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Research Article

Bounds for Certain Nonlinear Dynamic Inequalities on Time Scales

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We investigate some new nonlinear dynamic inequalities on time scales. Our results unify and extend some integral inequalities and their corresponding discrete analogues. The inequalities given here can be used to investigate the properties of certain dynamic equations on time scales.

1. Introduction

To unify the theory of continuous and discrete dynamic systems, in 1988, Hilger [1] first introduced the calculus on time scales. Motivated by the paper [1], many authors have expounded on various aspects of the theory of dynamic equations on time scales. For example, we refer the reader to the literatures [2–7] and the references cited therein. At the same time, a few papers [8–13] have studied the theory of dynamic inequalities on time scales.

The main purpose of this paper is to investigate some nonlinear dynamic inequalities on time scales, which unify and extend some integral inequalities and their corresponding discrete analogues. Our work extends some known results of dynamic inequalities on time scales.

Throughout this paper, a knowledge and understanding of time scales and time-scale notation is assumed. For an excellent introduction to the calculus on time scales, we refer the reader to monographes [6, 7].

2. Main Results

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the set of nonnegative integers, $C(M, S)$ denotes the class of all continuous functions defined on set M with range in the set S, $\mathbb{T}$ is an arbitrary time scale, C_{rd} denotes the set of rd-continuous functions, $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. We use the usual conventions that empty sums and products are taken to be 0 and 1, respectively. Throughout this paper, we always assume that $p \geq q > 0$, p and q are real constants, and $t \geq t_0$, $t_0 \in \mathbb{T}^\kappa$.

Firstly, we introduce the following lemmas, which are useful in our main results.

Lemma 2.1. *Let $a \geq 0$. Then*

$$a^{q/p} \leq \left(\frac{q}{p}K^{q-p/p}a + \frac{p-q}{p}K^{q/p}\right) \quad \textit{for any} \quad K > 0. \tag{2.1}$$

Proof. If $a = 0$, then we easily see that the inequality (2.1) holds. Thus we only prove that the inequality (2.1) holds in the case of $a > 0$.

Letting

$$f(K) = \frac{q}{p}K^{q-p/p}a + \frac{p-q}{p}K^{q/p}, \quad K > 0, \tag{2.2}$$

we have

$$f'(K) = \frac{q(p-q)}{p^2}K^{q-2p/p}(K-a). \tag{2.3}$$

It is easy to see that

$$\begin{aligned} f'(K) &\geq 0, \quad K > a, \\ f'(K) &= 0, \quad K = a, \\ f'(K) &\leq 0, \quad 0 < K < a. \end{aligned} \tag{2.4}$$

Therefore,

$$f(K) \geq f(a) = a^{q/p}. \tag{2.5}$$

The proof of Lemma 2.1 is complete. □

Lemma 2.2 (see [6]). *Let $t_0 \in \mathbb{T}^\kappa$ and $w : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ be continuous at (t,t), $t \in \mathbb{T}^\kappa$ with $t > t_0$. Assume that $w_1^\Delta(t,\cdot)$ is* rd*-continuous on $[t_0, \sigma(t)]$. If, for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\left|w(\sigma(t),\tau) - w(s,\tau) - w_1^\Delta(t,\tau)(\sigma(t)-s)\right| \leq \varepsilon|\sigma(t)-s| \quad \textit{for all } s \in U, \tag{2.6}$$

where w_1^{Δ} *denotes the derivative of* w *with respect to the first variable, then*

$$v(t) := \int_{t_0}^{t} w(t,\tau)\Delta\tau \tag{2.7}$$

implies

$$v^{\Delta}(t) = \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\Delta\tau + w(\sigma(t),t). \tag{2.8}$$

Lemma 2.3 (Comparison theorem [6]). *Suppose* $u, b \in C_{\mathrm{rd}}$, $a \in \mathcal{R}^+$. *Then*

$$u^{\Delta}(t) \le a(t)u(t) + b(t) \quad \text{for all } t \in \mathbb{T}^{\kappa} \tag{2.9}$$

implies

$$u(t) \le u(t_0)e_a(t,t_0) + \int_{t_0}^{t} e_a(t,\sigma(\tau))b(\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}^{\kappa}. \tag{2.10}$$

Next, we establish our main results.

