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Research Article

Fine Spectra of Upper Triangular Double-Band Matrices over the Sequence Space ℓ_p, $(1 < p < \infty)$

Ali Karaisa

Department of Mathematics, Faculty of Sciences, Konya Necmettin Erbakan University, Karacian Mahallesi, Ankara Caddesi 74, 42060 Konya, Turkey

Correspondence should be addressed to Ali Karaisa, alikaraisa@hotmail.com

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The operator $A(\tilde{r},\tilde{s})$ on sequence space on ℓ_p is defined $A(\tilde{r},\tilde{s})x = (r_k x_k + s_k x_{k+1})_{k=0}^{\infty}$, where $x = (x_k) \in \ell_p$, and $\tilde{r}$ and $\tilde{s}$ are two convergent sequences of nonzero real numbers satisfying certain conditions, where $(1 < p < \infty)$. The main purpose of this paper is to determine the fine spectrum with respect to the Goldberg's classification of the operator $A(\tilde{r},\tilde{s})$ defined by a double sequential band matrix over the sequence space ℓ_p. Additionally, we give the approximate point spectrum, defect spectrum, and compression spectrum of the matrix operator $A(\tilde{r},\tilde{s})$ over the space ℓ_p.

1. Introduction

Let X and Y be Banach spaces, and let $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, that is,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}. \tag{1.1}$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Given an operator $T \in B(X)$, the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T_\lambda = \lambda I - T \text{ is a bijection}\} \tag{1.2}$$

is called the *resolvent set* of T and its complement with respect to the complex plain

$$\sigma(T) := \mathbb{C} \setminus \rho(T) \tag{1.3}$$

is called the *spectrum* of T. By the closed graph theorem, the inverse operator

$$R(\lambda;T) := (\lambda I - T)^{-1}, \quad (\lambda \in \rho(T)) \tag{1.4}$$

is always bounded and is usually called *resolvent operator* of T at λ.

2. Subdivisions of the Spectrum

In this section, we give the definitions of the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum, and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

2.1. The Point Spectrum, Continuous Spectrum, and Residual Spectrum

The name *resolvent* is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we will be interested in the set of all λ's in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We will also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects. A *regular value* λ of T is a complex number such that T_λ^{-1} exists and bounded and whose domain is dense in X. For our investigation of T, T_λ, and T_λ^{-1}, we need some basic concepts in spectral theory, which are given as follows (see [1, pp. 370-371]):

The *resolvent set* $\rho(T,X)$ of T is the set of all regular values λ of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows.

The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that T_λ^{-1} does not exist. An $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_λ^{-1} exists and is unbounded and the domain of T_λ^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_λ^{-1} exists (and may be bounded or not), but the domain of T_λ^{-1} is not dense in X.

Therefore, these three subspectra form a disjoint subdivisions

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X). \tag{2.1}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem, which we will have to discuss. Indeed, it is well known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite-dimensional case.

2.2. The Approximate Point Spectrum, Defect Spectrum, and Compression Spectrum

In this subsection, following Appell et al. [2], we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum*, and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{\text{ap}}(T, X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - T\} \tag{2.2}$$

the *approximate point spectrum of* T. Moreover, the subspectrum

$$\sigma_{\delta}(T, X) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\} \tag{2.3}$$

is called defect spectrum of T.

The two subspectra given by (2.2) and (2.3) form a (not necessarily disjoint) subdivision

$$\sigma(T, X) = \sigma_{\text{ap}}(T, X) \cup \sigma_{\delta}(T, X) \tag{2.4}$$

of the spectrum. There is another subspectrum

$$\sigma_{\text{co}}(T, X) = \left\{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X\right\}, \tag{2.5}$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{\text{ap}}(T, X) \cup \sigma_{\text{co}}(T, X) \tag{2.6}$$

of the spectrum. Clearly, $\sigma_p(T, X) \subseteq \sigma_{\text{ap}}(T, X)$ and $\sigma_{\text{co}}(T, X) \subseteq \sigma_{\delta}(T, X)$. Moreover, comparing these subspectra with those in (2.1) we note that

$$\begin{gathered} \sigma_r(T, X) = \sigma_{\text{co}}(T, X) \setminus \sigma_p(T, X), \\ \sigma_c(T, X) = \sigma(T, X) \setminus \left[\sigma_p(T, X) \cup \sigma_{\text{co}}(T, X)\right]. \end{gathered} \tag{2.7}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints is also useful.

Proposition 2.1 (see [2, Proposition 1.3, p. 28]). *Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:*

(a) $\sigma(T^*, X^*) = \sigma(T, X)$,

(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{\text{ap}}(T, X)$,

(c) $\sigma_{\text{ap}}(T^*, X^*) = \sigma_\delta(T, X)$,

(d) $\sigma_\delta(T^*, X^*) = \sigma_{\text{ap}}(T, X)$,

(e) $\sigma_p(T^*, X^*) = \sigma_{\text{co}}(T, X)$,

(f) $\sigma_{\text{co}}(T^*, X^*) \supseteq \sigma_p(T, X)$,

(g) $\sigma(T, X) = \sigma_{\text{ap}}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{\text{ap}}(T^*, X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{\text{ap}}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite-dimensional spaces (see [2]).

2.3. Goldberg's Classification of Spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(A) $R(T) = X$,

(B) $R(T) \neq \overline{R(T)} = X$,

(C) $\overline{R(T)} \neq X$,

and

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3. If an operator is in state C_2, for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [3] and Figure 1).

If λ is a complex number such that $T_\lambda = \lambda I - T \in A_1$ or $T_\lambda = \lambda I - T \in B_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $A_2\sigma(T, X) = \emptyset$, $A_3\sigma(T, X)$, $B_2\sigma(T, X)$, $B_3\sigma(T, X)$, $C_1\sigma(T, X)$, $C_2\sigma(T, X)$, and $C_3\sigma(T, X)$. For example, if $T_\lambda = \lambda I - T$ is in a given state, C_2 (say), then we write $\lambda \in C_2\sigma(T, X)$.

By the definitions given above, we can illustrate the subdivisions (2.1) in Table 1.

Observe that the case in the first row and second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, that is, if λ is not an eigenvalue of T, we may always consider the resolvent operator T_λ^{-1} (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I - T$.

