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Research Article

On the Study of Chemostat Model with Pulsed Input in a Polluted Environment

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Chemostat model with pulsed input in a polluted environment is considered. By using the Floquet theorem, we find that the microorganism eradication periodic solution is globally asymptotically stable if the impulsive period T is more than a critical value. At the same time, we can find that the nutrient and microorganism are permanent if the impulsive period T is less than the critical value.

1. Introduction

A chemostat is a piece of laboratory apparatus frequently used for culturing microorganisms. It can be used for representing all kinds of microorganism systems such as lake, waste-water treatment, and reaches for commercial production of the advantage of being easily implementable in a laboratory, and hence the model has been studied by more and more people. Chemostat with period inputs are studied in [1–3], those with periodic washout rate in [4, 5] and those with periodic input and washout in [6]. However, existing theories on chemostat model largely ignore the effects of environmental pollution.

Environmental pollution by various industries and pesticide used in agriculture is one of the most important of present day social and ecological problems. Organisms are often exposed to a polluted environment and take up toxicant. Uncontrolled contribution of pollutant to the environment has led many species to extinction. In order to use and regulate toxic substance wisely, we must assess the risk of the population exposed to toxicant. Therefore, it is important to study the effects of toxicant on populations and to find a theoretical threshold value, which determines permanence or extinction of a population community.

In this paper, we consider the dynamics of the polluted chemostat with pulsed input:

$$\begin{aligned} \dot{S} = -QS - \frac{\mu x S}{\delta}, \quad \dot{x} = x(\mu S - Q - rc), \quad \dot{c} = -Qc + Qf, \quad \bar{t} \neq nT, \\ \Delta S = S^0 Q, \quad \Delta x = 0, \quad \Delta c = 0, \quad \bar{t} = nT, \end{aligned} \tag{1.1}$$

where $\Delta S = S(nT^+) - S(nT)$, $\Delta x = x(nT^+) - x(nT)$, $\Delta c = c(nT^+) - c(nT)$, $n \in Z_+$, $Z_+ = \{1,2,\ldots\}$. $S(\bar{t})$ denotes the concentration of the nutrient, and $x(\bar{t})$ denotes the concentration of the microorganism at time $\bar{t}$. $c(\bar{t})$ is the concentration of the toxicant in the organism at time $\bar{t}$. $r > 0$ is the decreasing rate of the intrinsic growth rate associated with the uptake of the toxicant. f represents the exogenous rate of toxicant input into the organism. S^0 represents the input concentration of the nutrient. Q $(0 < Q < 1)$ is referred to as the dilution rate. μ denote the predation constants of the predator. δ shows yield term. T is the period of the pulse.

The aim of this work is to study the dynamical behaviors of the polluted chemostat with pulsed input, and investigate how the impulsive perturbation affects the dynamical behaviors of unforced continuous system.

The variables in the above system may be rescaled by measuring $S = (Q/\mu)x_1$, $x = (\delta Q/\mu)x_2$, $c = (Q/r)x_3$, $t = Q\bar{t}$, then we have the following system:

$$\begin{aligned} \dot{x}_1 = -x_1 - x_1 x_2, \quad \dot{x}_2 = x_2(x_1 - 1 - x_3), \quad \dot{x}_3 = -x_3 + u, \quad t \neq nT, \\ \Delta x_1 = p, \quad \Delta x_2 = 0, \quad \Delta x_3 = 0, \quad t = nT, \end{aligned} \tag{1.2}$$

where $p = \mu S^0$, $u = fr/Q$, $\Delta x_i = x_i(nT^+) - x_i(nT)$, $i = 1,2,3$.

This paper is arranged as follows. In Section 2, we introduce some useful notations and definitions. In Section 3, by using Floquet theorem for the impulsive equation, small-amplitude perturbation skills and techniques of comparison, we get the local stability and global asymptotic stability of the microorganism eradication periodic solution. In Section 4, we show that the system is permanent if the impulsive period is less than some critical value. In Section 5, we give a brief discussion.

2. Preliminaries

In this section, we will give some definitions, notations, and some lemmas which will be useful for our main results.

