and* $L \geq 0$ *such that*

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -M(u_2 - u_1) - L(v_1 - v_2), \tag{3.1}$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

($H2$) *The impulsive function $I_k(\cdot,\cdot)$ satisfies*

$$I_k(u_1, v_1) \leq I_k(u_2, v_2), \quad k = 1, 2, \dots, m, \tag{3.2}$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

($H3$) *There exist a constant* $L_1 > 0$ *such that*

$$\alpha(\{f(t, u_n, v_n) + Mu_n\}) \leq L_1(\alpha(\{u_n\}) + \alpha(\{v_n\})), \tag{3.3}$$

for all $t \in J$, and increasing or decreasing monotonic sequences $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [v_0(t), w_0(t)]$.

($H4$) *The sequences $v_n(0)$ and $w_n(0)$ are convergent, where $v_n = Q(v_{n-1}, w_{n-1})$, $w_n = Q(w_{n-1}, v_{n-1})$, $n = 1, 2, \dots$.*

Then the IPBVP(1.1) has minimal and maximal coupled L-quasisolutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.

Proof. By the definition of Q and Lemma 2.5, $Q : [v_0, w_0]\times[v_0, w_0] \to \mathrm{PC}(J, E)$ is continuous, and the coupled L-quasisolutions of the IPBVP(1.1) is equivalent to the coupled fixed point of operator Q. Combining this with the assumptions ($H1$) and ($H2$), we know $Q : [v_0, w_0] \times [v_0, w_0] \to \mathrm{PC}(J, E)$ is a mixed monotone operator (about the mixed monotone operator, please see [6, 7]).

Next, we show $v_0 \leq Q(v_0, w_0)$, $Q(w_0, v_0) \leq w_0$. Let $h(t) = v_0'(t) + Mv_0(t)$, by (2.9), $h \in \mathrm{PC}(J, E)$ and $h(t) \leq f(t, v_0, w_0) + (L + M)v_0 - Lw_0$, $t \in J'$. By Lemma 2.5

$$\begin{aligned} v_0(t) = {} & \frac{e^{-Mt}}{1 - e^{-M\omega}}\left(\int_0^{\omega} e^{-M(\omega-s)}h(s)ds + \sum_{k=1}^{m} e^{-M(\omega-t_k)}\,\Delta v_0|_{t=t_k}\right) \\ & + \int_0^{t} e^{-M(t-s)}h(s)ds + \sum_{t_k<t} e^{-M(t-t_k)}\,\Delta v_0|_{t=t_k} \end{aligned}$$

$$\begin{aligned}&\leq e^{-Mt}S(v_0,w_0)+\int_0^t e^{-M(t-s)}\left[f(s,v_0(s),w_0(s))+(L+M)v_0(s)-Lw_0(s)\right]ds\\&\quad+\sum_{t_k<t}e^{-M(t-t_k)}I_k(v_0(t_k),w_0(t_k))\\&=Q(v_0,w_0)(t),\quad t\in J,\end{aligned}\tag{3.4}$$

namely, $v_0 \leq Q(v_0,w_0)$. Similarly, it can be show that $Q(w_0,v_0) \leq w_0$. So, $Q : [v_0,w_0] \times [v_0,w_0] \to [v_0,w_0]$.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0,w_0]$ by the iterative scheme

$$v_n = Q(v_{n-1},w_{n-1}),\quad w_n = Q(w_{n-1},v_{n-1}),\quad n=1,2,\ldots.\tag{3.5}$$

Then from the mixed monotonicity of Q, it follows that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0.\tag{3.6}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are uniformly convergent in J.