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_1}$ of all complex sequences which contains ϕ, the set of all finitely nonzero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write ℓ_∞, c, c_0, and bv for the spaces of all bounded, convergent, null, and bounded variation sequences, which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$ and $\|x\|_{bv} = \sum_{k=0}^{\infty} |x_k - x_{k+1}|$, while ϕ is not a Banach space with respect to any norm, respectively, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Also by ℓ_p, we denote the

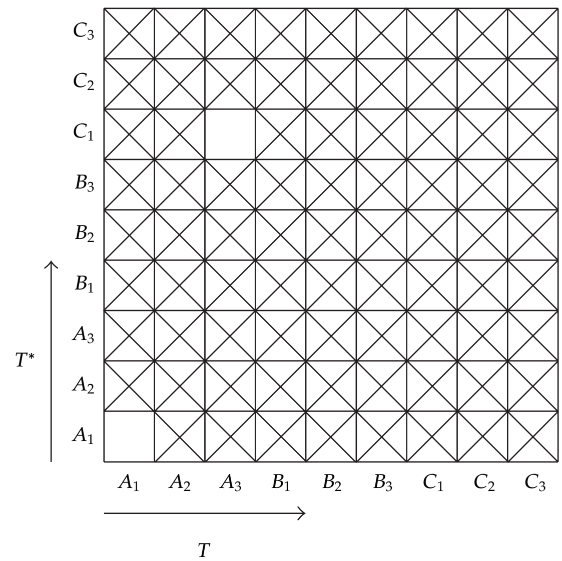

Figure 1: State diagram for $B(X)$ and $B(X^*)$ for a nonreflective Banach space X.

Table 1: Subdivisions of spectrum of a linear operator.

		1 T_λ^{-1} exists and is bounded	2 T_λ^{-1} exists and is unbounded	3 T_λ^{-1} does not exist
A	$R(\lambda I - T) = X$	$\lambda \in \rho(T, X)$	—	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$
B	$\overline{R(\lambda I - T)} = X$	$\lambda \in \rho(T, X)$	$\lambda \in \sigma_c(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$
C	$\overline{R(\lambda I - T)} \neq X$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$

space of all p-absolutely summable sequences, which is a Banach space with the norm $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$, where $1 \leqslant p < \infty$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$, and write

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N},\ x \in D_{00}(A)), \tag{2.8}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x \in w$ for which the sum exists as a finite sum. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞, and we will use the convention that any term with negative subscript is equal to naught. More generally if μ is a normed sequence space, we can write $D_\mu(A)$ for the $x \in w$ for which the sum in (2.8) converges in the norm of μ. We write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\} \tag{2.9}$$

for the space of those matrices which send the whole of the sequence space λ into μ in this sense.

We give a short survey concerning the spectrum and the fine spectrum of the linear operators defined by some particular triangle matrices over certain sequence spaces. The fine spectrum of the Cesàro operator of order one on the sequence space ℓ_p studied by González [19], where $1 < p < \infty$. Also, weighted mean matrices of operators on ℓ_p have been investigated by Cartlidge [20]. The spectrum of the Cesàro operator of order one on the sequence spaces bv_0 and bv investigated by Okutoyi [8, 21]. The spectrum and fine spectrum of the Rhally operators on the sequence spaces c_0, c, ℓ_p, bv, and bv_0 were examined by Yıldırım [9, 22–28]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c was studied by Altay and Başar [12]. The same authors also worked the fine spectrum of the generalized difference operator $B(r,s)$ over c_0 and c, in [29]. The fine spectra of Δ over ℓ_1 and bv studied by Kayaduman and Furkan [30]. Recently, the fine spectra of the difference operator Δ over the sequence spaces ℓ_p and bv_p studied by Akhmedov and Başar [31, 32], where bv_p is the space of p-bounded variation sequences and introduced by Başar and Altay [33] with $1 \leqslant p < \infty$. Also, the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv determined by Furkan et al. [34]. Recently, the fine spectrum of $B(r,s,t)$ over the sequence spaces c_0 and c has been studied by Furkan et al. [35]. Quite recently, de Malafosse [11] and Altay and Başar [12] have, respectively, studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k\in\mathbb{N}}(|x_k|/r^k)$, $(r > 0)$. Altay and Karakuş [36] have determined the fine spectrum of the Zweier matrix, which is a band matrix as an operator over the sequence spaces ℓ_1 and bv. Farés and de Malafosse [37] studied the spectra of the difference operator on the sequence spaces $\ell_p(\alpha)$, where (α_n) denotes the sequence of positive reals and $\ell_p(\alpha)$ is the Banach space of all sequences $x = (x_n)$ normed by $\|x\|_{\ell_p(\alpha)} = [\sum_{n=1}^{\infty}(|x_n|/\alpha_n)^p]^{1/p}$ with $p \geqslant 1$. Also the fine spectrum of the same operator over ℓ_1 and bv has been studied by Bilgiç and Furkan [13]. More recently the fine spectrum of the operator $B(r,s)$ over ℓ_p and bv_p has been studied by Bilgiç and Furkan [38]. In 2010, Srivastava and Kumar [16] have determined the spectra and the fine spectra of generalized difference operator Δ_ν on ℓ_1, where Δ_ν is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_n)$, and they have just generalized these results by the generalized difference operator Δ_{uv} defined by $\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$, (see [18]). Altun [39] has studied the fine spectra of the Toeplitz operators, which are represented by upper and lower triangular n-band infinite matrices, over the sequence spaces c_0 and c. Later, Karakaya and Altun have determined the fine spectra of upper triangular double-band matrices over the sequence spaces c_0 and c, in [40]. Quite recently, Akhmedov and El-Shabrawy [15] have obtained the fine spectrum of the generalized difference operator $\Delta_{a,b}$, defined as a double band matrix with the convergent sequences $\tilde{a} = (a_k)$ and $\tilde{b} = (b_k)$ having certain properties, over the sequence space c. Finally, the fine spectrum with respect to the Goldberg's classification of the operator $B(r,s,t)$ defined by a triple band matrix over the sequence spaces ℓ_p and bv_p

Table 2: Spectrum and fine spectrum of some triangle matrices in certain sequence spaces. In this paper, we study the fine spectrum of the generalized difference operator spectrum of the generalized difference operator defined by an upper double sequential band matrix acting on the sequence spaces ℓ_p with respect to the Goldberg's classification. Additionally, we give the approximate point spectrum, defect spectrum, and compression spectrum of the matrix operator $A(\widetilde{r},\widetilde{s})$ over the spaces ℓ_p. We quote some lemmas, which are needed in proving the theorems given in Section 3.