Let $R_+ = [0,\infty)$, let $R_+^3 = \{x = (x_1,x_2,x_3) \in R^3 : x > 0\}$, and let N be the set of all nonnegative integers. Denote by $f = (f_1, f_2, f_3)$ the map defined by the right-hand side of the first three equations of system (1.2). Let $V : R_+ \times R_+^3 \to R_+$, then V is said to belong to class V_0 if

(i) V is continuous in $(nT,(n+1)T] \times R_+^3$ and for each $x \in R_+^3$, $n \in Z_+$, $\lim_{(t,y)\to(nT^+,x)} V(t,y) = V(nT^+,x)$ exists.

(ii) V is locally Lipschitzian in x.

Definition 2.1. Let $V \in V_0$, then for $(t,x) \in (nT,(n+1)T)] \times R_+^3$, the upper right derivative of $V(t,x)$ with respect to the impulsive differential system (1.2) is defined as

$$D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,x+hf(t,x)) - V(t,x)]. \tag{2.1}$$

The solution of system (1.2) is a piecewise continuous function $x : R_+ \to R_+^3$, $x(t)$ is continuous on $(nT,(n+1)T]$, $n \in Z_+$, and $x(nT^+) = \lim_{t\to nT^+} x(t)$ exists, the smoothness properties of f guarantees the global existence and uniqueness of the solution (1.2), for details see [7, 8].

Definition 2.2. The microorganism x_2 of (1.2) is said to be permanent if there exist constants $0 < m < M$ and $T_0 > 0$ such that $m < x_2 < M$ for $t > T_0$ with initial condition $x_2(0) > 0$, that is, the system (1.2) is permanent.

Definition 2.3. The microorganism x_2 of (1.2) is said to be extinct if $\lim_{t\to\infty} x_2(t) = 0$.

LEMMA 2.4. *Suppose $\omega(t) = (x_1(t),x_2(t),x_3(t))$ is a solution of (1.2) subject to $\omega(0^+) \geq 0$, then $\omega(t) \geq 0$ for all $t \geq 0$, and further $\omega(t) > 0$, $t \geq 0$, if $\omega(0^+) > 0$.*

We will use a basic comparison result from [7, Theorem 3.1]. For convenience, we state it in our notations.

LEMMA 2.5. *Let $V : R_+ \times R^3 \to R_+$ and $V \in V_0$. Assume that*

$$\begin{aligned} D^+V(t,\omega) &\leq g(t,V(t,\omega)), \quad t \neq nT, \\ V(t,\omega(t^+)) &\leq \psi_n(V(t,\omega(t))), \quad t = nT, \end{aligned} \tag{2.2}$$

where $g : R_+ \times R_+ \to R$ is continuous in $(nT,(n+1)T] \times R_+$ and for $y \in R_+$, $n \in Z_+$, $\lim_{(t,y)\to(nT^+,y)} g(t,y)$ exists, $\psi_n : R^+ \to R^+$ is nondecreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{aligned} \frac{du}{dt} &= g(t,u), \quad t \neq nT, \\ u(t^+) &= \psi_n(u(t)), \quad t = nT, \\ u(0^+) &= u_0, \end{aligned} \tag{2.3}$$

existing on $[0,\infty)$. Then $V(0^+,\omega_0) \leq u_0$ implies that $V(t,\omega(t)) \leq r(t)$, $t \geq 0$, where $\omega(t)$ is any solution of system (1.2).

Similar result can be obtained when all conditions of the inequalities in the lemma are reversed. Note that if we have some smoothness conditions of $g(t,u)$ to guarantee the existence and uniqueness of the solutions for (2.3), then $r(t)$ is exactly the unique solution of (2.3).