For convenience, let $B = \{v_n \mid n \in \mathbb{N}\} + \{w_n \mid n \in \mathbb{N}\}$, $B_1 = \{v_n \mid n \in \mathbb{N}\}$, $B_2 = \{w_n \mid n \in \mathbb{N}\}$, $B_{10} = \{v_{n-1} \mid n \in \mathbb{N}\}$ and $B_{20} = \{w_{n-1} \mid n \in \mathbb{N}\}$. Since, $B_1 = Q(B_{10},B_{20})$ and $B_2 = Q(B_{20},B_{10})$, by (2.11) and the boundedness of B_{10} and B_{20}, we easy see that B_1 and B_2 is equicontinuous in every interval J'_k, so, B is equicontinuous in every interval J'_k, where $J'_1 = [0,t_1]$, $J'_k = (t_{k-1},t_k]$, $k = 2,3,\ldots,m+1$. From $B_{10} = B_1 \cup \{v_0\}$ and $B_{20} = B_2 \cup \{w_0\}$ it follows that $\alpha(B_{10}(t)) = \alpha(B_1(t))$ and $\alpha(B_{20}(t)) = \alpha(B_2(t))$ for $t \in J$. Let $\varphi(t) := \alpha(B(t))$, $t \in J$, by Lemma 2.3, $\varphi \in PC(J,\mathbb{R}^+)$. Going from J'_1 to J'_{m+1} interval by interval we show that $\varphi(t) \equiv 0$ in J.

For $t \in J'_1$, from (2.11), using Lemma 2.1 and assumption $(H3)$ and $(H4)$, we have

$$\begin{aligned}\varphi(t)&=\alpha(B(t))=\alpha(B_1(t)+B_2(t))=\alpha(Q(B_{10},B_{20})(t)+Q(B_{20},B_{10})(t))\\&=\alpha\Bigg(\Bigg\{e^{-Mt}S(v_{n-1},w_{n-1})\\&\qquad+\int_0^t e^{-M(t-s)}\left[f(s,v_{n-1}(s),w_{n-1}(s))+(L+M)v_{n-1}(s)-Lw_{n-1}(s)\right]ds\\&\qquad+e^{-Mt}S(w_{n-1},v_{n-1})\\&\qquad+\int_0^t e^{-M(t-s)}\left[f(s,w_{n-1}(s),v_{n-1}(s))+(L+M)w_{n-1}(s)-Lv_{n-1}(s)\right]ds\Bigg\}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha\Big(\Big\{e^{-Mt}v_n(0)\Big\}\Big)+\alpha\Big(\Big\{e^{-Mt}w_n(0)\Big\}\Big)\\
&\quad+2\int_0^t e^{-M(t-s)}\alpha(\{f(s,v_{n-1}(s),w_{n-1}(s))+(L+M)v_{n-1}(s)-Lw_{n-1}(s)\\
&\qquad\qquad+f(s,w_{n-1}(s),v_{n-1}(s))+(L+M)w_{n-1}(s)-Lv_{n-1}(s)\}\big)ds\\
&\leq 2\int_0^t 2L_1(\alpha(B_{10}(s))+\alpha(B_{20}(s)))ds\\
&\leq 8L_1\int_0^t \varphi(s)ds.
\end{aligned}
\tag{3.7}
$$

Hence by the Belman inequality, $\varphi(t)\equiv 0$ in J_1'. In particular, $\alpha(B_{10}(t_1))=0$, $\alpha(B_{20}(t_1))=0$, this means that $B_{10}(t_1)$ and $B_{20}(t_1)$ are precompact in E. Thus $I_1(B_{10}(t_1))$ and $I_1(B_{20}(t_1))$ are precompact in E, and $\alpha(I_1(B_{10}(t_1)))=0$, $\alpha(I_1(B_{20}(t_1)))=0$.

Now, for $t\in J_2'$, by (2.11) and the above argument for $t\in J_1'$, we have

$$
\begin{aligned}
\varphi(t)&=\alpha(B(t))=\alpha(B_1(t)+B_2(t))=\alpha(Q(B_{10},B_{20})(t)+Q(B_{20},B_{10})(t))\\
&\leq\alpha\Bigg(\Bigg\{e^{-Mt}S(v_{n-1},w_{n-1})\\
&\qquad+\int_0^t e^{-M(t-s)}\big[f(s,v_{n-1}(s),w_{n-1}(s))+(L+M)v_{n-1}(s)-Lw_{n-1}(s)\big]ds\\
&\qquad+e^{-Mt}S(w_{n-1},v_{n-1})\\
&\qquad+\int_0^t e^{-M(t-s)}\big[f(s,w_{n-1}(s),v_{n-1}(s))+(L+M)w_{n-1}(s)-Lv_{n-1}(s)\big]ds\Bigg\}\Bigg)\\
&\quad+\alpha(I_1(B_{10}(t_1)))+\alpha(I_1(B_{20}(t_1)))\\
&\leq 8L_1\int_0^t\varphi(s)ds=8L_1\int_{t_1}^t\varphi(s)ds.
\end{aligned}
\tag{3.8}
$$

Again by Belman inequality, $\varphi(t)\equiv 0$ in J_2', from which we obtain that $\alpha(B_{10}(t_2))=0$, $\alpha(B_{20}(t_2))=0$ and $\alpha(I_2(B_{10}(t_2)))=0$, $\alpha(I_2(B_{20}(t_2)))=0$.

