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Research Article

On the Behavior of a System of Rational Difference Equations $x_{n+1} = x_{n-1}/(y_n x_{n-1} - 1)$, $y_{n+1} = y_{n-1}/(x_n y_{n-1} - 1)$, $z_{n+1} = 1/x_n z_{n-1}$

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We are concerned with a three-dimensional system of rational difference equations with nonzero initial values. We present solutions of the system in an explicit way and obtain the asymptotical behavior of solutions.

1. Introduction

Difference equations, also referred to recursive sequence, is a hot topic. There has been an increasing interest in the study of qualitative analysis of difference equations and systems of difference equations. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economics, physics, computer sciences, and so on. Especially, Gu and Ding [1] have considered the state space models described by difference equations.

Particularly, there is a class of nonlinear difference equations, known as rational difference equations or fractional difference equations. A lot of work has been concentrated on it [2–12]. There is one way to study rational difference equations—giving the exact expression of solutions [4, 5]. Another way is studying the qualitative behavior such as asymptotical stability using the linearized method, semicycle analysis, and so on [2].

At the same time, more and more attention is paid to systems of rational difference equations composed by two or three rational difference equations [3, 6–12]. The single equation is simple, but the coupled ways of systems are various and thus such systems have no fixed ways to follow to investigate their behavior.

In [4, 5], Çinar has obtained the solutions of the following difference equations:

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \qquad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}. \tag{1.1}$$

In [6], Çinar has proved the periodicity of positive solutions of the following difference equation system:

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}. \tag{1.2}$$

In [7], Stevic has investigated the following system of difference equations:

$$x_{n+1} = \frac{a x_{n-1}}{b y_n x_{n-1} + c}, \qquad y_{n+1} = \frac{\alpha y_{n-1}}{\beta x_n y_{n-1} + \gamma}. \tag{1.3}$$

In fact, such a general system has no explicit solutions and the author has classified the parameters to give explicit solutions for 14 special cases.

In [8], Kurbanli et al. have studied the behavior of positive solutions of the system of the following rational difference equations:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}. \tag{1.4}$$

Based on it, other three-dimensional systems have been investigated in [9], [10], and [11], respectively,

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1}; \tag{1.5}$$

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{1}{y_n z_n}; \tag{1.6}$$

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{x_n}{y_n z_{n-1}}. \tag{1.7}$$

In [12], we improved the results on (1.5) of those in [9] and also investigated the system

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{z_{n-1}}{x_n z_{n-1} - 1}. \tag{1.8}$$

Some other results would be presented in [3].

In this paper, motivated by the above references and the references cited therein, we consider the following system:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{1}{x_n z_{n-1}}, \tag{1.9}$$

where the initial conditions are nonzero real numbers.

In next section, we express solutions of the system (1.9) and try to describe the behavior of solutions.

2. Main Results

Through the paper, we suppose the initial values to be

$$y_0 = a, \qquad x_0 = c, \qquad y_{-1} = b, \qquad x_{-1} = d, \qquad z_0 = e, \qquad z_{-1} = f. \tag{2.1}$$

Here, a, b, c, d, e, and f are real numbers such that $(ad - 1)(cb - 1) \neq 0$, $cdef \neq 0$. We call this to be the hypothesis H.

Theorem 2.1. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system* (1.9). *Then all solutions of* (1.9) *are*

$$x_n = \begin{cases} \dfrac{d}{(ad-1)^k}, & n = 2k-1, \\ c(cb-1)^k, & n = 2k, \end{cases} \qquad k = 1, 2, \dots, \tag{2.2}$$

$$y_n = \begin{cases} \dfrac{b}{(cb-1)^k}, & n = 2k-1, \\ a(ad-1)^k, & n = 2k, \end{cases} \qquad k = 1, 2, \dots, \tag{2.3}$$

$$z_n = \begin{cases} \dfrac{1}{cf(cb-1)^{k-1}}, & n = 4(k-1)+1, \\ \dfrac{(ad-1)^k}{de}, & n = 4(k-1)+2, \\ \dfrac{f}{(cb-1)^k}, & n = 4(k-1)+3, \\ e(ad-1)^k, & n = 4(k-1)+4. \end{cases} \qquad k = 1, 2, \dots. \tag{2.4}$$

Proof. It is obvious to obtain (2.2) and (2.3) and referred to [8]. Here, we only focus on (2.4).

First, for $k = 1$, from (1.9) and (2.2), we easily check that

$$
\begin{aligned}
z_1 &= \frac{1}{x_0 z_{-1}} = \frac{1}{cf}, \\
z_2 &= \frac{1}{x_1 z_0} = \frac{1}{(d/(ad-1))e} = \frac{ad-1}{de}, \\
z_3 &= \frac{1}{x_2 z_1} = \frac{f}{cb-1}, \\
z_4 &= \frac{1}{x_3 z_2} = e(ad-1).
\end{aligned}
\tag{2.5}
$$

Next, we assume the conclusion is true for k, that is, (2.4) holds.