For convenience, we give the basic properties of the following system:

$$\begin{aligned} \dot{x}_1 &= -x_1, \quad t \neq nT, \\ \Delta x_1 &= p, \quad t = nT, \\ x_1(0^+) &= x_{10} \geq 0. \end{aligned} \tag{2.4}$$

Lemma 2.6. *System (2.4) has a positive periodic solution $x_1^*(t)$ and for every solution $x_1(t)$ of (2.4) with initial value $x_{10} \geq 0$, $|x_1(t) - x_1^*(t)| \to 0$, as $t \to \infty$; moreover, $x_1(t) \geq x_1^*(t)$ if $x_{10} \geq p/(1 - e^{-T})$ and $x_1(t) < x_1^*(t)$ if $x_{10} < p/(1 - e^{-T})$, where $x_1^*(t) = pe^{-(t-nT)}/(1 - e^{-T})$, $t \in (nT, (n+1)T]$, $n \in Z_+$, $x_1^*(0^+) = p/(1 - e^{-T})$.*

Proof. Clearly $x_1^*(t)$ is a positive solution of (2.4). The solution $x_1(t) = (x_{10} - p/(1 - e^{-T}))e^{-t} + x_1^*(t)$, $t \in (nT, (n+1)T]$, $n \in N$. Hence, $|x_1(t) - x_1^*(t)| \to 0$ as $t \to \infty$. And $x_1(t) \geq x_1^*(t)$ if $x_{10} \geq p/(1 - e^{-T})$; $x_1(t) < x_1^*(t)$ if $x_{10} < p/(1 - e^{-T})$. The proof is complete. □

3. Extinction

In the section, we study the stability of the microorganism eradication periodic solution as a solution of the full system (1.2). Firstly, we present the Floquet theory for the linear T-periodic impulsive equation:

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, \quad t \neq \tau_k,\ t \in R, \\ \Delta x &= B_k x, \quad t = \tau_k,\ k \in Z. \end{aligned} \tag{3.1}$$

Then we introduce the following conditions:

(H_1) $A(\cdot) \in PC(R, C^{n\times n})$ and $A(t+T) = A(t)$ $(t \in R)$, where $PC(R, C^{n\times n})$ is a set of all piecewise continuous matrix functions which is left continuous at $t = \tau_k$, and $C^{n\times n}$ is a set of all $n \times n$ matrices.

(H_2) $B_k \in C^{n\times n}$, $\det(E + B_k) \neq 0$, $\tau_k < \tau_{k+1}$ $(k \in Z)$,

(H_3) There exists a $q \in \mathbf{N}$ such that

$$B_{k+q} = B_k, \quad \tau_{k+q} = \tau_k + T \quad (k \in Z). \tag{3.2}$$

Let $\Phi(t)$ be a fundamental matrix of (3.1), then there exists a unique nonsingular matrix $M \in C^{n\times n}$ such that

$$\Phi(t+T) = \Phi(t)M \quad (t \in R). \tag{3.3}$$

By equality (3.3) there corresponds to the fundamental matrix $\Phi(t)$ and the constant matrix M which we call the monodromy matrix of (3.1) (corresponding to the fundamental matrix of $\Phi(t)$).

All monodromy matrices of (3.1) are similar and have the same eigenvalues. The eigenvalues $\mu_1, \ldots, \mu_n$ of the monodromy matrices are called the Floquet multipliers of (3.1).

Lemma 3.1 [8] (Floquet theory). *Let conditions (H_1)–(H_3) hold. Then the linear T-periodic impulsive equation (3.1) is*

(1) *stable if and only if all multipliers μ_j $(j = 1, \ldots, n)$ of (3.1) satisfy the inequality $|\mu_j| \leq 1$, and moreover, to those μ_j for which $|\mu_j| = 1$, there correspond simple elementary divisors;*

(2) *asymptotically stable if and only if all multipliers μ_j $(j = 1, \ldots, n)$ of (3.1) satisfy the inequality $|\mu_j| < 1$;*

(3) *unstable if $|\mu_j| > 1$ for some $j = 1, \ldots, n$.*

Theorem 3.2. *Let $\omega(t) = (x_1(t), x_2(t), x_3(t))$ be any solution of system (1.2), then $(x_1^*(t), 0, u)$ is globally asymptotically stable, provided that $T > p/(1+u)$.*

Proof. Firstly, we prove locally asymptotically stable. The local stability of the periodic solution $(x_1^*(t), 0, u)$ may be determined by considering the behavior of small-amplitude perturbations of the solution. Define

$$x_1(t) = x_1^*(t) + y_1(t), \qquad x_2(t) = y_2(t), \qquad x_3(t) = u + y_3(t), \tag{3.4}$$

where y_1, y_2 and y_3 are small perturbations. Equation (1.2) can be expanded in a Taylor series: after neglecting higher-order terms, they may be written as