Theorem 2.4. *Assume that* $u, a, b, g, h \in C_{\mathrm{rd}}$, *and* $u(t), a(t), b(t), g(t)$ *and* $h(t)$ *are nonnegative. Then*

$$u^{p}(t) \le a(t) + b(t)\int_{t_0}^{t} \left[g(\tau)u^{q}(\tau) + h(\tau)\right]\Delta\tau \quad \text{for all } t \in \mathbb{T}^{\kappa} \tag{E1}$$

implies

$$u(t) \le \left\{a(t) + b(t)\int_{t_0}^{t} e_B(t,\sigma(\tau))F(\tau)\Delta\tau\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}, \tag{2.11}$$

where

$$F(t) = g(t)\left(\frac{p-q}{p}K^{q/p} + \frac{qa(t)}{pK^{(p-q)/p}}\right) + h(t), \tag{2.12}$$

and also

$$B(t) = \frac{qb(t)g(t)}{pK^{(p-q)/p}} \quad \text{for all } t \in \mathbb{T}^{\kappa}. \tag{2.13}$$

Proof. Obviously, if $t = t_0$, then the inequality (2.11) holds. Therefore, in the next proof, we always assume that $t > t_0$, $t \in \mathbb{T}^{\kappa}$.

Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} \left[g(\tau)u^q(\tau) + h(\tau)\right] \Delta\tau. \tag{2.14}$$

Then $(E1)$ can be restated as

$$u^p(t) \le a(t) + b(t)z(t). \tag{2.15}$$

Using Lemma 2.1, from (2.15), for any $K > 0$, we easily obtain

$$\begin{aligned} u^q(t) &\le (a(t) + b(t)z(t))^{q/p} \\ &\le \frac{p-q}{p}K^{q/p} + \frac{qa(t)}{pK^{(p-q)/p}} + \frac{qb(t)z(t)}{pK^{(p-q)/p}}. \end{aligned} \tag{2.16}$$

It follows from (2.14) and (2.16) that

$$\begin{aligned} z^{\Delta}(t) &\le g(t)\left(\frac{p-q}{p}K^{q/p} + \frac{qa(t)}{pK^{(p-q)/p}} + \frac{qb(t)z(t)}{pK^{(p-q)/p}}\right) + h(t) \\ &= F(t) + B(t)z(t), \end{aligned} \tag{2.17}$$

where $F(t)$ and $B(t)$ are defined as in (2.12) and (2.13), respectively. Using Lemma 2.3 and noting $z(t_0) = 0$, from (2.17) we have

$$z(t) \le \int_{t_0}^{t} e_B(t, \sigma(\tau))F(\tau)\Delta\tau, \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}. \tag{2.18}$$

Therefore, the desired inequality (2.11) follows from (2.15) and (2.18). This completes the proof of Theorem 2.4. □

Remark 2.5. By letting $p = q = 1$ in Theorem 2.4, it is easy to observe that the bound obtained in (2.11) reduces to the bound obtained in [9, Theorem 3.1].

As a particular case of Theorem 2.4, we immediately obtain the following result.

Corollary 2.6. *Assume that $u, g \in C_{\mathrm{rd}}$, and $u(t)$ and $g(t)$ are nonnegative. If $\alpha > 0$ is a constant, then*

$$u^p(t) \le \alpha + \int_{t_0}^{t} g(\tau)u^q(\tau)\Delta\tau \quad \textit{for all } t \in \mathbb{T}^{\kappa} \tag{$E'1$}$$

implies

$$u(t) \leq \left\{ \alpha + \int_{t_0}^{t} e_{\widehat{B}}(t, \sigma(\tau)) \widehat{F}(\tau) \Delta\tau \right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}, \tag{2.19}$$

where

$$\widehat{F}(t) = g(t) \left(\frac{p-q}{p} K^{q/p} + \frac{q\alpha}{pK^{(p-q)/p}} \right), \tag{2.20}$$

$$\widehat{B}(t) = \frac{qg(t)}{pK^{(p-q)/p}} \quad \text{for all } t \in \mathbb{T}^{\kappa}. \tag{2.21}$$

Remark 2.7. The result of Theorem 2.4 holds for an arbitrary time scale. Therefore, using Theorem 2.4, we immediately obtain many results for some peculiar time scales. For example, letting $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively, we have the following two results.