Continuing such a process interval by intervai up to J_{m+1}', we can prove that $\varphi(t)\equiv 0$ in every J_k', $k=1,2,\dots,m+1$.

For any J_k, if we modify the value of v_n, w_n at $t=t_{k-1}$ via $v_n(t_{k-1})=v_n(t_{k-1}^+)$, $w_n(t_{k-1})=w_n(t_{k-1}^+)$, $n\in\mathbb{N}$, then $\{v_n\}+\{w_n\}\subset C(J_k,E)$ and it is equicontinuous. Since $\alpha(\{v_n(t)\}+\{w_n(t)\})=0$, $\{v_n(t)\}+\{w_n(t)\}$ is precompact in E for every $t\in J_k$. By the Arzela-Ascoli theorem, $\{v_n\}+\{w_n\}$ is precompact in $C(J_k,E)$. Hence, $\{v_n\}+\{w_n\}$ has a convergent subsequence in $C(J_k,E)$. Combining this with the monotonicity (3.6), we easily prove that $\{v_n\}+\{w_n\}$ itself is convergent in $C(J_k,E)$. In particular, $\{v_n(t)\}+\{w_n(t)\}$ is uniformly convergent over the whole of J. Hence, $\{v_n\} + \{w_n\}$ is uniformly convergent in $\mathrm{PC}(J, E)$. Set

$$\underline{u} = \lim_{n\to\infty} v_n, \quad \overline{u} = \lim_{n\to\infty} w_n, \quad \text{in } \mathrm{PC}(J, E). \tag{3.9}$$

Letting $n \to \infty$ in (3.5) and (3.6), we see that $v_0 \le \underline{u} \le \overline{u} \le w_0$ and $\underline{u} = Q(\underline{u}, \overline{u})$, $\overline{u} = Q(\overline{u}, \underline{u})$. By the mixed monotonicity of Q, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal coupled fixed points of Q in $[v_0, w_0]$, and therefore, they are the minimal and maximal coupled L-quasisolutions of the IPBVP(1.1) in $[v_0, w_0]$, respectively. □

In Theorem 3.1, if E is weakly sequentially complete, condition $(H3)$ and $(H4)$ hold automatically. In fact, by Theorem 2.2 in [12], any monotonic and order-bounded sequence is precompact. By the monotonicity (3.6) and the same method in proof of Theorem 3.1, we can easily see that $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent on J. In particular, $\{v_n(0)\}$ and $\{w_n(0)\}$ are convergent. So, condition $(H4)$ holds. Let $\{u_n\}$ and $\{v_n\}$ be increasing or decreasing sequences obeying condition $(H3)$, then by condition $(H1)$, $\{f(t, u_n, v_n) + Mu_n\}$ is a monotonic and order-bounded sequence, so $\alpha(\{f(t, u_n, v_n) + Mu_n\}) = 0$. Hence, condition $(H3)$ holds. From Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let E be an ordered and weakly sequentially complete Banach space, whose positive* cone *P is normal, $f \in C(J \times E \times E, E)$ and $I_k \in C(E \times E, E)$, $k = 1, 2, \ldots, m$. If the IPBVP*(1.1) *has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0 \le w_0$, and conditions* $(H1)$ *and* $(H2)$ *are satisfied, then the IPBVP*(1.1) *has minimal and maximal coupled L-quasisolutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

If we replace the assumption $(H3)$ by the following assumption:

$(H5)$ There exist positive constants $\overline{M}$ and $\overline{L}$ such that

$$f(t, u_2, v_2) - f(t, u_1, v_1) \le \overline{M}(u_2 - u_1) + \overline{L}(v_1 - v_2), \tag{3.10}$$

for any $t \in J$, and $v_0(t) \le u_1 \le u_2 \le w_0(t)$, $v_0(t) \le v_2 \le v_1 \le w_0(t)$.