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix}, \tag{3.5}$$

where $\Phi(t)$ must satisfy

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} -1 & -x_1^*(t) & 0 \\ 0 & x_1^*(t) - 1 - u & 0 \\ 0 & 0 & -1 \end{pmatrix} \Phi(t), \tag{3.6}$$

with $\Phi(0) = I$, where I is the identity matrix. Hence, the fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} e^{-t} & * & 0 \\ 0 & e^{\int_0^t (x_1^*(t) - 1 - u)dt} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}, \tag{3.7}$$

there is no need to calculate the exact form $(*)$ as it is not required in the analysis that follows. The resetting impulsive conditions of (1.2) from the fourth to the sixth become

$$\begin{pmatrix} y_1(nT^+) \\ y_2(nT^+) \\ y_3(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(nT) \\ y_2(nT) \\ y_3(nT) \end{pmatrix}. \tag{3.8}$$

Thus, the monodromy matrix of (3.5) is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(T). \tag{3.9}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of M, then

$$\begin{gathered} \lambda_1 = \lambda_3 = e^{-T} < 1, \\ \lambda_2 = e^{\int_0^T (x_1^*(t) - 1 - u)dt}. \end{gathered} \tag{3.10}$$

Hence, according to Lemma 3.1, if absolute values of all eigenvalues of M are less than one, then T-periodic solution locally asymptotically stable. Thus, if and only if $T > p/(1+u)$, the solution $(x_1^*(t), 0, u)$ is locally asymptotically stable. The proof is complete. □

In the following, we prove the global attractivity. Choose $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$ such that

$$\delta = p + \varepsilon_1 T - T + \varepsilon_2 T - uT < 0. \tag{3.11}$$

Noting that $\dot{x}_1 \leq -x_1$, considering the following impulsive differential equation:

$$\begin{gathered} \dot{z}_1(t) = -z_1(t), \quad t \neq nT, \\ \Delta z_1 = p, \quad t = nT, \end{gathered} \tag{3.12}$$

we have $x_1(t) \leq z_1(t)$, and $z_1(t) \to z_1^*(t)$, as $t \to \infty$, where $z_1^*(t)$ is the periodic solution.

Then,

$$x_1(t) \leq z_1(t) < z_1^*(t) + \varepsilon_1, \tag{3.13}$$

for t large enough. Since $x_3(t) \to u$, as $t \to \infty$, therefore,

$$x_3(t) > u - \varepsilon_2, \tag{3.14}$$

for t large enough. For simplification, we may assume that (3.13), (3.14) hold for all $t \geq 0$. From (1.2), we can get

$$\dot{x}_2 \leq x_2 (z_1^*(t) + \varepsilon_1 - 1 - u + \varepsilon_2), \tag{3.15}$$

integrating (3.15) on $(nT, (n+1)T]$ yields

$$x_2((n+1)T) \leq x_2(nT^+) \exp \int_{nT}^{(n+1)T} (z_1^*(t) + \varepsilon_1 - 1 - u + \varepsilon_2)\,dt, \tag{3.16}$$

therefore, we have $x_2((n+1)T) \leq x_2(nT^+)\exp(\delta)$. Thus, $x_2(nT) \leq x_2(0^+)\exp(n\delta)$ and $x_2(nT) \to 0$ as $n \to \infty$. Since $0 < x_2(t) < x_2(nT)$, therefore, $x_2(t) \to 0$ as $t \to \infty$.

Next we prove that $x_1(t) \to x_1^*(t)$ as $t \to \infty$, if $\lim_{t\to\infty} x_2(t) = 0$. For $\varepsilon_3 > 0$, there exists a $\hat{T} > 0$ such that $0 < x_2(t) < \varepsilon_3$, for all $t \geq \hat{T}$.

