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Research Article

K-nacci Sequences in Finite Triangle Groups

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A *k-nacci sequence* in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by $x_n = x_0 x_1 \ldots x_{n-1}$, for $j \leq n < k$, and $x_n = x_{n-k} x_{n-k+1} \ldots x_{n-1}$, for $n \geq k$. We also require that the initial elements of the sequence, $x_0, x_1, x_2, \ldots, x_{j-1}$, generate the group, thus forcing the *k-nacci* sequence to reflect the structure of the group. The *K-nacci* sequence of a group generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, \ldots, x_{j-1})$ and its period is denoted by $P_k(G; x_0, x_1, \ldots, x_{j-1})$. In this paper, we obtain the period of *K-nacci* sequences in finite polyhedral groups and the extended triangle groups.

1. Introduction

The Fibonacci sequences and their related higher-order (tribonacci, quatranacci, *k-nacci*) are generally viewed as sequences of integers. In [1] the Fibonacci length of a 2-generator group is defined, thus extending the idea of forming a sequence of group elements based on a Fibonacci-like recurrence relation first introduced by Wall in [2]. There he considered the Fibonacci length of the cyclic group C_n. The concept of Fibonacci length for more than two generators has also been considered, see, for example [3, 4]. Also, the theory has been expanded to the nilpotent groups, see, for example [5–7]. Other works on Fibonacci length are discussed in, for example, [8–12]. Knox proved that the periods of *k-nacci* (k-step Fibonacci) sequences in dihedral groups are equal to $2k+2$ [13]. Campbell and Campbel, examined the behaviour of the Fibonacci length of the finite polyhedral, binary polyhedral groups, and related groups in [14].

This paper discusses the period of *k-nacci* Fibonacci sequences in the polyhedral groups $(2,2,2)$, $(n,2,2)$, $(2,n,2)$, $(2,2,n)$ for any n and in the extended triangle groups $E(2,2,2)$, $E(n,2,2)$, $E(2,n,2)$, $E(2,2,n)$ for any $n>2$. We consider polyhedral groups both as 2-generator and as 3-generator groups. A 2-step Fibonacci sequence in the integers modulo m can be written as $F_2(Z_m; 0, 1)$. A 2-step Fibonacci sequence of group elements is called a *Fibonacci sequence of a finite group*. A finite group G is *k-nacci sequenceable* if there exists a k-*nacci* sequence of G such that every element of the group appears in the sequence. A sequence of group elements is *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the *period of the sequence*. For example, the sequence $x_0, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, \ldots$ is periodic after the initial element x_0 and has period 4. A sequence of group elements is *simply periodic* with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $x_0, x_1, x_2, x_3, x_4, x_0, x_1, x_2, x_3, x_4, \ldots$ is simply periodic with period 5. It is important to note that the Fibonacci length depends on the chosen generating n- tuple for a group.

Definition 1.1. For a finitely generated group $G = \langle A \rangle$ where $A = \{a_1, a_2, \ldots, a_n\}$ the sequence $x_i = a_{i+1}$, $0 \leq i \leq n-1$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \geq 0$, is called the *Fibonacci orbit* of G with respect to the generating set A, denoted $F_A(G)$.

Notice that the orbit of a k-generated group is a k-*nacci* sequence. The orbits of $(n,2,2)$, $(2,n,2)$, $(2,2,n)$ for any $n > 2$ and $E(2,q,2)$ for any $q > 2$ are studied in [14].

2. The Groups $(2,2,2)$, $(n,2,2)$, $(2,n,2)$, and $(2,2,n)$

Definition 2.1. The *polyhedral group* (l,m,n) for $l,m,n > 1$ is defined by the presentation

$$\left\langle x, y, z : x^l = y^m = z^n = xyz = e \right\rangle \tag{2.1}$$

or

$$\left\langle x, y : x^l = y^m = (xy)^n = e \right\rangle. \tag{2.2}$$

The *polyhedral group* (l,m,n) is finite if and only if the number

$$\mu = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn \tag{2.3}$$

is positive, that is, in the case $(2,2,n)$, $(2,3,3)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$. Its order is $2lmn/\mu$. Using Tietze transformations, we may show that $(l,m,n) \cong (m,n,l) \cong (n,l,m)$. For more information on these groups see [15] and [16, pages 67–68]. The groups considered in Theorems 2.3 and 2.4 are the same group, namely, D_n, the dihedral group of $2n$ elements, except the generators x, y, and z are different from one theorem to the other.