$\sigma(A,\lambda)$	$\sigma_p(A,\lambda)$	$\sigma_c(A,\lambda)$	$\sigma_r(A,\lambda)$	refer to
$\sigma(C_1^p,c)$	—	—	—	[4]
$\sigma(W,c)$	—	—	—	[5]
$\sigma(C_1,c_0)$	—	—	—	[6]
$\sigma(C_1,c_0)$	$\sigma_p(C_1,c_0)$	$\sigma_c(C_1,c_0)$	$\sigma_r(C_1,c_0)$	[7]
$\sigma(C_1,bv)$	—	—	—	[8]
$\sigma(R,c_0)$	$\sigma_p(R,c_0)$	$\sigma_c(R,c_0)$	$\sigma_r(R,c_0)$	[9]
$\sigma(R,c)$	$\sigma_p(R,c)$	$\sigma_c(R,c)$	$\sigma_r(R,c)$	[9]
$\sigma(C_1^p,c_0)$	—	—	—	[10]
$\sigma(\Delta,s_r)$	—	—	—	[11]
$\sigma(\Delta,c_0)$	—	—	—	[11]
$\sigma(\Delta,c)$	—	—	—	[11]
$\sigma(\Delta^{(1)},c)$	$\sigma_p(\Delta^{(1)},c)$	$\sigma_c(\Delta^{(1)},c)$	$\sigma_r(\Delta^{(1)},c)$	[12]
$\sigma(\Delta^{(1)},c_0)$	$\sigma_p(\Delta^{(1)},c_0)$	$\sigma_c(\Delta^{(1)},c_0)$	$\sigma_r(\Delta^{(1)},c_0)$	[12]
$\sigma(B(r,s),\ell_p)$	$\sigma_p(B(r,s),\ell_p)$	$\sigma_c(B(r,s),\ell_p)$	$\sigma_r(B(r,s),\ell_p)$	[13]
$\sigma(B(r,s),bv_p)$	$\sigma_p(B(r,s),bv_p)$	$\sigma_c(B(r,s),bv_p)$	$\sigma_r(B(r,s),bv_p)$	[13]
$\sigma(B(r,s,t),\ell_p)$	$\sigma_p(B(r,s,t),\ell_p)$	$\sigma_c(B(r,s,t),\ell_p)$	$\sigma_r(B(r,s,t),\ell_p)$	[14]
$\sigma(B(r,s,t),bv_p)$	$\sigma_p(B(r,s,t),bv_p)$	$\sigma_c(B(r,s,t),bv_p)$	$\sigma_r(B(r,s,t),bv_p)$	[14]
$\sigma(\Delta_{a,b},c)$	$\sigma_p(\Delta_{a,b},c)$	$\sigma_c(\Delta_{a,b},c)$	$\sigma_r(\Delta_{a,b},c)$	[15]
$\sigma(\Delta_v,\ell_1)$	$\sigma_p(\Delta_v,\ell_1)$	$\sigma_c(\Delta_v,\ell_1)$	$\sigma_r(\Delta_v,\ell_1)$	[16]
$\sigma(\Delta_{uv}^2,c_0)$	$\sigma_p(\Delta_{uv}^2,c_0)$	$\sigma_c(\Delta_{uv}^2,c_0)$	$\sigma_r(\Delta_{uv}^2,c_0)$	[17]
$\sigma(\Delta_{uv},\ell_1)$	$\sigma_p(\Delta_{uv},\ell_1)$	$\sigma_c(\Delta_{uv},\ell_1)$	$\sigma_r(\Delta_{uv},\ell_1)$	[18]

with $1 < p < \infty$ has recently been studied by Furkan et al. [14]. At this stage, Table 2 may be useful.

Lemma 2.2 (see [41, p. 253, Theorem 34.16]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 2.3 (see [41, p. 245, Theorem 34.3]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the rows of A is bounded.*

Lemma 2.4 (see [41, p. 254, Theorem 34.18]). *Let $1 < p < \infty$ and $A \in (\ell_\infty : \ell_\infty) \cap (\ell_1 : \ell_1)$. Then, $A \in (\ell_p : \ell_p)$.*

Let $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ be sequences whose entries either constants or distinct real numbers satisfying the following conditions:

$$\begin{gathered}\lim_{k\to\infty} r_k = r > 0, \\ \lim_{k\to\infty} s_k = s; \quad |s| = r, \\ \sup_{k\in\mathbb{N}}|r_k| \leqslant r, \qquad s_k^2 \leqslant r_k^2.\end{gathered} \tag{2.10}$$

Then, we define the sequential generalized difference matrix $A(\widetilde{r},\widetilde{s})$ by

$$A(\widetilde{r},\widetilde{s}) = \begin{bmatrix} r_0 & s_0 & 0 & 0 & \cdots \\ 0 & r_1 & s_1 & 0 & \cdots \\ 0 & 0 & r_2 & s_2 & \cdots \\ 0 & 0 & 0 & r_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.11}$$

Therefore, we introduce the operator $A(\widetilde{r},\widetilde{s})$ from ℓ_p to itself by

$$A(\widetilde{r},\widetilde{s})x = (r_k x_k + s_k x_{k+1})_{k=0}^{\infty}, \quad \text{where } x = (x_k) \in \ell_p. \tag{2.12}$$

3. Fine Spectra of Upper Triangular Double-Band Matrices over the Sequence Space ℓ_p

Theorem 3.1. *The operator $A(\widetilde{r},\widetilde{s}) : \ell_p \to \ell_p$ is a bounded linear operator and*

$$\sup_{k\in\mathbb{N}}\left(|r_k|^p + |s_k|^p\right)^{1/p} \leqslant \|A(\widetilde{r},\widetilde{s}\|_{\ell_p} \leqslant \sup_{k\in\mathbb{N}}|r_k| + \sup_{k\in\mathbb{N}}|s_k|. \tag{3.1}$$