Then, we have

$$-x_1 - \varepsilon_3 x_1 \leq \dot{x}_1 \leq -x_1, \tag{3.17}$$

by Lemma 2.5 we obtain $z_2(t) \leq x_1(t) \leq z_1(t)$, and $z_1(t) \to x_1^*(t)$ and $z_2(t) \to z_2^*(t)$ as $t \to \infty$, where $z_2(t)$ is the solution of

$$\begin{gathered} \dot{z}_2 = -z_2(1 + \varepsilon_3), \quad t \neq nT, \\ \Delta z_2 = p, \quad t = nT, \end{gathered} \tag{3.18}$$

and $z_2^*(t) = p\exp((-1-\varepsilon_3)(t-nT))/(1-\exp((-1-\varepsilon_3)T))$, $nT < t \leq (n+1)T$, therefore $z_2^*(t) - \varepsilon_4 < x_1(t) < x_1^*(t) + \varepsilon_4$, $\varepsilon_4 > 0$, for t large enough. Let $\varepsilon_3 \to 0$, we get $z_2^*(t) \to x_1^*(t)$. Hence, $x_1(t) \to x_1^*(t)$ as $t \to \infty$. This completes the proof.

4. Permanence

First, we show that all solutions of (1.2) are uniformly ultimately bounded.

THEOREM 4.1. *There exists a constant $M > 0$ such that $x_1(t) \le M$, $x_2(t) \le M$, $x_3(t) \le M$ for each positive solution $\omega(t) = (x_1(t), x_2(t), x_3(t))$ of (1.2) with t large enough.*

Proof. Define a function $V(t,\omega(t)) = x_1 + x_2 + x_3$, then $V(t,\omega(t)) \in V_0$ and the upper right derivative of $V(t,\omega(t))$ along a solution of (1.2) is described as

$$\begin{gathered} D^+V(t,\omega(t)) = -(x_1+x_2+x_3) - x_2x_3 + u \le -V + u, \quad t \ne nT, \\ \Delta V = p, \quad t = nT, \end{gathered} \tag{4.1}$$

we obtain

$$V(t) \le V(0^+)e^{-t} + u(1-e^{-t}) + p\frac{e^{-(t-T)}}{1-e^T} + \frac{pe^T}{e^T-1} \longrightarrow \frac{pe^T}{e^T-1} + u, \quad t \longrightarrow \infty. \tag{4.2}$$

By the definition of $V(t,\omega(t))$, we obtain that each positive solution of (1.2) is uniformly ultimately bounded. □

Next we give the conditions of permanence.

THEOREM 4.2. *System (1.2) is permanent if $T < p/(1+u)$.*

Proof. Suppose $\omega(t) = (x_1(t), x_2(t), x_3(t))$ is a solution of (1.2) with positive initial value. From Theorem 4.1 we may assume $x_1(t) \le M$, $x_2(t) \le M$, $x_3(t) \le M$, $t \ge 0$, and $M > 0$. From system (1.2), we can see that

$$\dot{x}_1 \ge -x_1(1+M). \tag{4.3}$$

Considering the comparison

$$\begin{gathered} \dot{w} = -w(1+M), \quad t \ne nT, \\ \Delta w = p, \quad t = nT, \end{gathered} \tag{4.4}$$

let $m_1 = pe^{-(1+M)T}/(1-e^{-(1+M)T}) - \varepsilon_5 > 0$, $\varepsilon_5 > 0$. From Lemma 2.5, clearly we have $x_1(t) \ge w(t) > m_1$ for t large enough.

In the following, we want to find $m_3 > 0$ such that $x_2(t) \ge m_3$ for t large enough. We will do it in the following two steps for convenience.

Step 1. Let $m_3 > 0$ and ε_6, ε_7 be small enough. Such that

$$\rho = p - \varepsilon_6 T - T - uT - \varepsilon_7 T > 0. \tag{4.5}$$

We will prove that $x_2(t) < m_3$ cannot hold for all $t \ge 0$. Otherwise,

$$\frac{dx_1}{dt} \ge -(1+m_3)x_1, \tag{4.6}$$

from Lemma 2.5, we have $x_1(t) \geq v(t)$ and $v(t) \to v^*(t)$, $t \to \infty$. Where $v(t)$ is the solution of

$$\frac{dv}{dt} = -(1+m_3)v, \quad t \neq nT, \qquad \Delta v = p, \quad t = nT, \tag{4.7}$$

$$v^*(t) = \frac{p\exp((-1-m_3)(t-nT))}{1-\exp((-1-m_3)T)}, \quad t \in (nT,(n+1)T], \tag{4.8}$$