Theorem 2.2. *Let G_2 be the group defined by the presentation $G_2 = \langle x, y, z : x^2 = y^2 = z^2 = xyz = e \rangle$. Then $P_k(G_2, x, y, z) = k + 1$.*

Proof. Firstly, let us consider the 2-generator case. Notice that G_2 is $Z_2 \oplus Z_2$ and $P_k(Z_2; 0, 1) = k + 1$. Under these identifications, since the period of a Fibonacci sequence in a direct product of groups is the least common multiple of the periods in each the factors we get $P_k(G_2; x, y) = k + 1$. On the other hand, since $z = xy$ the formulas in the “three generator case” with recurrences of period $k+1$ are the same as the formulas the two generator case as long as $k \geq 4$. □

Theorem 2.3. *Let $G_n, n > 2$, be the group defined by the presentation $\langle x, y, z : x^n = y^2 = z^2 = xyz = e\rangle$. Then $P_k(G_n; x, y, z) = 2k + 2$.*

Proof. Let us consider the 3-generator case. We first note that the orders of x, y, and z are $n, 2, 2$, respectively. If $k = 2$, we have the sequence

$$x, y, z, yz, zyz, z, x, y, \ldots, \tag{2.4}$$

which has period 6. If $k = 3$, we have the sequence

$$x, y, z, xyz = e, yz, zyz, z, e, x, y, z, \ldots, \tag{2.5}$$

which has period 8. If $k \geq 4$, the first k elements of sequence are

$$x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = xyz, \quad x_4 = (xyz)^2, \ldots, \quad x_{k-1} = (xyz)^{2^{k-3}}. \tag{2.6}$$

Thus, using the above information the sequence reduces to

$$x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = e, \quad x_4 = e, \ldots, e, \tag{2.7}$$

where $x_j = e$ for $3 \leq j \leq k-1$. Thus,

$$\begin{gathered} x_k = \prod_{i=0}^{k-1} x_i = (xy)^{2^{k-2}} = e, \qquad x_{k+1} = \prod_{i=1}^{k} x_i = yz = x^{n-1}, \qquad x_{k+2} = \prod_{i=2}^{k+1} x_i = zyz = xz, \\ x_{k+3} = \prod_{i=3}^{k+2} x_i = z, \qquad x_{k+4} = \prod_{i=4}^{k+3} x_i = e, \ldots, e. \end{gathered} \tag{2.8}$$

It follows that $x_{k+j} = e$ for $4 \leq j \leq k$. We also have,

$$\begin{gathered} x_{k+k+1} = \prod_{i=k+1}^{k+k} x_i = e, \qquad x_{k+k+2} = \prod_{i=k+2}^{k+k+1} x_i = x, \\ x_{k+k+3} = \prod_{i=k+3}^{k+k+2} x_i = y, \qquad x_{k+k+4} = \prod_{i=k+4}^{k+k+3} x_i = z. \end{gathered} \tag{2.9}$$

Since the elements succeeding $x_{2k+2}, x_{2k+3}, x_{2k+4}$, depend on x, y, and z for their values, the cycle begins again with the $2k + 2$nd element; that is, $x_0 = x_{2k+2}, x_1 = x_{2k+3}, x_2 = x_{2k+4}, \ldots$. Thus, $P_k(G_n; x, y, z) = 2k + 2$.

Similarly, it is easy to show that for 2-generator, $P_k(G_n; x, y, z) = 2k + 2$ in $(n, 2, 2)$, and it can be shown that $P_k(G_n; x, y, z) = 2k + 2$ for $(2, n, 2)$.

Because of $(n, 2, 2) \cong (2, n, 2) \cong (2, 2, n) \cong D_n$ for any $n > 2$ and using Tietze transformations we can obtain the same presentation for this groups, it is easy to show that for 2-generator $P_k(G_n; x, y) = 2k + 2$ in the groups $(n, 2, 2)$, $(2, n, 2)$, and $(2, 2, n)$. □

Theorem 2.4. *Let G_n, $n > 2$, be the group defined by the presentation $\langle x, y, z : x^2 = y^2 = z^n = xyz = e\rangle$*

(i) $P_2(G_n; y, x, z) = 6$:

(ii)

$$P_4(G_n; x, y, z) = \begin{cases} n\binom{5}{2}, & n \equiv 0 \bmod 4, \\ 5n, & n \equiv 2 \bmod 4, \\ 10n, & \text{otherwise}, \end{cases} \tag{2.10}$$

(iii) $k \geq 5$.

