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Research Article

The Form of the Solutions and Periodicity of Some Systems of Difference Equations

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This paper is devoted to get the form of the solutions and the periodic nature of the following systems of rational difference equations $x_{n+1} = x_{n-5}/(-1 + x_{n-5}y_{n-2})$, $y_{n+1} = y_{n-5}/(\pm 1 \pm y_{n-5}x_{n-2})$, where the initial conditions are real numbers.

1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential equations. They have many applications in biology, ecology, economy, and physics. So, recently, there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions, see [1–23] and the references cited therein.

Periodic solutions of a difference equations have been investigated by many researchers, and various methods have been proposed for the existence and qualitative properties of the solution.

The periodicity of the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}} \tag{1.1}$$

was studied by Çinar in [5].

Elsayed [11] has obtained the solution of the following system of the difference equations:

$$x_{n+1} = \frac{1}{y_{n-k}}, \qquad y_{n+1} = \frac{y_{n-k}}{x_n y_n}. \tag{1.2}$$

The behavior of the positive solution of the following system:

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1} y_n}, \qquad y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1} x_n}. \tag{1.3}$$

has been studied by Kurbanli et al. [22].

Özban [24] has investigated the positive solution of the system of rational difference equations as

$$x_{n+1} = \frac{1}{y_{n-k}}, \qquad y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m-k}}. \tag{1.4}$$

Özban [25] has investigated the solution of the following system:

$$x_{n+1} = \frac{a}{y_{n-3}}, \qquad y_{n+1} = \frac{b y_{n-3}}{x_{n-q} y_{n-q}}. \tag{1.5}$$

In [26] Yalcinkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations:

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \qquad t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}. \tag{1.6}$$

Also, Yalcinkaya [27] has obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations as

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \qquad y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}. \tag{1.7}$$

Yang et al. [28] has investigated the positive solution of the system following:

$$x_n = \frac{a}{y_{n-p}}, \qquad y_n = \frac{b y_{n-p}}{x_{n-q} y_{n-q}}. \tag{1.8}$$

Similar nonlinear systems of rational difference equations were investigated [26–41].

In this paper, we investigate the behavior of the solutions of the difference equations systems as

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-5} y_{n-2}}, \qquad y_{n+1} = \frac{y_{n-5}}{\pm 1 \pm y_{n-5} x_{n-2}}, \tag{1.9}$$

where the initial conditions x_i, y_i for $i = -5, -4, -3, -2, -1, 0$ are real numbers.

2. The First System: $x_{n+1} = x_{n-5}/(-1 + y_{n-2}x_{n-5})$, $y_{n+1} = y_{n-5}/(1 + x_{n-2}y_{n-5})$

In this section, we investigate the solution of the system of two difference equations as

$$x_{n+1} = \frac{x_{n-5}}{-1 + y_{n-2}x_{n-5}}, \qquad y_{n+1} = \frac{y_{n-5}}{1 + x_{n-2}y_{n-5}}, \tag{2.1}$$

where the initial conditions are arbitrary real numbers with $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0 \neq 1, \neq 1/2$, and $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0y_{-3} \neq \pm 1$.

The following theorem is devoted to the form of the solutions of system (2.1).

Theorem 2.1. *Suppose that $\{x_n, y_n\}$ are solutions of system (2.1). Also, assume that the initial conditions $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$ and y_0 are arbitrary real numbers and let $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$, $y_{-5} = s$, $y_{-4} = r$, $y_{-3} = q$, $y_{-2} = p$, $y_{-1} = h$, $y_0 = g$. Then for $n = 0, 1, 2, \ldots$, one has*

$$x_{12n-5} = \frac{(-1)^n f(-1+2pf)^n}{(-1+pf)^{2n}}, \qquad x_{12n-4} = \frac{(-1)^n e(-1+2eh)^n}{(-1+eh)^{2n}},$$

$$x_{12n-3} = \frac{(-1)^n d(-1+2dg)^n}{(-1+dg)^{2n}}, \qquad x_{12n-2} = c(1+sc)^n(1-sc)^n,$$

$$x_{12n-1} = b(1+br)^n(1-br)^n, \qquad x_{12n} = a(1+aq)^n(1-aq)^n,$$

$$x_{12n+1} = \frac{(-1)^n f(-1+2pf)^n}{(-1+pf)^{2n+1}}, \qquad x_{12n+2} = \frac{(-1)^n e(-1+2eh)^n}{(-1+eh)^{2n+1}},$$