therefore, there exists $n_1 > 0$, $t > n_1T$ such that $x_1(t) \geq v(t) > v^*(t) - \varepsilon_6$. Since $x_3(t) \to u$, as $t \to \infty$, we have $x_3(t) < u + \varepsilon_7$ for $t > n_2T$. At the same time, we have

$$\dot{x}_2(t) \geq x_2(v^*(t) - \varepsilon_6 - 1 - u - \varepsilon_7), \tag{4.9}$$

integrating (4.9) on $t \in (nT,(n+1)T]$, $n > n_2 > n_1$, we obtain

$$x_2((n+1)T) \geq x_2(nT)\exp\int_{nT}^{(n+1)T}(v^*(t) - \varepsilon_6 - 1 - u - \varepsilon_7)\,dt, \tag{4.10}$$

therefore,

$$x_2((n+1)T) \geq x_2(nT)\exp(\rho). \tag{4.11}$$

Then, $x_2((N_1+k)T) \geq x_2(N_1T)\exp(k\rho) \to \infty$ as $k \to \infty$, which is contradiction to the boundedness of $x_2(t)$. Therefore, there is a $t_1 > 0$ such that $x_2(t_1) > m_3$. If $x_2(t) \geq m_3$ for all $t > t_1$, then our aim is obtained. Otherwise, there exists a $\bar{t}_1 > t_1$ such that $x_2(\bar{t}_1) < m_3$. Setting $t^* = \inf_{t>t_1}\{x_2(t) < m_3\}$, then we have $x_2(t) \geq m_3$ for $t \in [t, t^*)$, and $x_2(t^*) = m_3$.

Step 2. Since $x_2(t)$ is continuous, suppose that $t^* \in (n_1T,(n_1+1)T]$, $n_1 \in N$, select $n_2 \in N$, $n_3 \in N$, such that

$$n_2T > \frac{1}{-1-m_3}\ln\frac{\varepsilon_6}{M+p}, \qquad \exp(\eta(n_2+1)T)\exp(n_3\rho) > 1, \tag{4.12}$$

where $\eta = -1 + m_1 - u - \varepsilon_7 < 0$, let $T' = n_2T + n_3T$, we claim that there must exist a $t' \in ((n_1+1)T,(n_1+1)T+T']$ such that $x_2(t) \geq m_3$; otherwise, $x_2(t) < m_3$, $t \in ((n_1+1)T,(n_1+1)T+T']$. Consider (4.7) with $v((n_1+1)T^+) = x_1((n_1+1)T^+)$, we have

$$v(t) = \left(v((n_1+1)T^+) - \frac{p}{1-e^{-(1+m_3)T}}\right)e^{-(1+m_3)(t-(n_1+1)T)} + v^*(t), \tag{4.13}$$

for $t \in (nT,(n+1)T]$, $n_1+1 < n \leq n_1+n_2+n_3+1$. Then,

$$|v(t) - v^*(t)| \leq (M+p)e^{-(1+m_3)(t-(n_1+1)T)} < \varepsilon_6, \tag{4.14}$$

and $x_1(t) \geq v(t) > v^*(t) - \varepsilon_6$, for $(n_1+1+n_2)T \leq t \leq (n_1+1)T+T'$ which implies that

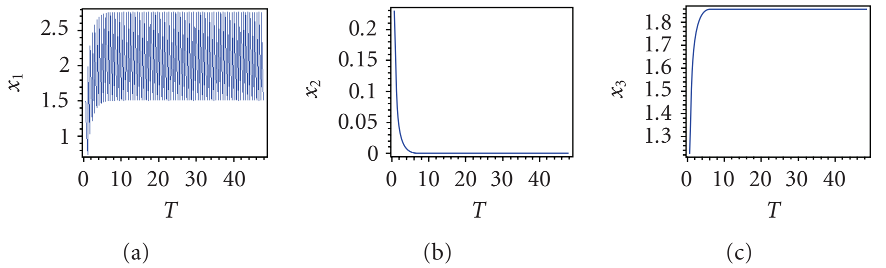

Figure 5.1. Extinct of system (1.2): (a) time series of the nutrient, (b) time series of the microorganism, (c) time series of the toxicant in the organism, $p = 1.25$, $q = 1.86$, $T = 0.6$.