(1) *If there is no $t \in [3, k-2]$ such that t is a odd factor of n, then*

$$P_k(G_n; x, y, z) = \begin{cases} n\binom{k+1}{2}, & n \equiv 0 \bmod 4, \\ n(k+1), & n \equiv 2 \bmod 4, \\ 2n(k+1), & \text{otherwise}. \end{cases} \tag{2.11}$$

(2) *Let α be the biggest odd factor of n in $[3, k-2]$. Then two cases occur:*

(i′) *if $\alpha 3^j \notin [3, k-2]$ for $j \in N$, then*

$$P_k(G_n; x, y, z) = \begin{cases} \alpha\left(n\binom{k+1}{2}\right), & n \equiv 0 \bmod 4, \\ \alpha(n(k+1)), & n \equiv 2 \bmod 4, \\ \alpha(2n(k+1)), & \text{otherwise}; \end{cases} \tag{2.12}$$

(ii′) *if β is the biggest odd number which is in $[3, k-2]$ and $\beta = \alpha 3^j$ for $j \in N$, then*

$$P_k(G_n; x, y, z) = \begin{cases} \beta\left(n\binom{k+1}{2}\right), & n \equiv 0 \bmod 4, \\ \beta(n(k+1)), & n \equiv 2 \bmod 4, \\ \beta(2n(k+1)), & \text{otherwise}. \end{cases} \tag{2.13}$$

Proof. We consider G_n as D_n, the dihedral group of $2n$ elements. Now D_n being the group of symmetries of the regular polygon with n elements admits a presentation as the group generated by the two matrices:

$$a := \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix}, \qquad b := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.14}$$

Under these identifications, we can take $z = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $x = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ -\sin(2\pi/n) & -\cos(2\pi/n) \end{pmatrix}$.

(i) If $k = 2$, we have the sequence

$$x_0 = y, \quad x_1 = x, \quad x_2 = z, \quad x_3 = \begin{pmatrix} \cos\left(\frac{4\pi}{n}\right) & -\sin\left(\frac{4\pi}{n}\right) \\ -\sin\left(\frac{4\pi}{n}\right) & -\cos\left(\frac{4\pi}{n}\right) \end{pmatrix} = xz, \quad x_4 = x,$$
$$x_5 = \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix} = xy, \quad x_6 = y, \quad x_7 = x, \quad x_8 = z, \ldots. \tag{2.15}$$

Thus we get $P_2(G_n; y, x, z) = 6$.

(ii) If $k = 4$, we have the sequence

$$x, y, z, xyz = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e, \qquad (xyz)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e, \qquad \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ -\sin\left(\frac{2\pi}{n}\right) & -\cos\left(\frac{2\pi}{n}\right) \end{pmatrix} = x,$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = y, \qquad \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix} = xy, \qquad \begin{pmatrix} \cos\left(\frac{4\pi}{n}\right) & \sin\left(\frac{4\pi}{n}\right) \\ -\sin\left(\frac{4\pi}{n}\right) & \cos\left(\frac{4\pi}{n}\right) \end{pmatrix} = z^{-2},$$
$$\begin{pmatrix} \cos\left(\frac{6\pi}{n}\right) & \sin\left(\frac{6\pi}{n}\right) \\ \sin\left(\frac{6\pi}{n}\right) & -\cos\left(\frac{6\pi}{n}\right) \end{pmatrix} = z^4 x, \qquad \begin{pmatrix} \cos\left(\frac{14\pi}{n}\right) & \sin\left(\frac{14\pi}{n}\right) \\ \sin\left(\frac{14\pi}{n}\right) & -\cos\left(\frac{14\pi}{n}\right) \end{pmatrix} = z^8 x,$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = y, \qquad \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix} = z, \qquad \begin{pmatrix} \cos\left(\frac{8\pi}{n}\right) & -\sin\left(\frac{8\pi}{n}\right) \\ \sin\left(\frac{8\pi}{n}\right) & \cos\left(\frac{8\pi}{n}\right) \end{pmatrix} = z^4,$$
$$\begin{pmatrix} \cos\left(\frac{24\pi}{n}\right) & -\sin\left(\frac{24\pi}{n}\right) \\ \sin\left(\frac{24\pi}{n}\right) & \cos\left(\frac{24\pi}{n}\right) \end{pmatrix} = z^{12}, \qquad \begin{pmatrix} \cos\left(\frac{34\pi}{n}\right) & -\sin\left(\frac{34\pi}{n}\right) \\ -\sin\left(\frac{34\pi}{n}\right) & -\cos\left(\frac{34\pi}{n}\right) \end{pmatrix} = xz^{16}, \ldots. \tag{2.16}$$