Proof. Since the linearity of the operator $A(\widetilde{r},\widetilde{s})$ is not difficult to prove, we omit the detail. Now we prove that (3.1) holds for the operator $A(\widetilde{r},\widetilde{s})$ on the space ℓ_p. It is trivial that $A(\widetilde{r},\widetilde{s})e^{(k)} = (0,0,\ldots,s_{k-1},r_k,0,\ldots,0,\ldots)$ for $e^{(k)} \in \ell_p$. Therefore, we have

$$\|A(\widetilde{r},\widetilde{s})\|_{\ell_p} \geqslant \frac{\left\|A(\widetilde{r},\widetilde{s})e^{(k)}\right\|_{\ell_p}}{\left\|e^{(k)}\right\|_{\ell_p}} = \left(|r_k|^p + |s_{k-1}|^p\right)^{1/p}, \tag{3.2}$$

which implies that

$$\|A(\widetilde{r},\widetilde{s})\|_{\ell_p} \geqslant \sup_{k\in\mathbb{N}}\left(|r_k|^p + |s_k|^p\right)^{1/p}. \tag{3.3}$$

Let $x = (x_k) \in \ell_p$, where $p > 1$. Then, since $(s_k x_{k+1}), (r_k x_k) \in \ell_p$ it is easy to see by Minkowski's inequality that

$$\begin{aligned}
\|A(\tilde{r},\tilde{s})x\|_{\ell_p} &= \left(\sum_{k=0}^{\infty}|s_k x_{k+1} + r_k x_k|^p\right)^{1/p} \\
&\leqslant \left(\sum_{k=0}^{\infty}|s_k x_{k+1}|^p\right)^{1/p} + \left(\sum_{k=0}^{\infty}|r_k x_k|^p\right)^{1/p} \\
&\leqslant \sup_{k\in\mathbb{N}}|r_k|\left(\sum_{k=0}^{\infty}|x_k|^p\right)^{1/p} + \sup_{k\in\mathbb{N}}|s_k|\left(\sum_{k=0}^{\infty}|x_{k+1}|^p\right)^{1/p} \\
&= \sup_{k\in\mathbb{N}}|r_k|\|x\|_{\ell_p} + \sup_{k\in\mathbb{N}}|s_k|\|x\|_{\ell_p} \\
&= \left(\sup_{k\in\mathbb{N}}|r_k| + \sup_{k\in\mathbb{N}}|s_k|\right)\|x\|_{\ell_p},
\end{aligned} \tag{3.4}$$

which leads us to the result that

$$\|A(\tilde{r},\tilde{s})\|_{\ell_p} \leqslant \sup_{k\in\mathbb{N}}|r_k| + \sup_{k\in\mathbb{N}}|s_k|. \tag{3.5}$$

Therefore, by combining the inequalities in (3.3) and (3.5) we have (3.1), as desired. □

Lemma 3.2 (see [42, p. 115, Lemma 3.1]). *Let* $1 < p < \infty$. *If*

$$\alpha \in \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}, \tag{3.6}$$

then the series

$$\sum_{k=1}^{\infty}\left|\frac{(r_{k-1}-\alpha)(r_{k-2}-\alpha)\cdots(r_1-\alpha)(r_0-\alpha)}{s_{k-1}s_{k-2}\cdots s_1 s_0}\right|^p \tag{3.7}$$

is not convergent.

Throughout the paper, by $\mathcal{C}$ and $\mathcal{SD}$, we denote the set of constant sequences and the set of sequences of distinct real numbers, respectively.

Theorem 3.3.

$$\sigma_p(A(\tilde{r},\tilde{s}),\ell_p) = \begin{cases} \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\}, & \tilde{r} \in \mathcal{C}, \\ \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\} \cup \{(r_k)_{k\in\mathbb{N}}\}, & \tilde{r},\tilde{s} \in \mathcal{SD}. \end{cases} \tag{3.8}$$

Proof. Let $A(\tilde{r},\tilde{s})x = \alpha x$ for $\theta \neq x \in \ell_p$ Then, by solving linear equation

$$\begin{aligned} r_0x_0 + s_0x_1 &= \alpha x_0, \\ r_1x_1 + s_1x_2 &= \alpha x_1, \\ r_2x_2 + s_2x_3 &= \alpha x_2, \\ &\vdots \\ r_{k-1}x_{k-1} + s_{k-1}x_k &= \alpha x_k, \\ &\vdots \end{aligned} \tag{3.9}$$

$x_k = ((\alpha - r_k)/s_{k-1})x_{k-1}$ for all $k \geqslant 1$ and

$$x_k = \left[\frac{(r_{k-1}-\alpha)(r_{k-2}-\alpha)\cdots(r_1-\alpha)(r_0-\alpha)}{s_{k-1}s_{k-2}\cdots s_1s_0}\right]x_0. \tag{3.10}$$

Part 1. Assume that $\tilde{r},\tilde{s} \in \mathcal{C}$. Let $r_k = r$ and $s_k = s$ For all $k \in \mathbb{N}$. We observe that $x_k = ((\alpha - r)/s)^k x_0$. This shows that $x \in \ell_p$ if and if only $|\alpha - r| < |s|$, as asserted.

Part 2. Assume that $\tilde{r},\tilde{s} \in \mathcal{SD}$. We must take $x_0 \neq 0$, since $x \neq 0$. It is clear that, for all $k \in \mathbb{N}$, the vector $x = (x_0, x_1, \ldots, x_k, 0, 0, \ldots)$ is an eigenvector of the operator $A(\tilde{r},\tilde{s})$ corresponding to the eigenvalue $\alpha = r_k$, where $x_0 \neq 0$ and $x_n = ((\alpha - r_n)/s_{n-1})x_{n-1}$, for $1 \leqslant n \leqslant k$. Thus $\{r_k : k \in \mathbb{N}\} \subseteq \sigma_p(A(\tilde{r},\tilde{s}),\ell_p)$. If $r_k \neq \alpha$, for all $k \in \mathbb{N}$, then $x_k \neq 0$. If we take $|\alpha - r| < |s|$, since $\lim_{k\to\infty}|x_{k+1}/x_k|^p = \lim_{k\to\infty}|(r_k - \alpha)/s_k|^p = |(r-\alpha)/s|^p < 1$, $x \in \ell_p$. Hence $\{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \subseteq \sigma_p(A(\tilde{r},\tilde{s}),\ell_p)$. Conversely, let $\alpha \in \sigma_p(A(\tilde{r},\tilde{s}),\ell_p)$. Then, there exists $x = (x_0, x_1, x_2, \ldots)$ in ℓ_p and we have $x_k = ((\alpha - r_k)/s_{k-1})x_{k-1}$, for all $k \geqslant 1$. Since $x \in \ell_p$, we can use ratio test. And so $\lim_{k\to\infty}|x_{k+1}/x_k|^p = \lim_{k\to\infty}|(r_k - \alpha)/s_k|^p = |(r-\alpha)/s|^p < 1$ or $\alpha \in \{r_k : k \in \mathbb{C}\}$. If $|\alpha - r| = |s|$, by Lemma 3.2 $x \notin \ell_p$. This completes the proof. □