Then, for $k+1$, we confirm it. In fact, from (1.9), (2.2), and (2.4), we have the following:

$$
\begin{aligned}
z_{4k+1} &= \frac{1}{x_{4k} z_{4(k-1)+3}} = \frac{1}{c(cb-1)^{2k} \times \left(f/(cb-1)^{k}\right)} = \frac{1}{cf(cb-1)^{k}}, \\
z_{4k+2} &= \frac{1}{x_{4k+1} z_{4k}} = \frac{1}{\left(d/(ad-1)^{2k+1}\right) \times e(ad-1)^{k}} = \frac{(ad-1)^{k+1}}{de}, \\
z_{4k+3} &= \frac{1}{x_{4k+2} z_{4k+1}} = \frac{1}{c(cb-1)^{2k+1} \times \left(1/cf(cb-1)^{k}\right)} = \frac{f}{(cb-1)^{k+1}}, \\
z_{4k+4} &= \frac{1}{x_{4k+3} z_{4k+2}} = \frac{1}{\left(d/(ad-1)^{2k+2}\right) \times \left((ad-1)^{k+1}/de\right)} = e(ad-1)^{k+1},
\end{aligned}
\tag{2.6}
$$

and complete the proof. □

By Theorem 2.1, the expressions of (2.2), (2.3), and (2.4) will greatly help us to investigate the asymptotical behavior of solutions of (2.4).

Corollary 2.2. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system* (1.9)*. Also, if $ad = cb = 2$, then all solutions of* (1.9) *are four periodic.*

Proof. In this case, from (2.2), (2.3), and (2.4), we have the following:

$$
x_n = \begin{cases} d, & n = 2k-1, \\ c, & n = 2k, \end{cases} \quad k = 1, 2, \ldots
$$

$$y_n = \begin{cases} b, & n = 2k-1, \\ a, & n = 2k, \end{cases} \quad k = 1, 2, \ldots$$

$$z_n = \begin{cases} \dfrac{1}{cf}, & n = 4(k-1)+1, \\ \dfrac{1}{de}, & n = 4(k-1)+2, \\ f, & n = 4(k-1)+3, \\ e, & n = 4(k-1)+4, \end{cases} \quad k = 1, 2, \ldots, \tag{2.7}$$

and complete the proof. □

Corollary 2.3. *Suppose that the hypothesis H holds and let* $\{x_n, y_n, z_n\}$ *be a solution of the system* (1.9). *Also, if* $ad, cb \in (1, 2)$, *and* $c > 0$, *then all solutions of* (1.9) *satisfy*

$$\lim_{n\to\infty} \left(x_{2n-1}, y_{2n-1}, z_{2n-1}\right) = (\infty, \infty, \infty),$$
$$\lim_{n\to\infty} \left(x_{2n}, y_{2n}, z_{2n}\right) = (0, 0, 0). \tag{2.8}$$

Proof. From the hypothesis and $ad, cb \in (1, 2)$, and $d > c$, we obtain that $0 < ad - 1 < 1$, $0 < cb - 1 < 1$ and thus, $(ad-1)^n$ and $(cb-1)^n$ tend to zero as n tends to ∞.

First, from (2.2), we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \cdot \infty = \begin{cases} -\infty, & d < 0, \\ +\infty, & d > 0. \end{cases} \tag{2.9}$$

Similarly, from (2.3), we have

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b \cdot \infty = \begin{cases} -\infty, & b < 0, \\ +\infty, & b > 0. \end{cases} \tag{2.10}$$

As far as z_{2n-1} is concerned, from (2.4) we could consider z_{4k+1} and z_{4k+3} for $n = k+1$, respectively,

$$\lim_{n\to\infty} z_{4k+1} = \lim_{n\to\infty} \frac{1}{cf(cb-1)^k} = \frac{1}{cf} \cdot \infty = \begin{cases} -\infty, & f < 0,\ c > 0 \\ +\infty, & f > 0, \end{cases}$$
$$\lim_{n\to\infty} z_{4k+3} = \lim_{n\to\infty} \frac{f}{(cb-1)^{k+1}} = f \cdot \infty = \begin{cases} -\infty, & f < 0, \\ +\infty, & f > 0. \end{cases} \tag{2.11}$$

Thus,

$$\lim_{n\to\infty} z_{2n-1} = \begin{cases} -\infty, & f < 0, \\ +\infty, & f > 0. \end{cases} \tag{2.12}$$

Therefore,

$$\lim_{n\to\infty} \left(x_{2n-1}, y_{2n-1}, z_{2n-1}\right) = (\infty, \infty, \infty). \tag{2.13}$$

Next, from (2.2) and (2.3), we have

$$\begin{aligned} \lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} c(cb-1)^n = 0, \\ \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} a(ad-1)^n = 0. \end{aligned} \tag{2.14}$$