$$x_{12n+3} = \frac{(-1)^n d(-1+2dg)^n}{(-1+dg)^{2n+1}}, \qquad x_{12n+4} = -c(1+sc)^{n+1}(1-sc)^n,$$

$$x_{12n+5} = -b(1+br)^{n+1}(1-br)^n, \qquad x_{12n+6} = -a(1+aq)^{n+1}(1-aq)^n,$$

$$y_{12n-5} = \frac{s}{(1+sc)^n(1-sc)^n}, \qquad y_{12n-4} = \frac{r}{(1+br)^n(1-br)^n},$$

$$y_{12n-3} = \frac{q}{(1+aq)^n(1-aq)^n}, \qquad y_{12n-2} = \frac{(-1)^n p(-1+pf)^{2n}}{(-1+2pf)^n},$$

$$y_{12n-1} = \frac{(-1)^n h(-1+eh)^{2n}}{(-1+2eh)^n}, \qquad y_{12n} = \frac{(-1)^n g(-1+dg)^{2n}}{(-1+2dg)^n},$$

$$y_{12n+1} = \frac{s}{(1+sc)^{n+1}(1-sc)^{n}}, \qquad y_{12n+2} = \frac{r}{(1+br)^{n+1}(1-br)^{n}},$$
$$y_{12n+3} = \frac{q}{(1+aq)^{n+1}(1-aq)^{n}}, \qquad y_{12n+4} = \frac{(-1)^{n}p(-1+pf)^{2n+1}}{(-1+2pf)^{n+1}},$$
$$y_{12n+5} = \frac{(-1)^{n}h(-1+eh)^{2n+1}}{(-1+2eh)^{n+1}}, \qquad y_{12n+6} = \frac{(-1)^{n}g(-1+dg)^{2n+1}}{(-1+2dg)^{n+1}}. \tag{2.2}$$

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$, that is,

$$x_{12n-17} = \frac{(-1)^{n-1}f(-1+2pf)^{n-1}}{(-1+pf)^{2n-2}}, \qquad x_{12n-16} = \frac{(-1)^{n-1}e(-1+2eh)^{n-1}}{(-1+eh)^{2n-2}},$$
$$x_{12n-15} = \frac{(-1)^{n-1}d(-1+2dg)^{n-1}}{(-1+dg)^{2n-2}}, \qquad x_{12n-14} = c(1+sc)^{n-1}(1-sc)^{n-1},$$
$$x_{12n-13} = b(1+br)^{n-1}(1-br)^{n-1}, \qquad x_{12n-12} = a(1+aq)^{n-1}(1-aq)^{n-1},$$
$$x_{12n-11} = \frac{(-1)^{n-1}f(-1+2pf)^{n-1}}{(-1+pf)^{2n-1}}, \qquad x_{12n-10} = \frac{(-1)^{n-1}e(-1+2eh)^{n-1}}{(-1+eh)^{2n-1}},$$
$$x_{12n-9} = \frac{(-1)^{n-1}d(-1+2dg)^{n-1}}{(-1+dg)^{2n-1}}, \qquad x_{12n-8} = -c(1+sc)^{n}(1-sc)^{n-1},$$
$$x_{12n-7} = -b(1+br)^{n}(1-br)^{n-1}, \qquad x_{12n-6} = -a(1+aq)^{n}(1-aq)^{n-1},$$
$$y_{12n-17} = \frac{s}{(1+sc)^{n-1}(1-sc)^{n-1}}, \qquad y_{12n-16} = \frac{r}{(1+br)^{n-1}(1-br)^{n-1}}, \tag{2.3}$$
$$y_{12n-15} = \frac{q}{(1+aq)^{n-1}(1-aq)^{n-1}}, \qquad y_{12n-14} = \frac{(-1)^{n-1}p(-1+pf)^{2n-2}}{(-1+2pf)^{n-1}},$$
$$y_{12n-13} = \frac{(-1)^{n-1}h(-1+eh)^{2n-2}}{(-1+2eh)^{n-1}}, \qquad y_{12n-12} = \frac{(-1)^{n-1}g(-1+dg)^{2n-2}}{(-1+2dg)^{n-1}},$$
$$y_{12n-11} = \frac{s}{(1+sc)^{n}(1-sc)^{n-1}}, \qquad y_{12n-10} = \frac{r}{(1+br)^{n}(1-br)^{n-1}},$$
$$y_{12n-9} = \frac{q}{(1+aq)^{n}(1-aq)^{n-1}}, \qquad y_{12n-8} = \frac{(-1)^{n-1}p(-1+pf)^{2n-1}}{(-1+2pf)^{n}},$$
$$y_{12n-7} = \frac{(-1)^{n-1}h(-1+eh)^{2n-1}}{(-1+2eh)^{n}}, \qquad y_{12n-6} = \frac{(-1)^{n-1}g(-1+dg)^{2n-1}}{(-1+2dg)^{n}}.$$