Now we consider what happens to the 4-*nacci* sequence when we have a section of the form $\ldots,\ z^{\tau}x,\ zx,\ z,\ldots:$

$$\begin{gathered} z^{\tau}x, zx=y, z, z^{\varepsilon}, z^{\tau+\varepsilon}, xz^{2\varepsilon+\tau}, y, xzx=xy, xz^{\varepsilon+2}x=z^{-(\varepsilon+2)}, \\ xz^{3\varepsilon+\tau+4}x=z^{-(3\varepsilon+\tau+4)}, z^{4\varepsilon+\tau+8}x, zx=y, z,\ldots. \end{gathered} \tag{2.17}$$

The 4-*nacci* sequence can be said to form layers of length 10. Using the above, the 4-*nacci* sequence becomes

$$\begin{gathered} x_0=x,\quad x_1=y,\quad x_2=z,\quad x_3=e,\ldots, \\ x_{10}=z^{8}x,\quad x_{11}=zx=y,\quad x_{12}=z,\quad x_{13}=z^{4},\ldots, \\ x_{20}=z^{32}x,\quad x_{21}=zx=y,\quad x_{22}=z,\quad x_{23}=z^{8},\ldots, \\ x_{10i}=z^{8i^2}x,\quad x_{10i+1}=zx=y,\quad x_{10i+2}=z,\quad x_{10i+3}=z^{4i},\ldots, \end{gathered} \tag{2.18}$$

where $z^{8i^2}=\begin{pmatrix}\cos 8i^2(2\pi/n) & -\sin 8i^2(2\pi/n)\\ \sin 8i^2(2\pi/n) & \cos 8i^2(2\pi/n)\end{pmatrix}$ and $z^{4i}=\begin{pmatrix}\cos 4i(2\pi/n) & -\sin 4i(2\pi/n)\\ \sin 4i(2\pi/n) & \cos 4i(2\pi/n)\end{pmatrix}$.

So, we need the smallest $i\in N$ such that $8i^2=nv_1$ and $4i=nv_2$ for $v_1,v_2\in N$.

If $n\equiv 0\pmod 4$, $z^{8i^2}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $z^{4i}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ for $i=n/4$.

Thus, $10i=(5/2)n$ and $P_4=(G_n;\ x,\ y,\ z)=n((k+1)/2)=(5/2)n$.

If $n\equiv 2\pmod 4$, $z^{8i^2}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $z^{4i}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ for $i=n/2$.

Thus, $10i=5n$ and $P_4=(G_n;x,y,z)=n(k+1)=5n$.

If $n\equiv 1\pmod 4$ or $n\equiv 3\pmod 4$, $z^{8i^2}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $z^{4i}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ for $i=n$.

Thus, $10i=10n$ and $P_4=(G_n;x,y,z)=2n(k+1)=10n$.

(iii) If $k\geq 5$, the first $k+1$ elements of the sequence are

$$x_0=x,\quad x_1=y,\quad x_2=z,\quad x_3=z^{n},\quad x_4=z^{2n},\ldots,\quad x_k=z^{2^{k-3}n}. \tag{2.19}$$

Thus, using the above information, the sequence reduces

$$x_0=x,\quad x_1=y,\quad x_2=z,\quad x_3=e,\quad x_4=e,\ldots,e, \tag{2.20}$$

where $x_j=e$ for $3\leq j\leq k$.