Corollary 2.8. *Let $\mathbb{T} = \mathbb{R}$ and assume that $u(t), a(t), b(t), g(t), h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. Then the inequality*

$$u^p(t) \leq a(t) + b(t) \int_0^t \left[g(s) u^q(s) + h(s) \right] ds, \quad t \in \mathbb{R}_+ \tag{2.22}$$

implies

$$u(t) \leq \left[a(t) + b(t) \int_0^t F(\theta) \exp\left(\int_\theta^t B(s) ds \right) d\theta \right]^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{R}_+, \tag{2.23}$$

where $F(t)$ and $B(t)$ are defined as in Theorem 2.4.

Corollary 2.9. *Let $\mathbb{T} = \mathbb{Z}$ and assume that $u(t), a(t), b(t), g(t)$, and $h(t)$ are nonnegative functions defined for $t \in \mathbb{N}_0$. Then the inequality*

$$u^p(t) \leq a(t) + b(t) \sum_{s=0}^{t-1} \left[g(s) u^q(s) + h(s) \right], \quad t \in \mathbb{N}_0 \tag{2.24}$$

implies

$$u(t) \leq \left[a(t) + b(t) \sum_{\theta=0}^{t-1} F(\theta) \prod_{s=\theta+1}^{t-1} (1 + B(s)) \right]^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{N}_0, \tag{2.25}$$

where $F(t)$ and $B(t)$ are defined as in Theorem 2.4.

Investigating the proof procedure of Theorem 2.4 carefully, we can obtain the following result.

Theorem 2.10. *Assume that* $u, a, b, g_i, h \in C_{\mathrm{rd}}$, *and* $u(t), a(t), b(t), g_i(t)$, *and* $h(t)$ *are nonnegative,* $i = 1, 2, \ldots, n$. *If there exists a series of positive real numbers* $q_1, q_2, \ldots, q_n$ *such that* $p \geq q_i > 0,\ i = 1, 2, \ldots, n$, *then*

$$u^p(t) \leq a(t) + b(t)\int_{t_0}^{t}\left[\sum_{i=1}^{n} g_i(\tau)u^{q_i}(\tau) + h(\tau)\right]\Delta\tau \quad \text{for all } t \in \mathbb{T}^{\kappa} \tag{$E''1$}$$

implies

$$u(t) \leq \left\{a(t) + b(t)\int_{t_0}^{t} e_{B^*}(t, \sigma(\tau))F^*(\tau)\Delta\tau\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}, \tag{2.26}$$

where

$$F^*(t) = \sum_{i=1}^{n} g_i(t)\left(\frac{p - q_i}{p}K^{q_i/p} + \frac{q_i a(t)}{pK^{(p-q_i)/p}}\right) + h(t), \tag{2.27}$$

$$B^*(t) = \sum_{i=1}^{n} \frac{q_i b(t) g_i(t)}{pK^{(p-q_i)/p}} \quad \text{for all } t \in \mathbb{T}^{\kappa}. \tag{2.28}$$

Theorem 2.11. *Assume that* $u, a, b, f, g, m \in C_{\mathrm{rd}}$, $u(t), a(t), b(t), f(t), g(t)$, *and* $m(t)$ *are nonnegative, and* $w(t,s)$ *is defined as in Lemma 2.2 such that* $w(t,s) \geq 0$ *and* $w_1^{\Delta}(t,s) \geq 0$ *for* $t, s \in \mathbb{T}$ *with* $s \leq t$. *If, for any* $\varepsilon > 0$, *there exists a neighborhood* U *of* t, *independent of* $\tau \in [t_0, \sigma(t)]$, *such that for all* $s \in U$,

$$\left|\left[w(\sigma(t), \tau) - w(s, \tau) - w_1^{\Delta}(t, \tau)(\sigma(t) - s)\right]\left[f(\tau)u^p(\tau) + g(\tau)u^q(\tau) + m(\tau)\right]\right| \leq \varepsilon|\sigma(t) - s|, \tag{2.29}$$

then

$$u^p(t) \leq a(t) + b(t)\int_{t_0}^{t} w(t, \tau)\left[f(\tau)u^p(\tau) + g(\tau)u^q(\tau) + m(\tau)\right]\Delta\tau, \quad t \in \mathbb{T}^{\kappa} \tag{$E2$}$$

implies

$$u(t) \leq \left\{a(t) + b(t)\int_{t_0}^{t} e_A(t, \sigma(\tau))G(\tau)\Delta(\tau)\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}, \tag{2.30}$$

where

$$A(t) = w(\sigma(t),t)b(t)\left(f(t) + \frac{qg(t)}{pK^{(p-q)/p}}\right) + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)b(\tau)\left(f(\tau) + \frac{qg(\tau)}{pK^{(p-q)/p}}\right)\Delta\tau, \quad (2.31)$$