Theorem 3.4.

$$\sigma_p\left(A(\tilde{r},\tilde{s})^*,\ell_p^*\right) = \begin{cases} \emptyset, & \tilde{r} \in \mathcal{C}, \\ \mathcal{B}, & \tilde{r} \in \mathcal{SD}, \end{cases} \quad \textit{where } \mathcal{B} = \{r_k : k \in \mathbb{N},\ |r - r_k| > |s|\}. \tag{3.11}$$

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r},\widetilde{s}\in\mathcal{C}$. Consider $A(\widetilde{r},\widetilde{s})^*f=\alpha f$ for $f\neq\theta=(0,0,0,\ldots)$ in $\ell_p^*=\ell_q$. Then, by solving the system of linear equations

$$\begin{aligned} r_0f_0&=\alpha f_0,\\ s_0f_0+r_1f_1&=\alpha f_1,\\ s_1f_1+r_2f_2&=\alpha f_2,\\ &\vdots\\ s_{k-1}f_{k-1}+r_kf_k&=\alpha f_k,\\ &\vdots \end{aligned} \tag{3.12}$$

we find that $f_0=0$ if $\alpha\neq r=r_k$ and $f_1=f_2=\cdots=0$ if $f_0=0$, which contradicts $f\neq\theta$. If f_{n_0} is the first nonzero entry of the sequence $f=(f_n)$ and $\alpha=r$, then we get $s_{n_0}f_{n_0}+rf_{n_0+1}=\alpha f_{n_0+1}$ that implies $f_{n_0}=0$, which contradicts the assumption $f_{n_0}\neq 0$. Hence, the equation $A(\widetilde{r},\widetilde{s})^*f=\alpha f$ has no solution $f\neq\theta$.

Part 2. Assume that $\widetilde{r},\widetilde{s}\in\mathcal{SD}$. Then, by solving the equation $A(\widetilde{r},\widetilde{s})^*f=\alpha f$ for $f\neq\theta=(0,0,0,\ldots)$ in ℓ_q, we obtain $(r_0-\alpha)f_0=0$ and $(r_{k+1}-\alpha)f_{k+1}+s_kf_k=0$ for all $k\in\mathbb{N}$. Hence, for all $\alpha\notin\{r_k:k\in\mathbb{N}\}$, we have $f_k=0$ for all $k\in\mathbb{N}$, which contradicts our assumption. So, $\alpha\notin\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_q)$. This shows that $\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_q)\subseteq\{r_k:k\in\mathbb{N}\}\setminus\{r\}$. Now, we prove that

$$\alpha\in\sigma_p\left(A(\widetilde{r},\widetilde{s})^*,\ell_q\right)\quad\text{iff }\alpha\in\mathcal{B}. \tag{3.13}$$

If $\alpha\in\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_q)$, then, by solving the equation $A(\widetilde{r},\widetilde{s})^*f=\alpha f$ for $f\neq\theta=(0,0,0,\ldots)$ in ℓ_q with $\alpha=r_0$,

$$f_k=\frac{s_0s_1s_2\cdots s_{k-1}}{(r_0-r_k)(r_0-r_{k-1})(r_0-r_{k-2})\cdots(r_0-r_1)}f_0\quad\forall k\geqslant 1, \tag{3.14}$$

which can expressed by the recursion relation

$$\left|f_k\right|=\left|\frac{s_0s_1s_2\cdots s_{k-1}}{(r_0-r_1)(r_0-r_2)\cdots(r_0-r_k)}\right|\left|f_0\right|. \tag{3.15}$$

Using ratio test,

$$\lim_{k\to\infty}\left|\frac{f_k}{f_{k-1}}\right|^q=\lim_{k\to\infty}\left|\frac{s_{k-1}}{r_k-r_0}\right|^q=\left|\frac{s}{r-r_0}\right|^q\leqslant 1. \tag{3.16}$$

But $|s/(r-r_0)|\neq 1$. Hence,

$$\alpha=r_0\in\{r_k:k\in\mathbb{N},\ |r_k-r|>|s|\}=\mathcal{B}. \tag{3.17}$$

If we choose $\alpha = r_k \neq r$ for all $k \in \mathbb{N}_1$, then we get $f_0 = f_1 = f_2 = \cdots = f_{k-1} = 0$ and

$$f_{n+1} = \frac{s_n s_{n-1} s_{n-2} \cdots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} f_k \quad \forall n \geqslant k, \tag{3.18}$$

which can expressed by the recursion relation

$$\left| f_{n+1} \right| = \left| \frac{s_{n-1} s_{n-2} s_{n-2} \cdots s_k}{(r_k - r_{n+1})(r_k - r_{n-1})(r_k - r_{n-2}) \cdots (r_k - r_{k+1})} \right| \left| f_k \right|. \tag{3.19}$$

Using ratio test,

$$\lim_{n\to\infty} \left| \frac{f_{n+1}}{f_n} \right|^q = \lim_{n\to\infty} \left| \frac{s_n}{r_{n+1} - r_k} \right|^q = \left| \frac{s}{r - r_k} \right|^q \leqslant 1. \tag{3.20}$$

But $|s/(r - r_k)| \neq 1$. So we have

$$\alpha = r_k \in \{ r_k : k \in \mathbb{N},\ |r_k - r| > |s| \} = \mathcal{B}. \text{ Hence, } \sigma_p\left(A(\widetilde{r}, \widetilde{s})^*, \ell_q \right) \subseteq \mathcal{B}. \tag{3.21}$$