We have the following result.

Theorem 3.3. *Let E be an ordered Banach space, whose positive* cone *P is normal, $f \in C(J \times E \times E, E)$ and $I_k \in C(E \times E, E)$, $k = 1, 2, \ldots, m$. If the IPBVP*(1.1) *has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0 \le w_0$, and conditions* $(H1)$, $(H2)$, $(H4)$ *and* $(H5)$ *hold, then the IPBVP*(1.1) *has minimal and maximal coupled L-quasisolutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Proof. For $t \in J$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ be a increasing sequence and $\{v_n\} \subset [v_0(t), w_0(t)]$ be a decreasing sequence. For $m, n \in \mathbb{N}$ with $m > n$, by $(H1)$ and $(H5)$,

$$\begin{aligned} \theta &\le f(t, u_m, v_m) - f(t, u_n, v_n) + M(u_m - u_n) + L(v_n - v_m) \\ &\le \left(M + \overline{M}\right)(u_m - u_n) + \left(L + \overline{L}\right)(v_n - v_m). \end{aligned} \tag{3.11}$$

By this and the normality of cone P, we have

$$\begin{aligned}
&\left\| f(t,u_m,v_m) - f(t,u_n,v_n) + M(u_m - u_n) \right\| \\
&\quad \le N\left\| \left(M+\overline{M}\right)(u_m - u_n) + \left(L+\overline{L}\right)(v_n - v_m) \right\| + L\|v_n - v_m\| \\
&\quad \le N\left(M+\overline{M}\right)\|u_m - u_n\| + \left(N\left(L+\overline{L}\right) + L\right)\|v_n - v_m\|.
\end{aligned} \tag{3.12}$$

From this inequality and the definition of the measure noncompactness, it follows that

$$\begin{aligned}
\alpha\left(\{ f(t,u_n,v_n) + Mu_n \}\right) &\le N\left(M+\overline{M}\right)\alpha(\{u_n\}) + \left(N\left(L+\overline{L}\right) + L\right)\alpha(\{v_n\}) \\
&\le L_1(\alpha(\{u_n\}) + \alpha(\{v_n\})),
\end{aligned} \tag{3.13}$$

where $L_1 = N(M+\overline{M}) + N(L+\overline{L}) + L$. If $\{u_n\}$ is a increasing sequence and $\{v_n\}$ is a decreasing sequence, the above inequality is also valid. Hence $(H3)$ holds.

Therefore, by Theorem 3.1, the IPBVP(1.1) has minimal and maximal coupled L-quasisolutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively. □

Now, we discuss the existence of the solution to the IPBVP(1.1) between the minimal and maximal coupled L-quasisolutions $\underline{u}$ and $\overline{u}$. If we replace the assumptions $(H2)$ and $(H3)$ by the following assumptions:

$(H2)^*$ The impulsive function $I_k(\cdot,\cdot)$ satisfies

$$I_k(u_1,v_1) \le I_k(u_2,v_2), \quad k = 1,2,\ldots,m, \tag{3.14}$$

for any $t \in J$, and $v_0(t) \le u_1 \le u_2 \le w_0(t)$, $v_0(t) \le v_2 \le v_1 \le w_0(t)$; and there exist $M_k > 0$, $\sum_{k=1}^{m} M_k \le ((8L_1\omega - 1) + (1 - 16L_1\omega)e^{M\omega})/2(2e^{M\omega} - 1)$, such that

$$\alpha(I_k(\{u_n(t_k)\} \times \{v_n(t_k)\})) \le M_k[\alpha(u_n(t_k)) + \alpha(v_n(t_k))], \tag{3.15}$$

for any countable sets $\{u_n\}$ and $\{v_n\}$ in $[v_0(t), w_0(t)]$.

$(H3)^*$ There exist a constant $L_1 > 0$ such that

$$\alpha\left(f(t, D_1 \times D_2) + MD_1\right) \le L_1(\alpha(D_1) + \alpha(D_2)), \tag{3.16}$$

for any $t \in J$, where $D_1 = \{v_n\}$ and $D_2 = \{w_n\}$ are countable sets in $[v_0(t), w_0(t)]$.