Now we consider what happens to the k-*nacci* sequence when we have a section of the form $\ldots,z^{\tau}x,zx,z,\ldots:$

$$\begin{gathered} x_{2k+2}=\prod_{i=k+2}^{2k+1}x_i=z^{\tau}x,\quad x_{2k+2+1}=\prod_{i=k+3}^{2k+2}x_i=zx=y,\quad x_{2k+2+2}=\prod_{i=k+4}^{2k+3}x_i=z, \\ x_{2k+2+3}=\prod_{i=k+5}^{2k+4}x_i=z^{\varepsilon},\quad x_{2k+2+4}=\prod_{i=k+6}^{2k+5}x_i=z^{c},\quad x_{2k+2+5}=\prod_{i=k+7}^{2k+6}x_i=z^{u_1},\ldots, \\ x_{2k+2+k}=\prod_{i=2k+2}^{3k+1}x_i=z^{u_{k-4}},\ldots. \end{gathered} \tag{2.21}$$

The *k-nacci* sequence can be said to form layers of length $(2k+2)$. Using the above, the *k-nacci* sequence becomes

$$\begin{gathered} x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = e, \ldots, \quad x_k = z^{2^{k-3}n} = e, \ldots, \\ x_{i(2k+2)} = z^{\tau} x, \quad x_{i(2k+2)+1} = zx, \quad x_{i(2k+2)+2} = z, \quad x_{i(2k+2)+3} = z^{4i}, \\ x_{i(2k+2)+4} = z^{8i^2+4i}, \quad x_{i(2k+2)+5} = z^{u_1}, \ldots, \quad x_{i(2k+2)+k} = z^{u_{k-4}}, \ldots. \end{gathered} \tag{2.22}$$

So, we need the smallest $i \in N$ such that $4i = nv_1$ and $8i^2 + 4i = nv_2$ for $v_1, v_2 \in N$.

(1) If there is no $t \in [3, k-2]$ such that t is an odd factor of n, there are 3 subcases.

Case 1. If $n \equiv 0 \pmod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = n/4$. So, we get $P_k = (G_n; x, y, z) = n((k+1)/2)$ since $i(2k+2) = n((k+1)/2)$ (where by $n \mid \tau$ we mean that n divides τ).

Case 2. If $n \equiv 2 \pmod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = n/2$. So, we get $P_k = (G_n; x, y, z) = n(k+1)$ since $i(2k+2) = n(k+1)$.

Case 3. If $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = n$. So, we get $P_k = (G_n; x, y, z) = 2n(k+1)$ since $i(2k+2) = 2n(k+1)$.

(2) Let α odd be the biggest factor of n in $[3,\ k-2]$. Then two cases occur:

(i′) If $\alpha 3^j \notin [3, k-2]$ for $j \in N$, then there are 3 subcases.

Case 1. If $n \equiv 0 \pmod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = \alpha(n/4)$. So, we get $P_k = (G_n; x, y, z) = \alpha(n((k+1)/2))$ since $i(2k+2) = \alpha(n((k+1)/2))$.

Case 2. If $n \equiv 2 \pmod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = \alpha(n/2)$. So, we get $P_k = (G_n; x, y, z) = \alpha(n(k+1))$ since $i(2k+2) = \alpha(n(k+1))$.

Case 3. If $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = \alpha n$. So, we get $P_k = (G_n; x, y, z) = \alpha(2n(k+1))$ since $i(2k+2) = \alpha(2n(k+1))$.

(ii′) If β is the biggest odd number which is in $[3, k-2]$ and $\beta = \alpha 3^j$ for $j \in N$, then there are 3 subcases.

Case 1. If $n \equiv 0 \pmod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = \beta(n/4)$. So, we get $P_k = (G_n; x, y, z) = \beta(n((k+1)/2))$ since $i(2k+2) = \beta(n((k+1)/2))$.

Case 2. If $n \equiv 2 \mod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = \beta(n/2)$. So, we get $P_k = (G_n; x, y, z) = \beta(n(k+1))$ since $i(2k+2) = \beta(n(k+1))$.

Case 3. If $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ and $n \mid \tau$, $n \mid u_1, \ldots, n \mid u_{k-4}$, $z^{4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $z^{8i^2+4i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = \beta n$. So, we get $P_k = (G_n; x, y, z) = \beta(2n(k+1))$ since $i(2k+2) = \beta(2n(k+1))$.

This completes the proof. □

In the case of 2-generator the group has the presentation $\langle x, y : x^2 = y^2 = (xy)^n = e\rangle$ and the period is the same as in the 3-generator case and proof is similar.