Conversely, let $\alpha \in \mathcal{B}$. Then exist $k \in \mathbb{N}$, $\alpha = r_k \neq r$, and

$$\lim_{n\to\infty} \left| \frac{f_n}{f_{n-1}} \right|^q = \lim_{n\to\infty} \left| \frac{s_n}{r_{n+1} - r_k} \right|^q = \left| \frac{s}{r - r_k} \right|^q < 1. \tag{3.22}$$

That is, $f \in \ell_q$. So we have $\mathcal{B} \subseteq \sigma_p(A(\widetilde{r}, \widetilde{s})^*, \ell_q)$. This completes the proof. □

Lemma 3.5 (see [3, p. 60]). *The adjoint operator T^* of T is onto if and only if T is a bounded operator.*

Theorem 3.6. $\sigma_r(A(\widetilde{r}, \widetilde{s}), \ell_p) = \sigma_p(A(\widetilde{r}, \widetilde{s})^*, \ell_p^*) \setminus \sigma_p(A(\widetilde{r}, \widetilde{s}), \ell_p)$

Proof. The proof is obvious so is omitted. □

Theorem 3.7. *Let $(r_k), (s_k)$ in $\mathcal{SD}$ and $\mathcal{C}$. $\sigma_r(A(\widetilde{r}, \widetilde{s}), \ell_p) = \emptyset$.*

Proof. By Theorems 3.4 and 3.6, $\sigma_r(A(\widetilde{r}, \widetilde{s}), \ell_p) = \emptyset$. □

Theorem 3.8. *Let $\mathcal{A} = \{ \alpha \in \mathbb{C} : |r - \alpha| \leqslant |s| \}$ and $\mathcal{B} = \{ r_k : k \in \mathbb{N}, |r - r_k| > |s| \}$. Then, the set $\mathcal{B}$ is finite and $\sigma(A(\widetilde{r}, \widetilde{s}), \ell_p) = \mathcal{A} \cup \mathcal{B}$.*

Proof. We will show that $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is onto, for $|r - \alpha| > |s|$. Thus, for every $y \in \ell_q$, we find $x \in \ell_q$. $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is triangle so it has an inverse. Also equation $A_\alpha(\widetilde{r}, \widetilde{s})^* x = y$ gives

$[A_\alpha(\widetilde{r},\widetilde{s})^*]^{-1}y = x$. It is sufficient to show that $[A_\alpha(\widetilde{r},\widetilde{s})^*]^{-1} \in (\ell_q : \ell_q)$. We can calculate that $A = (a_{nk}) = [A_\alpha(\widetilde{r},\widetilde{s})^*]^{-1}$ as follows:

$$(a_{nk}) = \begin{bmatrix} \dfrac{1}{r_0-\alpha} & 0 & 0 & \cdots \\ \dfrac{-s_0}{(r_1-\alpha)(r_0-\alpha)} & \dfrac{1}{r_1-\alpha} & 0 & \cdots \\ \dfrac{s_0 s_1}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)} & \dfrac{-s_1}{(r_2-\alpha)(r_1-\alpha)} & \dfrac{1}{r_2-\alpha} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.23}$$

Therefore, the supremum of the ℓ_1 norms of the rows of $[A_\alpha(\widetilde{r},\widetilde{s})^*]^{-1}$ is S_k, where

$$\begin{aligned} S_k = {} & \left|\frac{1}{r_k-\alpha}\right| + \left|\frac{s_{k-1}}{(r_{k-1}-\alpha)(r_k-\alpha)}\right| + \left|\frac{s_{k-1}s_{k-2}}{(r_{k-2}-\alpha)(r_{k-1}-\alpha)(r_k-\alpha)}\right| \\ & + \cdots + \left|\frac{s_0 s_1 \cdots s_{k-1}}{(r_0-\alpha)(r_1-\alpha)\cdots(r_k-\alpha)}\right|. \end{aligned} \tag{3.24}$$

Now, we prove that $(S_k) \in \ell_\infty$. Since $\lim_{k\to\infty}|s_k/(r_k-\alpha)| = |s/(r-\alpha)| = p < 1$, then there exists $k_0 \in \mathbb{N}$ such that $|s_k/(r_k-\alpha)| < p_0$ with $p_0 < 1$, for all $k \geqslant k_0 + 1$,

$$\begin{aligned} S_k = {} & \frac{1}{|r_k-\alpha|}\left[1 + \left|\frac{s_{k-1}}{r_{k-1}-\alpha}\right| + \left|\frac{s_{k-1}s_{k-2}}{(r_{k-1}-\alpha)(r_{k-2}-\alpha)}\right| \right. \\ & \left. + \cdots + \left|\frac{s_{k-1}s_{k-2}\cdots s_{k_0+1}s_{k_0}\cdots s_0}{(r_{k-1}-\alpha)(r_{k-2}-\alpha)\cdots(r_{k_0+1}-\alpha)(r_{k_0}-\alpha)\cdots(r_0-\alpha)}\right|\right] \\ \leqslant {} & \frac{1}{|r_k-\alpha|}\left[1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0} + p_0^{k-k_0}\frac{|s_{k_0-1}|}{|r_{k_0-1}-\alpha|} \right. \\ & \left. + \cdots + p_0^{k-k_0}\left|\frac{s_{k_0-1}s_{k_0-2}\cdots s_0}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)\cdots(r_0-\alpha)}\right|\right]. \end{aligned} \tag{3.25}$$

Therefore,

$$S_k \leqslant \frac{1}{|r_k-\alpha|}\left(1 + p_0 + p_0^2 + \cdots p_0^{k-k_0} + p_0^{k-k_0} Mk_0\right), \tag{3.26}$$

where

$$Mk_0 = 1 + \left|\frac{s_{k_0-1}}{r_{k_0-1}-\alpha}\right| + \left|\frac{s_{k_0-1}s_{k_0-2}}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)}\right| + \cdots + \left|\frac{s_{k_0-1}s_{k_0-2}\cdots s_0}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)\cdots(r_0-\alpha)}\right|. \tag{3.27}$$

Then, $Mk_0 \geqslant 1$ and so

$$S_k \leqslant \frac{Mk_0}{|r_k - \alpha|}\left(1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0}\right). \tag{3.28}$$