We have the following existence result.

Theorem 3.4. *Let E be an ordered Banach space, whose positive* cone *P is normal, $f \in C(J\times E\times E, E)$ and $I_k \in C(E\times E, E)$, $k = 1,2,\ldots,m$. If the IPBVP(1.1) has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0 \le w_0$, such that assumptions $(H1)$, $(H2)^*$, $(H3)^*$ and $(H4)$ hold, then the IPBVP(1.1) has minimal and maximal coupled L-quasisolutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, and at least has one solution between $\underline{u}$ and $\overline{u}$.*

Proof. We can easily see that $(H2)^* \Rightarrow (H2)$, $(H3)^* \Rightarrow (H3)$. Hence, by the Theorem 3.1, the IPBVP(1.1) has minimal and maximal coupled L-quasisolutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0. Next, we prove the existence of the solution of the equation between $\underline{u}$ and $\overline{u}$. Let $Au = Q(u,u)$, clearly, $A : [v_0, w_0] \to [v_0, w_0]$ is continuous and the solution of the IPBVP(1.1) is equivalent to the fixed point of operator A. Since $A(D)$ is bounded and equicontinuous for any $D \subset [v_0, w_0]$, by Lemma 2.2, there exist a countable set $D_0 = \{u_n\}$, such that

$$\alpha(A(D)) \leq 2\alpha(A(D_0)). \tag{3.17}$$

By assumptions $(H2)^*$ and $(H3)^*$ and Lemma 2.1,

$$\begin{aligned}
&\alpha(A(D_0(t))) \\
&\quad = \alpha\Bigg(\Bigg\{\frac{e^{-M\omega}}{1-e^{-M\omega}}\Bigg[\int_0^\omega e^{-M(t-s)}\big(f(s,u_n(s),u_n(s)) + Mu_n(s)\big)ds \\
&\qquad\qquad + \sum_{k=1}^m e^{-M(t-t_k)} I_k(u_n(t_k),u_n(t_k))\Bigg] \\
&\qquad\qquad + \int_0^t e^{-M(t-s)}\big(f(s,u_n(s),u_n(s)) + Mu_n(s)\big)ds \\
&\qquad + \sum_{t_k<t} e^{-M(t-t_k)} I_k(u_n(t_k),u_n(t_k))\Bigg\}\Bigg) \\
&\quad \leq \frac{1}{e^{M\omega}-1}\Bigg[2\int_0^\omega e^{-M(t-s)}\alpha\big(f(s,D_0(s),D_0(s)) + MD_0(s)\big)ds \\
&\qquad\qquad + \sum_{k=1}^m e^{-M(t-t_k)}\alpha(I_k(D_0(t_k),D_0(t_k)))\Bigg] \\
&\qquad + 2\int_0^t e^{-M(t-s)}\alpha\big(f(s,D_0(s),D_0(s)) + MD_0(s)\big)ds + \sum_{t_k<t} e^{-M(t-t_k)}\alpha(I_k(D_0(t_k),D_0(t_k))) \\
&\quad < \frac{e^{M\omega}}{e^{M\omega}-1}\Bigg[4L_1\int_0^\omega \alpha(D_0(s))ds + 2\sum_{k=1}^m M_k\alpha(D_0(t_k))\Bigg] \\
&\qquad + 4L_1\int_0^t \alpha(D_0(s))ds + 2\sum_{t_k<t} M_k\alpha(D_0(t_k)) \\
&\quad \leq \frac{2e^{M\omega}-1}{e^{M\omega}-1}\left(4L_1\omega + 2\sum_{k=1}^m M_k\right)\alpha(D).
\end{aligned} \tag{3.18}$$

Since $A(D_0)$ is equicontinuous, by Lemma 2.3, $\alpha(A(D_0)) = \max_{t\in J}\alpha(A(D_0)(t))$. Combing (3.17) and $(H2)^*$.

We have

$$\alpha(A(D)) < \frac{4e^{M\omega}-2}{e^{M\omega}-1}\left(4L_1\omega+2\sum_{k=1}^{m}M_k\right)\alpha(D)\leq\alpha(D). \tag{3.19}$$

Hence, the operator $A:[v_0,w_0]\rightarrow[v_0,w_0]$ is condensing, by the Lemma 2.4, A has fixed point u in $[v_0,w_0]$.