Now, it follows from (2.1) that

$$
\begin{aligned}
x_{12n-5} &= \frac{x_{12n-11}}{-1+y_{12n-8}x_{12n-11}} \\
&= \frac{(-1)^{n-1}f(-1+2pf)^{n-1}/(-1+pf)^{2n-1}}{\left(-1+\left((-1)^{n-1}p(-1+pf)^{2n-1}/(-1+2pf)^{n}\right)\left((-1)^{n-1}f(-1+2pf)^{n-1}/(-1+pf)^{2n-1}\right)\right)} \\
&= \frac{(-1)^{n-1}f(-1+2pf)^{n-1}}{(-1+pf)^{2n-1}(-1+pf/(-1+2pf))}\left(\frac{-1+2pf}{-1+2pf}\right) \\
&= \frac{(-1)^{n-1}f(-1+2pf)^{n-1}(-1+2pf)}{(-1+pf)^{2n-1}(1-2pf+pf)} = \frac{(-1)^{n}f(-1+2pf)^{n}}{(-1+pf)^{2n}}, \\
y_{12n-5} &= \frac{y_{12n-11}}{1+x_{12n-8}y_{12n-11}} \\
&= \frac{s/(1+sc)^{n}(1-sc)^{n-1}}{\left(1-c(1+sc)^{n}(1-sc)^{n-1}\left(s/(1+sc)^{n}(1-sc)^{n-1}\right)\right)} \\
&= \frac{s}{(1+sc)^{n}(1-sc)^{n-1}(1-sc)} = \frac{s}{(1+sc)^{n}(1-sc)^{n}}, \\
x_{12n-4} &= \frac{x_{12n-10}}{-1+y_{12n-7}x_{12n-10}} \\
&= \frac{(-1)^{n-1}e(-1+2eh)^{n-1}/(-1+eh)^{2n-1}}{\left(-1+\left((-1)^{n-1}h(-1+eh)^{2n-1}/(-1+2eh)^{n}\right)\left((-1)^{n-1}e(-1+2eh)^{n-1}/(-1+eh)^{2n-1}\right)\right)} \\
&= \frac{(-1)^{n-1}e(-1+2eh)^{n-1}}{(-1+eh)^{2n-1}(-1+he/(-1+2eh))}\left(\frac{-1+2eh}{-1+2eh}\right) \\
&= \frac{(-1)^{n-1}e(-1+2eh)^{n}}{(-1+eh)^{2n-1}(1-2eh+he)} = \frac{(-1)^{n}e(-1+2eh)^{n}}{(-1+eh)^{2n}}, \\
y_{12n-4} &= \frac{y_{12n-10}}{1+x_{12n-7}y_{12n-10}} = \frac{r/(1+br)^{n}(1-br)^{n-1}}{\left(1-b(1+br)^{n}(1-br)^{n-1}\left(r/(1+br)^{n}(1-br)^{n-1}\right)\right)} \\
&= \frac{r}{(1+br)^{n}(1-br)^{n-1}(1-br)} = \frac{r}{(1+br)^{n}(1-br)^{n}}.
\end{aligned}
\tag{2.4}
$$

Similarly, we can prove the other relations. □

Lemma 2.2. *Let $\{x_n, y_n\}$ be a positive solution of system (2.1), then $\{y_n\}$ is bounded and converges to zero.*