But there exist $k_1 \in \mathbb{N}$ and a real number p_1 such that $1/|r_k - \alpha| < p_1$ for all $k \geqslant k_1$. Then, $S_k \leqslant (Mp_1k_0)/(1-p_0)$ for all $k > \max\{k_0, k_1\}$. Hence, $\sup_{k\in\mathbb{N}} S_k < \infty$. This shows that $[A^*(\tilde{r},\tilde{s}) - \alpha I]^{-1} \in (\ell_\infty : \ell_\infty)$. Similarly, we can show that $[(A(\tilde{r},\tilde{s}) - \alpha I)^*]^{-1} \in (\ell_1 : \ell_1)$. By Lemma 2.4, we have

$$\left[(A(\tilde{r},\tilde{s}) - \alpha I)^*\right]^{-1} \in \left(\ell_q : \ell_q\right) \text{ for } \alpha \in \mathbb{C} \text{ with } |r - \alpha| > |s|. \tag{3.29}$$

Hence, $A_\alpha(\tilde{r},\tilde{s})^*$ is onto. By Lemma 3.5, $A_\alpha(\tilde{r},\tilde{s})$ is bounded inverse. This means that

$$\sigma_c\left(A(\tilde{r},\tilde{s}), \ell_p\right) \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \leqslant |s|\}. \tag{3.30}$$

Combining this with Theorem 3.3 and Theorem 3.7, we get

$$\sigma\left(A(\tilde{r},\tilde{s}), \ell_p\right) \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \leqslant |s|\} \cup \mathcal{B} \tag{3.31}$$

and again from Theorem 3.3 $\{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \subseteq \sigma(A(\tilde{r},\tilde{s}), \ell_p)$ and $\mathcal{B} \subseteq \sigma(A(\tilde{r},\tilde{s}), \ell_p)$. Since the spectrum of any bounded operator is closed, we have

$$\{\alpha \in \mathbb{C} : |r - \alpha| \leqslant |s|\} \cup \mathcal{B} \subseteq \sigma\left(A(\tilde{r},\tilde{s}), \ell_p\right). \tag{3.32}$$

Combining (3.31) and (3.32), we get

$$\sigma\left(A(\tilde{r},\tilde{s}), \ell_p\right) = \mathcal{A} \cup \mathcal{B}. \tag{3.33}$$

□

Theorem 3.9. *Let $(r_k), (s_k)$ in $\mathcal{SD}$ or $\mathcal{C}$. $\sigma_c(A(\tilde{r},\tilde{s}), \ell_p) = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.*

Proof. The proof follows of immediately from Theorems 3.3, 3.7, and 3.8 because the parts $\sigma_c(A(\tilde{r},\tilde{s}), \ell_p)$, $\sigma_r(A(\tilde{r},\tilde{s}), \ell_p)$, and $\sigma_p(A(\tilde{r},\tilde{s}), \ell_p)$ are pairwise disjoint sets and union of these sets is $\sigma(A(\tilde{r},\tilde{s}), \ell_p)$. □

Theorem 3.10. *Let $(r_k), (s_k) \in \mathcal{SD}$ and $\mathcal{C}$. If $|\alpha - r| < |s|$, $\alpha \in \sigma(A(\tilde{r},\tilde{s}), \ell_p)A_3$.*

Proof. From Theorem 3.3, $\alpha \in \sigma_p(A(\tilde{r},\tilde{s}), \ell_p)$. Thus, $(A(\tilde{r},\tilde{s}) - \alpha I)^{-1}$ does not exist. It is sufficient to show that the operator $(A(\tilde{r},\tilde{s}) - \alpha I)$ is onto, that is, for given $y = (y_k) \in \ell_p$, we have to find $x = (x_k) \in \ell_p$ such that $(A(\tilde{r},\tilde{s}) - \alpha I)x = y$. Solving the linear equation $(A(\tilde{r},\tilde{s}) - \alpha I)x = y$,

$$[A(\tilde{r},\tilde{s}) - \alpha I]x = \begin{bmatrix} r_0-\alpha & s_0 & 0 & 0 & \cdots \\ 0 & r_1-\alpha & s_1 & 0 & \cdots \\ 0 & 0 & r_2-\alpha & s_2 & \cdots \\ 0 & 0 & 0 & r_3-\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}, \tag{3.34}$$

let

$$\begin{aligned} x_0 &= 0, \\ x_1 &= \frac{y_0}{s_0}, \\ x_2 &= \frac{(\alpha - r_1)y_0}{s_1 s_0} + \frac{y_1}{s_1}, \\ &\vdots \\ x_k &= \frac{(\alpha - r_1)(\alpha - r_2)\cdots(\alpha - r_{k-1})y_0}{s_0 s_1 \cdots s_{k-1}} + \cdots + \frac{(r_{k-2} - \alpha)y_{k-2}}{s_{k-1}s_{k-2}} + \frac{y_{k-1}}{s_{k-1}}. \end{aligned} \tag{3.35}$$

Then, $\sum_k |x_k|^p \leqslant \sup_k (R_k)^p \sum_k |y_k|^p$, where

$$\begin{aligned} R_k &= \left|\frac{1}{s_k}\right| + \left|\frac{(r_{k+1} - \alpha)}{s_k s_{k+1}}\right| + \left|\frac{(r_{k+1} - \alpha)(r_{k+2} - \alpha)}{s_k s_{k+1} s_{k+2}}\right| + \cdots, \\ R_k^n &= \left|\frac{1}{s_k}\right| + \left|\frac{(r_{k+1} - \alpha)}{s_k s_{k+1}}\right| + \left|\frac{(r_{k+1} - \alpha)(r_{k+2} - \alpha)}{s_k s_{k+1} s_{k+2}}\right| + \cdots, \\ &\quad + \left|\frac{(r_{k+1} - \alpha)(r_{k+2} - \alpha)\cdots(r_{k+n} - \alpha)}{s_k s_{k+1} \cdots s_{n+k}}\right| \end{aligned} \tag{3.36}$$

for all $k, n \in \mathbb{N}$. Then, since

$$R^n = \lim_{k\to\infty} R_n^k = \left|\frac{1}{s}\right| + \left|\frac{(r-\alpha)}{s^2}\right| + \left|\frac{(r-\alpha)^2}{s^3}\right| + \cdots + \left|\frac{(r-\alpha)^{n+1}}{s^{n+2}}\right|, \tag{3.37}$$

we have

$$R = \lim_{n\to\infty} R^n = \left|\frac{1}{s}\right| \left(1 + \left|\frac{r-\alpha}{s}\right|^1 + \cdots\right) < \infty. \tag{3.38}$$