and also

$$G(t) = w(\sigma(t),t)\left[a(t)f(t) + g(t)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(t)}{pK^{(p-q)/p}}\right) + m(t)\right] + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\left[a(\tau)f(\tau) + g(\tau)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(\tau)}{pK^{(p-q)/p}}\right) + m(\tau)\right]\Delta\tau. \quad (2.32)$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} k(t,\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}^{\kappa}, \quad (2.33)$$

where

$$k(t,\tau) = w(t,\tau)\left[f(\tau)u^p(\tau) + g(\tau)u^q(\tau) + m(\tau)\right]. \quad (2.34)$$

Then $z(t_0) = 0$. As in the proof of Theorem 2.4, we easily obtain (2.15) and (2.16).

It follows from (2.34) that

$$k(\sigma(t),t) = w(\sigma(t),t)\left[f(\tau)u^p(\tau) + g(\tau)u^q(\tau) + m(\tau)\right], \quad (2.35)$$

and also

$$k_1^{\Delta}(t,\tau) = w_1^{\Delta}(t,\tau)\left[f(\tau)u^p(\tau) + g(\tau)u^q(\tau) + m(\tau)\right]. \quad (2.36)$$

Therefore, noting the condition (2.29), using Lemma 2.2 and combining (2.33)–(2.36), (2.15), and (2.16), we have

$$\begin{aligned}
z^{\Delta}(t) &= k(\sigma(t),t) + \int_{t_0}^{t} k_1^{\Delta}(t,\tau)\Delta\tau \\
&= w(\sigma(t),t)\left[f(t)u^p(t) + g(t)u^q(t) + m(t)\right] \\
&\quad + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\left[f(\tau)u^p(\tau) + g(\tau)u^q(\tau) + m(\tau)\right]\Delta\tau \\
&\leq w(\sigma(t),t)\Bigg[a(t)f(t) + g(t)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(t)}{pK^{(p-q)/p}}\right) + m(t) \\
&\qquad\qquad + b(t)\left(f(t) + \frac{qg(t)}{pK^{(p-q)/p}}\right)z(t)\Bigg] \\
&\quad + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\Bigg[a(\tau)f(\tau) + g(\tau)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(\tau)}{pK^{(p-q)/p}}\right) + m(\tau) \\
&\qquad\qquad + b(\tau)\left(f(\tau) + \frac{qg(\tau)}{pK^{(p-q)/p}}\right)z(\tau)\Bigg]\Delta\tau \\
&\leq \left[w(\sigma(t),t)b(t)\left(f(t) + \frac{qg(t)}{pK^{(p-q)/p}}\right) + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)b(\tau)\left(f(\tau) + \frac{qg(\tau)}{pK^{(p-q)/p}}\right)\Delta\tau\right]z(t) \\
&\quad + w(\sigma(t),t)\left[a(t)f(t) + g(t)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(t)}{pK^{(p-q)/p}}\right) + m(t)\right] \\
&\quad + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\left[a(\tau)f(\tau) + g(\tau)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(\tau)}{pK^{(p-q)/p}}\right) + m(\tau)\right]\Delta\tau \\
&= A(t)z(t) + G(t) \quad \text{for all } t \in \mathbb{T}^{\kappa},
\end{aligned} \tag{2.37}$$

where $A(t)$ and $G(t)$ are defined as in (2.31) and (2.32), respectively. Therefore, using Lemma 2.3 and noting $z(t_0) = 0$, we get

$$z(t) \leq \int_{t_0}^{t} e_A(t,\sigma(\tau))G(\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}^{\kappa}. \tag{2.38}$$

It is easy to see that the desired inequality (2.30) follows from (2.15) and (2.38). This completes the proof of Theorem 2.11. □

Remark 2.12. Letting $p = q = 1$, $f(t) = 0$ in Theorem 2.11, we easily obtain [9, Theorem 3.10].

The following two corollaries are easily established by using Theorem 2.11.