(4.9) holds. For $(n_1+1+n_2)T \le t \le (n_1+1)T+T'$, as in Step 1, we have $x_2((n_1+n_2+n_3+1)T) \ge x_2((n_1+n_2+1)T)\exp(n_3\rho)$, there are two possible cases for $t \in (t^*,(n_1+1)T]$.

Case 1. If $x_2(t) < m_3$ for $t \in (t^*,(n_1+1)T]$, then $x_2(t) < m_3$ for all $t \in (t^*,(n_1+1+n_2)T]$, system (1.2) gives

$$\dot{x}_2(t) \ge x_2(t)\big(-1+m_1-u-\varepsilon_7\big) = \eta x_2(t), \tag{4.15}$$

integrating (4.15) on $(t^*,(n_1+1+n_2)T]$, which yields $x_2((n_1+n_2+1)T) \ge m_3\exp(\eta(n_2+1))$, then $x_2((n_1+1+n_2+n_3)T) \ge m_3\exp(\eta(n_2+1)T)\exp(n_3\rho) > m_3$, which is a contradiction. Let $\bar{t} = \inf_{t>t^*}\{x_2(t) \ge m_3\}$, then $x_2(\bar{t}) = m_3$ and (4.15) holds for $t \in [t^*,\bar{t})$. Then integrating (4.15) on $[t^*,\bar{t})$ yields $x_2(t) \ge x_2(t^*)\exp(\eta(t-t^*)) \ge m_3\exp(\eta(1+n_2+n_3)T) \triangleq \overline{m}_3$. For $t > \bar{t}$, the same argument can be continued since $x_2(\bar{t}) \ge m_3$. Hence, $x_2(t) \ge \overline{m}_3$ for all $t > t_1$.

Case 2. There exists a $t' \in (t^*,(n_1+1)T]$ such that $x_2(t') \ge m_3$. Let $\hat{t} = \inf_{t>t^*}\{x_2(t) \ge m_3\}$, then $x_2(t) < m_3$ for $t \in [t^*,\hat{t})$ and $x_2(\hat{t}) = m_3$. For $t \in [t^*,\hat{t})$, (4.15) holds and integrating (4.15) on $[t^*,\hat{t})$, we have $x_2(t) \ge x_2(t^*)\exp(\eta(t-t^*)) \ge m_3\exp(\eta T) > \overline{m}_3$. This process can be continued since $x_2(\hat{t}) \ge m_3$, and we have $x_2(t) \ge \overline{m}_3$ for $t > t_1$. Thus, in both cases, we conclude that $x_2(t) \ge \overline{m}_3$ for all $t > t_1$.

Incorporating into Theorem 4.1, the proof is complete. □

5. Discussion

In this paper, we have investigated the model for a polluted chemostat with impulsive input. We have proved that microorganism eradication periodic solution $(x_1^*(t),0,u)$ is globally asymptotically stable if $T > p/(1+u)$, which is showed in Figure 5.1. We can see that the variables $x_1(t)$, $x_3(t)$ oscillate in a stable periodical cycle, in contrast $x_2(t)$ rapidly decrease to zero. At the same time we also have proved that the system (1.2) is permanent if $T < p/(1+u)$, which is simulated in Figure 5.2. The variables $x_1(t)$, $x_2(t)$, $x_3(t)$ oscillate in a stable periodical cycle, respectively. So we can find that $T = p/(1+u)$ is a threshold. In

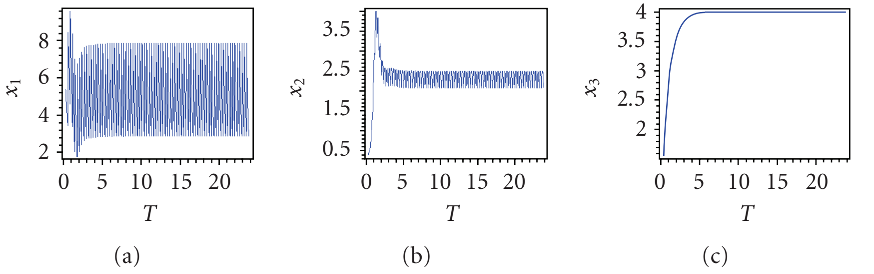

Figure 5.2. Permanence of system (1.2): (a) time series of the nutrient, (b) time series of the microorganism, (c) time series of the toxicant in the organism, $p = 5$, $q = 4$, $T = 0.299$.