3. The Groups $E(2,2,2)$, $E(n,2,2)$, $E(2,n,2)$, and $E(2,2,n)$

Definition 3.1. The *extended triangle group* $E(p,q,r)$, for $p,q,r > 1$, is defined by the presentation

$$\left\langle x, y, z : x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = e \right\rangle. \tag{3.1}$$

The *extended triangle groups* are a very important class of groups closely linked to automorphism groups of regular maps, see [17]. The *triangle groups (polyhedral groups)*, (p,q,r) are index two subgroups of *extended triangle groups*. To see this, let $X = xy$, $Y = yz$ and $Z = zx$ in $E(p,q,r)$ and then use the obvious epimorphism. We get the following three cases for $E(p,q,r)$:

(1) the Euclidean case if $1/p + 1/q + 1/r = 1$,

(2) the elliptic case if $1/p + 1/q + 1/r > 1$,

(3) the hyperbolic case if $1/p + 1/q + 1/r < 1$.

The group $E(p,q,r)$ is finite if and only if $1/p + 1/q + 1/r > 1$.

For more information on these groups, see [14, 18].

Theorem 3.2. *Let E_2 be the group defined by the presentation $\langle x, y, z : x^2 = y^2 = z^2 = (xy)^2 = (yz)^2 = (zx)^2 = e\rangle$. Then $P_k(E_2; x, y, z) = k + 1$ for $k > 2$.*

Proof. Since E_2 can be identified with $Z_2 \oplus Z_2 \oplus Z_2$ and x, y, z with $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively, from a similar argument applied to Theorem 2.2, we get $P_k(E_2; x, y, z) = k + 1$. □

Theorem 3.3. *Let $E_n, n > 2$, be the group defined by the presentation $\langle x, y, z : x^2 = y^2 = z^2 = (xy)^2 = (yz)^2 = (zx)^n = e\rangle$*

(i)

$$P_{4,5}(E_n; x, y, z) = \begin{cases} n\left(\dfrac{k+1}{2}\right), & n \equiv 0 \bmod 4, \\ n(k+1), & n \equiv 2 \bmod 4, \\ 2n(k+1), & \text{otherwise}; \end{cases} \tag{3.2}$$

(ii) *let $k \geq 6$.*

(1) *If there is no* $t \in [3, k-3]$ *such that* t *is an odd factor of* n*, then*

$$P_k(E_n; x, y, z) = \begin{cases} n\left(\dfrac{k+1}{2}\right), & n \equiv 0 \bmod 4, \\ n(k+1), & n \equiv 2 \bmod 4, \\ 2n(k+1), & \text{otherwise}. \end{cases} \tag{3.3}$$

(2) *Let* α *be the biggest odd factor of* n *in* $[3, k-3]$*. Then two cases occur:*

(i′) *if* $\alpha 3^j \notin [3, k-3]$ *for* $j \in N$*, then*

$$P_k(E_n; x, y, z) = \begin{cases} \alpha\left(n\left(\dfrac{k+1}{2}\right)\right), & n \equiv 0 \bmod 4, \\ \alpha(n(k+1)), & n \equiv 2 \bmod 4, \\ \alpha(2n(k+1)), & \text{otherwise;} \end{cases} \tag{3.4}$$

(ii′) *if* β *is be the biggest odd number which is in* $[3, k-3]$ *and* $\beta = \alpha 3^j$ *for* $j \in N$*, then*

$$P_k(E_r; x, y, z) = \begin{cases} \beta\left(n\left(\dfrac{k+1}{2}\right)\right), & n \equiv 0 \bmod 4, \\ \beta(n(k+1)), & n \equiv 2 \bmod 4, \\ \beta(2n(k+1)), & \text{otherwise.} \end{cases} \tag{3.5}$$

Proof. Since y has order 2 and commutes with x and z it follows that $E_n = Z_2 \oplus D_n$. As a group of matrices, the can be identified with a group of 3×3 matrices of form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \mathrm{a} \end{pmatrix}, \tag{3.6}$$

where a is a 2×2 matrix in dihedral group generated by a and b shown at (2.14). Here,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\dfrac{2\pi}{n}\right) & -\sin\left(\dfrac{2\pi}{n}\right) \\ 0 & -\sin\left(\dfrac{2\pi}{n}\right) & -\cos\left(\dfrac{2\pi}{n}\right) \end{pmatrix}, \quad y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.7}$$

Now, since the period of a Fibonacci sequence in a direct product of groups is the least common multiple of the periods in each the factors and from a similar argument applied to Theorem 2.4 the proof is done. □

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