Corollary 2.13. *Let $\mathbb{T} = \mathbb{R}$ and assume that $u(t), a(t), b(t), f(t), g(t), m(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. If $w(t,s)$ and its partial derivative $(\partial/\partial t)w(t,s)$ are real-valued nonnegative continuous functions for $t, s \in \mathbb{R}_+$ with $s \leq t$, then the inequality*

$$u^p(t) \leq a(t) + b(t)\int_0^t w(t,s)\left[f(s)u^p(s) + g(s)u^q(s) + m(s)\right]ds, \quad t \in \mathbb{R}_+ \tag{2.39}$$

implies

$$u(t) \leq \left\{a(t) + b(t)\int_0^t \overline{G}(s)\exp\left(\int_s^t \overline{A}(\tau)d\tau\right)ds\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{R}_+, \tag{2.40}$$

where

$$\overline{A}(t) = w(t,t)b(t)\left(f(t) + \frac{qg(t)}{pK^{(p-q)/p}}\right) + \int_0^t \frac{\partial}{\partial t}w(t,s)b(s)\left(f(s) + \frac{qg(s)}{pK^{(p-q)/p}}\right)ds, \tag{2.41}$$

and also

$$\begin{aligned}\overline{G}(t) = {}& w(t,t)\left[a(t)f(t) + g(t)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(t)}{pK^{(p-q)/p}}\right) + m(t)\right] \\ &+ \int_0^t \frac{\partial}{\partial t}w(t,s)\left[a(s)f(s) + g(s)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(s)}{pK^{(p-q)/p}}\right) + m(s)\right]ds.\end{aligned} \tag{2.42}$$

Remark 2.14. Letting $p = q = 1$, $f(t) = 0$ in Corollary 2.13, we easily obtain [14, Theorem 1.4.3].

Corollary 2.15. *Let $\mathbb{T} = \mathbb{Z}$ and assume that $u(t), a(t), b(t), f(t), g(t)$ and $m(t)$ are nonnegative functions defined for $t \in \mathbb{N}_0$. If $w(t,s)$ and $\Delta_1 w(t,s)$ are real-valued nonnegative functions for $t, s \in \mathbb{N}_0$ with $s \leq t$, then the inequality*

$$u^p(t) \leq a(t) + b(t)\sum_{s=0}^{t-1} w(t,s)\left[f(s)u^p(s) + g(s)u^q(s) + m(s)\right], \quad t \in \mathbb{N}_0, \tag{2.43}$$

implies

$$u(t) \leq \left\{a(t) + b(t)\sum_{s=0}^{t-1}\widetilde{G}(s)\prod_{\tau=s+1}^{t-1}\left(1 + \widetilde{A}(\tau)\right)\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{N}_0, \tag{2.44}$$

where $\mathbf{\Delta}_1 w(t,s) = w(t+1,s) - w(t,s)$ *for* $t, s \in \mathbb{N}_0$ *with* $s \le t$,

$$\widetilde{A}(t) = w(t+1,t)b(t)\left(f(t) + \frac{qg(t)}{pK^{(p-q)/p}} \right) + \sum_{s=0}^{t-1} \mathbf{\Delta}_1 w(t,s)b(s)\left(f(s) + \frac{qg(s)}{pK^{(p-q)/p}} \right), \quad (2.45)$$

$$\begin{aligned}\widetilde{G}(t) &= w(t+1,t)\left[a(t)f(t) + g(t)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(t)}{pK^{(p-q)/p}} \right) + m(t) \right] \\ &\quad + \sum_{s=0}^{t-1} \mathbf{\Delta}_1 w(t,s)\left[a(s)f(s) + g(s)\left(\frac{(p-q)K^{q/p}}{p} + \frac{qa(s)}{pK^{(p-q)/p}} \right) + m(s) \right].\end{aligned} \quad (2.46)$$

Remark 2.16. By letting $p = q = 1$, $f(t) = 0$ in Corollary 2.15, it is very easy to obtain [15, Theorem 1.3.4].

Corollary 2.17. *Suppose that* $u(t), a(t)$, *and* $w(t,s)$ *are defined as in Theorem 2.11, and let* $a(t)$ *be nondecreasing for all* $t \in \mathbb{T}^\kappa$. *If, for any* $\varepsilon > 0$, *there exists a neighborhood* U*of* t, *independent of* $\tau \in [t_0, \sigma(t)]$, *such that for all* $s \in U$,

$$\left| u^q(\tau)\left[w(\sigma(t),\tau) - w(s,\tau) - w_1^\Delta(t,\tau)(\sigma(t) - s) \right] \right| \le \varepsilon |\sigma(t) - s|, \quad (2.47)$$