At last, for z_{2n}, we have

$$\begin{aligned} \lim_{n\to\infty} z_{4k+2} &= \lim_{n\to\infty} \frac{(ad-1)^{k+1}}{de} = 0, \\ \lim_{n\to\infty} z_{4k+4} &= \lim_{n\to\infty} e(ad-1)^{k+1} = 0. \end{aligned} \tag{2.15}$$

Thus,

$$\lim_{n\to\infty} z_{2n} = 0 \tag{2.16}$$

and complete the proof. □

Corollary 2.4. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system* (1.9)*. Also, if a, b, c, $d \in (0,1)$, then all solutions of* (1.9) *satisfy*

$$\begin{aligned} \lim_{n\to\infty} \left(x_{2n-1}, y_{2n-1}, z_{2n-1}\right) &= (\infty, \infty, \infty), \\ \lim_{n\to\infty} \left(x_{2n}, y_{2n}, z_{2n}\right) &= (0,0,0). \end{aligned} \tag{2.17}$$

Proof. From a, b, c, $d \in (0,1)$, we have $-1 < ad-1 < 0$, $-1 < cb-1 < 0$. The remainder is similar to that of Corollary 2.3 and we omit here. □

Corollary 2.5. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system* (1.9)*. Also, if $ad, cb \in (2, +\infty)$, and $d > 0$, then all solutions of* (1.9) *satisfy*

$$\begin{aligned} \lim_{n\to\infty} \left(x_{2n-1}, y_{2n-1}, z_{2n-1}\right) &= (0,0,0), \\ \lim_{n\to\infty} \left(x_{2n}, y_{2n}, z_{2n}\right) &= (\infty, \infty, \infty). \end{aligned} \tag{2.18}$$

Corollary 2.6. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system* (1.9)*. Also, if $ad, cb \in (-\infty, 0)$, and $d > 0$, then all solutions of* (1.9) *satisfy*

$$\lim_{n\to\infty}(x_{2n-1}, y_{2n-1}, z_{2n-1}) = (0,0,0),$$
$$\lim_{n\to\infty}(x_{2n}, y_{2n}, z_{2n}) = (\infty,\infty,\infty). \tag{2.19}$$

The above theorems describe the asymptotical behavior of solutions in case of the initial values lying in different intervals. At last, we describe the behavior in another way.

Corollary 2.7. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system* (1.9)*. If one of the following holds:*

(1) $1 < ad < cb$;

(2) $cb < ad < 1$;

(3) $ad < 1 < cb$ *and* $ad + cb > 2$;

(4) $cb < 1 < ad$ *and* $ad + cb < 2$,

then all solutions of (1.9) *satisfy*

$$\lim_{n\to\infty} x_{2n}y_{2n-1} = cb,$$
$$\lim_{n\to\infty} x_{2n-1}y_{2n} = ad, \tag{2.20}$$
$$\lim_{n\to\infty} z_{2n-1}z_{2n} = 0.$$

Proof. In view of (2.2), (2.3), and (2.4), we have

$$\lim_{n\to\infty} x_{2n}y_{2n-1} = \lim_{n\to\infty}\left(c(cb-1)^n \times \frac{b}{(cb-1)^n}\right) = cb,$$
$$\lim_{n\to\infty} x_{2n-1}y_{2n} = \lim_{n\to\infty}\left(\frac{d}{(ad-1)^n} \times a(ad-1)^n\right) = ad. \tag{2.21}$$

As far as z_{2n-1} and z_{2n} are concerned, from (2.4) we could consider z_{4k+1} and z_{4k+2}, z_{4k+3} and z_{4k+4} for $n = k+1$, respectively. In fact, we have

$$z_{4k+1}z_{4k+2} = \frac{1}{cf(cb-1)^k} \times \frac{(ad-1)^{k+1}}{de} = \frac{ad-1}{cdef}\left(\frac{ad-1}{cb-1}\right)^k,$$
$$z_{4k+3}z_{4k+4} = \frac{f}{(cb-1)^{k+1}} \times e(ad-1)^{k+1} = ef\left(\frac{ad-1}{cb-1}\right)^{k+1}. \tag{2.22}$$

If one of the four conditions holds, we obtain $|(ad-1)/(cb-1)| < 1$ and the conclusion is apparent. □

Corollary 2.8. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system* (1.9). *If one of the following holds:*

(1) $1 < cb < ad$;

(2) $ad < cb < 1$;

(3) $ad < 1 < cb$ *and* $ad + cb < 2$;

(4) $cb < 1 < ad$ *and* $ad + cb > 2$

and $(ad - 1)/cd > 0$, *then all solutions of* (1.9) *satisfy*

$$\lim_{n\to\infty} x_{2n}y_{2n-1} = cb,$$
$$\lim_{n\to\infty} x_{2n-1}y_{2n} = ad, \tag{2.23}$$
$$\lim_{n\to\infty} z_{2n-1}z_{2n} = \infty.$$

The proof is omitted here. In fact, we could obtain $|(ad - 1)/(cb - 1)| > 1$ if one of the four conditions holds and the condition of $(ad - 1)/cd > 0$ is to keep the sign.

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