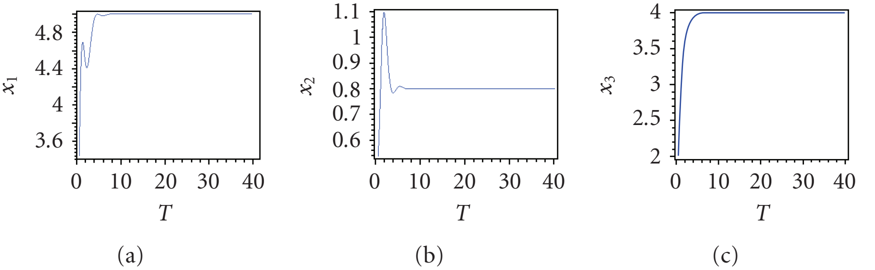

Figure 5.3. Time series of the system (5.1), $p = 4.5$, $q = 4$, $T = 0.5$.

fact, when the period of pulses is less than the threshold, the nutrient and microorganism coexist. If the period is more than the threshold, the microorganism will become extinct.

If we replace the pulse input in system (1.2) with continuous input, the system (1.2) becomes

$$\begin{aligned} \dot{x}_1 &= \frac{p}{T} - x_1 - x_1x_2, \\ \dot{x}_2 &= x_2(x_1 - 1 - x_3), \\ \dot{x}_3 &= -x_3 + u, \end{aligned} \tag{5.1}$$

there also exists a microorganism eradication equilibrium for system (5.1), that is, $(p/T, 0, u)$ which is globally asymptotically stable if $T > p/(1+u)$ (see the appendix). The result is the same as our system (1.2), the result is simulated in Figure 5.3. We can obtain that impulsive input effect is the same as the continuous input.

Appendix

In this appendix, we will prove that the microorganism eradication equilibrium $(p/T, 0, u)$ is globally stable with respect to $\operatorname{int} R^3_+$ if $T > p/(1+u)$, where $R^3_+ = \{(x_1, x_2, x_3) : x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$.

Let us denote the Jacobian matrix system (5.1) evaluated at the equilibrium point $E(p/T,0,u)$ as

$$J(E)=\begin{pmatrix} -1 & \dfrac{p}{T} & 0 \\ 0 & \dfrac{p}{T}-1-u & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

It is easy to see that all eigenvalues of $J(E)$ are negative if and only if $T > p/(1+u)$. Therefore, $E(p/T,0,u)$ is locally stable if $T > p/(1+u)$.

Since $x_3(t) \to u$, as $t \to \infty$, we only consider the following equation:

$$\begin{aligned} \dot{x}_1 &= \frac{p}{T} - x_1 - x_1x_2, \\ \dot{x}_2 &= x_2(x_1 - 1 - u). \end{aligned} \tag{A.2}$$

Let $V(x_1,x_2)$ be positive definite function about (x_1,x_2), given by

$$V(x_1,x_2) = c_1\left(x_1 - x_1^* - x_1^* \ln\frac{x_1}{x_1^*}\right) + c_2x_2, \tag{A.3}$$

where $c_i > 0$, $i = 1,2$. Then, the derivative of V along solution of the system (A.2) is

$$\dot{V} = -c_1\frac{(x_1 - x_1^*)^2}{x_1} + (c_2 - c_1)x_2(x_1 - x_1^*) + c_2x_2(x_1^* - 1 - u), \tag{A.4}$$

we choose c_i $(i = 1,2)$ such that $c_1 = c_2$. Then, $\dot{V}$ is negative definite in $\operatorname{int}R_+^2$ if and only if $T > p/(1+u)$. Therefore, the equilibrium $(p/T,0,u)$ is globally stable in $\operatorname{int}R_+^3$ if $T > (p/1+u)$.

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