then

$$u^p(t) \le a(t) + \int_{t_0}^{t} w(t,\tau)u^q(\tau)\Delta\tau \quad \textit{for all } t \in \mathbb{T}^\kappa \quad (E'2)$$

implies

$$u(t) \le \left\{ \frac{1}{q}\left[\left(K(p-q) + qa(t) \right) e_{\widetilde{\widetilde{A}}}(t,t_0) - K(p-q) \right] \right\}^{1/p} \quad \textit{for any } K > 0,\ t \in \mathbb{T}^\kappa, \quad (2.48)$$

where

$$\widetilde{\widetilde{A}}(t) = \frac{q}{pK^{(p-q)/p}}\left(w(\sigma(t),t) + \int_{t_0}^{t} w_1^\Delta(t,\tau)\Delta\tau \right). \quad (2.49)$$

Proof. Letting $b(t) = 1$, $f(t) = 0$, $g(t) = 1$, and $m(t) = 0$ in Theorem 2.11, we obtain

$$A(t) = \frac{q}{pK^{(p-q)/p}}\left(w(\sigma(t),t) + \int_{t_0}^{t} w_1^\Delta(t,\tau)\Delta\tau \right) := \widetilde{\widetilde{A}}(t), \quad (2.50)$$

and also

$$\begin{aligned}
G(t) &= \frac{1}{pK^{(p-q)/p}}\left\{w(\sigma(t),t)\left[K(p-q)+qa(t)\right]+\int_{t_0}^{t} w_1^{\Delta}(t,\tau)\left[K(p-q)+qa(\tau)\right]\Delta\tau\right\} \\
&\leq \frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\left\{w(\sigma(t),t)+\int_{t_0}^{t} w_1^{\Delta}(t,\tau)\Delta\tau\right\} \\
&= \frac{1}{q}\left[K(p-q)+qa(t)\right]\widetilde{\widetilde{A}}(t) \quad \text{for any } K>0,\ t\in\mathbb{T}^{\kappa},
\end{aligned} \tag{2.51}$$

where the inequality holds because $a(t)$ is nondecreasing for all $t \in \mathbb{T}^{\kappa}$. Therefore, using Theorem 2.11 and noting (2.50) and (2.51), we easily have

$$\begin{aligned}
u(t) &\leq \left\{a(t)+\int_{t_0}^{t} e_A(t,\sigma(\tau))G(\tau)\Delta\tau\right\}^{1/p} \\
&\leq \left\{a(t)+\frac{1}{q}\int_{t_0}^{t} e_{\widetilde{\widetilde{A}}}(t,\sigma(\tau))\left[K(p-q)+qa(\tau)\right]\widetilde{\widetilde{A}}(\tau)\Delta\tau\right\}^{1/p} \\
&\leq \left\{a(t)+\frac{1}{q}\left[K(p-q)+qa(t)\right]\int_{t_0}^{t} e_{\widetilde{\widetilde{A}}}(t,\sigma(\tau))\widetilde{\widetilde{A}}(\tau)\Delta\tau\right\}^{1/p} \\
&= \left\{a(t)+\frac{1}{q}\left[K(p-q)+qa(t)\right],\left[e_{\widetilde{\widetilde{A}}}(t,t_0)-e_{\widetilde{\widetilde{A}}}(t,t)\right]\right\}^{1/p} \\
&= \left\{\frac{1}{q}\left[\left(K(p-q)+qa(t)\right)e_{\widetilde{\widetilde{A}}}(t,t_0)-K(p-q)\right]\right\}^{1/p} \quad \text{for any } K>0,\ t\in\mathbb{T}^{\kappa}.
\end{aligned} \tag{2.52}$$

The proof of Corollary 2.17 is complete. □

Remark 2.18. In Corollary 2.17, letting $w(t,s)=w(s)$, $p=q=1$, we immediately obtain [12, Theorem 3.1].

From the proof procedure of Theorem 2.11, we can obtain the following result.