Since $|r-\alpha| < |s|$, (R_k) is a convergent sequence of positive real numbers with limit R. Hence, (R_k) bounded and we have $\sup_k (R_k)^p < \infty$. Therefore,

$$\sum_k |x_k|^p \leqslant \sup_k (R_k)^p \sum_k |y_k|^p < \infty. \tag{3.39}$$

This shows that $x = (x_k) \in \ell_p$. Thus $(A(\tilde{r},\tilde{s}) - \alpha I)$ is onto. So we have $\alpha \in \sigma(A(\tilde{r},\tilde{s}),\ell_p)A_3$. □

Theorem 3.11. *Let $(r_k),(s_k) \in \mathcal{C}$ with $r_k = r$, $s_k = s$ for all $k \in \mathbb{N}$. Then, the following statements hold:*

(i) $\sigma_{\text{ap}}(A(\tilde{r},\tilde{s}),\ell_p) = \sigma(A(\tilde{r},\tilde{s}),\ell_p)$,

(ii) $\sigma_{\delta}(A(\tilde{r},\tilde{s}),\ell_p) = \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\}$,

(iii) $\sigma_{\text{co}}(A(\tilde{r},\tilde{s}),\ell_p) = \emptyset$.

Proof. (i) Since from Table 1,

$$\sigma_{\text{ap}}\big(A(\tilde{r},\tilde{s}),\ell_p\big) = \sigma(A(\tilde{r},\tilde{s}),\ell_1) \setminus \sigma\big(A(\tilde{r},\tilde{s}),\ell_p\big)C_1, \tag{3.40}$$

we have by Theorem 3.7

$$\sigma\big(A(\tilde{r},\tilde{s}),\ell_p\big)C_1 = \sigma\big(A(\tilde{r},\tilde{s}),\ell_p\big)C_2 = \emptyset. \tag{3.41}$$

Hence,

$$\sigma_{\text{ap}}\big(A(\tilde{r},\tilde{s}),\ell_p\big) = \mathcal{A}. \tag{3.42}$$

(ii) Since the following equality:

$$\sigma_{\delta}\big(A(\tilde{r},\tilde{s}),\ell_p\big) = \sigma\big(A(\tilde{r},\tilde{s}),\ell_p\big) \setminus \sigma\big(A(\tilde{r},\tilde{s}),\ell_p\big)A_3 \tag{3.43}$$

holds from Table 1, we derive by Theorems 3.8 and 3.10 that $\sigma_{\delta}(A(\tilde{r},\tilde{s}),\ell_p) = \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\}$.

(iii) From Table 1, we have

$$\sigma_{\text{co}}\big(A(\tilde{r},\tilde{s}),\ell_p\big) = \sigma\big(A(\tilde{r},\tilde{s}),\ell_p\big)C_1 \cup \sigma\big(A(\tilde{r},\tilde{s}),\ell_p\big)C_2 \cup \sigma(A(\tilde{r},\tilde{s}),c_0)C_3. \tag{3.44}$$

By Theorem 3.4, it is immediate that $\sigma_{\text{co}}(A(\tilde{r},\tilde{s}),\ell_p) = \emptyset$. □

Theorem 3.12. *Let $(r_k) \in \mathcal{SD}$. Then*

$$\begin{aligned} \sigma_{ap}\big(A(\tilde{r},\tilde{s}),\ell_p\big) &= \mathcal{A} \cup \mathcal{B}, \\ \sigma_{\delta}\big(A(\tilde{r},\tilde{s}),\ell_p\big) &= \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\} \cup \mathcal{B}, \\ \sigma_{co}\big(A(\tilde{r},\tilde{s}),\ell_p\big) &= \mathcal{B}. \end{aligned} \tag{3.45}$$

Proof. We have by Theorem 3.4 and Part (e) of Proposition 2.1 that

$$\sigma_p\left(A(\widetilde{r},\widetilde{s})^*,\ell_p^*\right) = \sigma_{\text{co}}\left(A(\widetilde{r},\widetilde{s}),\ell_p\right) = \mathcal{B}. \tag{3.46}$$

By Theorems 3.7 and 3.4, we must have

$$\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_p\right)C_1 = \sigma\left(A(\widetilde{r},\widetilde{s}),\ell_p\right)C_2 = \emptyset. \tag{3.47}$$

Hence, $\sigma(A(\widetilde{r},\widetilde{s}),\ell_p)C_3 = \{r_k\}$. Additionally, since $\sigma(A(\widetilde{r},\widetilde{s}),\ell_p)C_1 = \emptyset$.

Therefore, we derive from Table 1, Theorems 3.8, and 3.10 that

$$\begin{aligned} \sigma_{\text{ap}}\left(A(\widetilde{r},\widetilde{s}),\ell_p\right) &= \sigma\left(A(\widetilde{r},\widetilde{s}),\ell_p\right) \setminus \sigma\left(A(\widetilde{r},\widetilde{s}),\ell_p\right)C_1 = \sigma(A(\widetilde{r},\widetilde{s}),\ell_1), \\ \sigma_\delta\left(A(\widetilde{r},\widetilde{s}),\ell_p\right) &= \sigma\left(A(\widetilde{r},\widetilde{s}),\ell_p\right) \setminus \sigma\left(A(\widetilde{r},\widetilde{s}),\ell_p\right)A_3 = \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\} \cup \mathcal{B}. \end{aligned} \tag{3.48}$$

□

4. Conclusion

In the present work, as a natural continuation of Akhmedov and El-Shabrawy [15] and Srivastava and Kumar [18], we have determined the spectrum and the fine spectrum of the double sequential band matrix $A(\widetilde{r},\widetilde{s})$ on the space ℓ_p. Many researchers determine the spectrum and fine spectrum of a matrix operator in some sequence spaces. In addition to this, we add the definition of some new divisions of spectrum called as approximate point spectrum, defect spectrum, and compression spectrum of the matrix operator and give the related results for the matrix operator $A(\widetilde{r},\widetilde{s})$ on the space ℓ_p, which is a new development for this type works giving the fine spectrum of a matrix operator on a sequence space with respect to the Goldberg's classification.

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