Lastly, since $u=Au=Q(u,u)$, $v_0\leq u\leq w_0$, by the mixed monotonity of Q

$$v_1=Q(v_0,w_0)\leq Q(u,u)\leq Q(w_0,v_0)=w_1. \tag{3.20}$$

Similarly, $v_2\leq u\leq w_2$, in general, $v_n\leq u\leq w_n$, letting $n\rightarrow\infty$, we get $\underline{u}\leq u\leq\overline{u}$. Therefore, the IPBVP(1.1) at least has one solution between $\underline{u}$ and $\overline{u}$. □

Remark 3.5. If $f(t,u,u)=f(t,u)$ and $I_k(u,u)=I_k(u)$, then Theorems 3.1, 3.3 and 3.4 are generalizations of the main results of [5] in Banach spaces.

Remark 3.6. If $f(t,u,u)=f(t,u)$ and $I_k\equiv 0$, then Theorems 3.1, 3.3 and 3.4 are generalizations of the Theory 4.1 of [4] in Banach spaces.

4. An Example

Consider the PBVP of infinite system for nonlinear impulsive differential equations:

$$\begin{gathered}
u_n'(t)=u_n(t)+\frac{1}{u_{n+2}(t)},\quad 0\leq t\leq\pi,\ t\neq\frac{\pi}{2},\\
\Delta u_n|_{t=\pi/2}=3u_n\left(\frac{\pi}{2}\right)-2u_{n+1}\left(\frac{\pi}{2}\right),\\
u_n(0)=u_n(\omega)\quad(n=1,2,\ldots).
\end{gathered} \tag{4.1}$$

4.1. Conclusion

IPBVP(4.1) has minimal and maximal coupled L-quasisolutions.

Proof. Let $\omega=\pi$, $E=l^2=\{u=(u_1,\ldots,u_n,\ldots)\mid\sum_{n=1}^{\infty}|u_n|^2<\infty\}$ with norm $\|u\|=(\sum_{n=1}^{\infty}|u_n|^2)^{1/2}$ and $P=\{u=(u_1,\ldots,u_n,\ldots)\in l^2\mid u_n\geq 0, n=1,2,\ldots\}$. Then E is a weakly sequentially complete Banach space and P is normal cone in E. IPBVP(4.1) can be regarded as an PBVP of the form (1.1) in E. In this case, $J=[0,\pi]$, $v=(v_1,\ldots,v_n,\ldots)$, $w=(w_1,\ldots,w_n,\ldots)$ and $f=(f_1,\ldots,f_n,\ldots)$, in which

$$f_n(t,u,v)=u_n(t)+\frac{1}{v_{n+2}(t)},\quad n=1,2,\ldots, \tag{4.2}$$

$k=1$, $t_1=\pi/2$ and $I_{1n}(u,v)=3u_n-2v_{n+1}$, $n=1,2,\ldots$.

Evidently, $f \in C(J \times E \times E, E)$, $I_1 \in C(E \times E, E)$. Let

$$v_0(t) = \begin{cases} \left(-\cos t, \dots, -\dfrac{\cos t}{n}, \dots\right), & 0 \le t \le \dfrac{\pi}{2}, \\ \left(-1 - \cos t, \dots, -\dfrac{1 + \cos t}{n}, \dots\right), & \dfrac{\pi}{2} < t \le \pi, \end{cases}$$
$$w_0(t) = \begin{cases} \left(\cos t, \dots, \dfrac{\cos t}{n}, \dots\right), & 0 \le t \le \dfrac{\pi}{2}, \\ \left(1 + \cos t, \dots, \dfrac{1 + \cos t}{n}, \dots\right), & \dfrac{\pi}{2} < t \le \pi, \end{cases} \tag{4.3}$$

$L = 1/2$, $M = 1$. Then it is easy to verify that v_0, w_0 are coupled lower and upper $1/2$-quasisolutions of the IPBVP(4.1), and conditions (H_1), (H_2) hold. Hence, our conclusion follows from Corollary 3.2. □

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