Theorem 2.19. *Assume that $u, a, b, f, g_i, m \in C_{\mathrm{rd}}$, $u(t), a(t), b(t), f(t), g_i(t)$, and $m(t)$ are nonnegative, $i=1,2,\ldots,n$, and there exists a series of positive real numbers $q_1, q_2, \ldots, q_n$ such that $p \geq q_i > 0$, $i=1,2,\ldots,n$. Let $w(t,s)$ be defined as in Lemma 2.2 such that $w(t,s)\geq 0$ and $w_1^{\Delta}(t,s)\geq 0$ for $t,s\in\mathbb{T}$ with $s\leq t$. If, for any $\varepsilon>0$, there exists a neighborhood U of t, independent of $\tau\in[t_0,\sigma(t)]$, such that for all $s\in U$,*

$$\left|\left[w(\sigma(t),\tau)-w(s,\tau)-w_1^{\Delta}(t,\tau)(\sigma(t)-s)\right]\left[f(\tau)u^{p}(\tau)+\sum_{i=1}^{n} g_i(\tau)u^{q_i}(\tau)+m(\tau)\right]\right| \leq \varepsilon|\sigma(t)-s|, \tag{2.53}$$

then

$$u^p(t) \leq a(t) + b(t)\int_{t_0}^{t} w(t,\tau)\left[f(\tau)u^p(\tau) + \sum_{i=1}^{n} g_i(\tau)u^{q_i}(\tau) + m(\tau)\right]\Delta\tau, \quad t \in \mathbb{T}^\kappa \qquad (E''2)$$

implies

$$u(t) \leq \left\{a(t) + b(t)\int_{t_0}^{t} e_{A^*}(t,\sigma(\tau))G^*(\tau)\Delta(\tau)\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^\kappa, \qquad (2.54)$$

where

$$\begin{aligned} A^*(t) &= w(\sigma(t),t)b(t)\left(f(t) + \sum_{i=1}^{n}\frac{q_i g_i(t)}{pK^{(p-q_i)/p}}\right) \\ &\quad + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)b(\tau)\left(f(\tau) + \sum_{i=1}^{n}\frac{q_i g_i(\tau)}{pK^{(p-q_i)/p}}\right)\Delta\tau, \end{aligned} \qquad (2.55)$$

$$\begin{aligned} G^*(t) &= w(\sigma(t),t)\left[a(t)f(t) + \sum_{i=1}^{n} g_i(t)\left(\frac{(p-q_i)K^{q_i/p}}{p} + \frac{q_i a(t)}{pK^{(p-q_i)/p}}\right) + m(t)\right] \\ &\quad + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\left[a(\tau)f(\tau) + \sum_{i=1}^{n} g_i(\tau)\left(\frac{(p-q_i)K^{q_i/p}}{p} + \frac{q_i a(\tau)}{pK^{(p-q_i)/p}}\right) + m(\tau)\right]\Delta\tau. \end{aligned} \qquad (2.56)$$

Remark 2.20. Using our main results, we can obtain many dynamic inequalities for some peculiar time scales. Due to limited space, their statements are omitted here.

3. An Application

In this section, we present an application of Corollary 2.6 to obtain the explicit estimates on the solutions of a dynamic equation on time scales.

Example 3.1. Consider the dynamic equation

$$(u^p(t))^{\Delta} = H(t,u(t)), \quad u(t_0) = C,\ t \in \mathbb{T}^\kappa, \qquad (3.1)$$

where p and C are constants, $p > 0$, and $H : \mathbb{T}^\kappa \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Assume that

$$|H(t,u(t))| \leq g(t)|u^q(t)|, \qquad (3.2)$$

where $g(t) \in C_{rd}$, $g(t)$ is nonnegative, and $0 < q \le p$ is a constant. If $u(t)$ is a solution of (3.1), then

$$|u(t)| \le \left\{ |C|^p + \int_{t_0}^{t} e_{\widehat{B}}(t, \sigma(\tau)) J(\tau) \Delta\tau \right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}, \tag{3.3}$$

where $\widehat{B}(t)$ is defined as in (2.21), and

$$J(t) = g(t) \left(\frac{p-q}{p} K^{q/p} + \frac{q|C|^p}{pK^{(p-q)/p}} \right) \quad \text{for all } t \in \mathbb{T}^{\kappa}. \tag{3.4}$$

In fact, the solution $u(t)$ of (3.1) satisfies the following equivalent equation:

$$u^p(t) = C^p + \int_{t_0}^{t} H(\tau, u(\tau)) \Delta\tau, \quad t \in \mathbb{T}^{\kappa}. \tag{3.5}$$

Using the assumption (3.2), we have

$$|u(t)|^p \le |C|^p + \int_{t_0}^{t} g(\tau) |u(\tau)|^q \Delta\tau, \quad t \in \mathbb{T}^{\kappa}. \tag{3.6}$$

Now a suitable application of Corollary 2.6 to (3.6) yields (3.3).

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