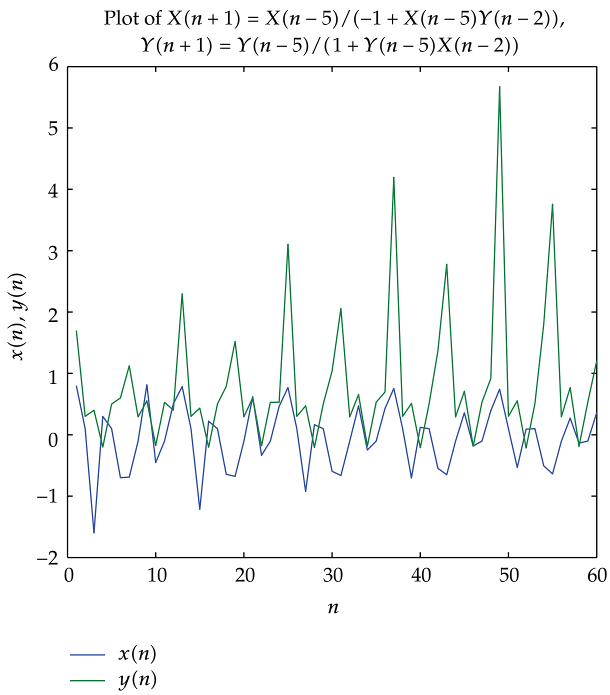

Figure 1

Proof. It follows from (2.1) that

$$y_{n+1} = \frac{y_{n-5}}{1 + x_{n-2}y_{n-5}} \leq y_{n-5}. \tag{2.5}$$

Then, the subsequences $\{y_{6n-5}\}_{n=0}^{\infty}$, $\{y_{6n-4}\}_{n=0}^{\infty}$, $\{y_{6n-3}\}_{n=0}^{\infty}$, $\{y_{6n-2}\}_{n=0}^{\infty}$, $\{y_{6n-1}\}_{n=0}^{\infty}$, and $\{y_{6n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0\}$. □

Example 2.3. We consider interesting numerical example for the difference system (2.1) with the initial conditions, where $x_{-5} = 0.8$, $x_{-4} = 0.1$, $x_{-3} = -1.6$, $x_{-2} = 0.3$, $x_{-1} = 0.1$, $x_0 = -0.7$, $y_{-5} = 1.7$, $y_{-4} = 0.3$, $y_{-3} = 0.4$, $y_{-2} = -0.2$, $y_{-1} = 0.5$, and $y_0 = 0.6$ (see Figure 1).

3. The Second System: $x_{n+1} = x_{n-5}/(-1 + y_{n-2}x_{n-5})$, $y_{n+1} = y_{n-5}/(-1 + x_{n-2}y_{n-5})$

In this section, we study the solution of the following system of the difference equations:

$$x_{n+1} = \frac{x_{n-5}}{-1 + y_{n-2}x_{n-5}}, \qquad y_{n+1} = \frac{y_{n-5}}{-1 + x_{n-2}y_{n-5}}, \tag{3.1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers such that $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0$, $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0y_{-3} \neq 1$.

Theorem 3.1. *Assume that $\{x_n, y_n\}$ are solutions of system* (3.1). *Then for $n = 0, 1, 2, \ldots$, one has*

$$\begin{gathered}
x_{6n-5} = \frac{f}{(-1+pf)^n}, \qquad x_{6n-4} = \frac{e}{(-1+eh)^n}, \qquad x_{6n-3} = \frac{d}{(-1+dg)^n}, \\
x_{6n-2} = c(-1+sc)^n, \qquad x_{6n-1} = b(-1+br)^n, \qquad x_{6n} = a(-1+aq)^n, \\
y_{6n-5} = \frac{s}{(-1+sc)^n}, \qquad y_{6n-4} = \frac{r}{(-1+br)^n}, \qquad y_{6n-3} = \frac{q}{(-1+aq)^n}, \\
y_{6n-2} = p(-1+pf)^n, \qquad y_{6n-1} = h(-1+eh)^n, \qquad y_{6n} = g(-1+dg)^n.
\end{gathered} \tag{3.2}$$

Proof. For $n = 0$, the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$, that is,

$$\begin{gathered}
x_{6n-11} = \frac{f}{(-1+pf)^{n-1}}, \qquad x_{6n-10} = \frac{e}{(-1+eh)^{n-1}}, \qquad x_{6n-9} = \frac{d}{(-1+dg)^{n-1}}, \\
x_{6n-8} = c(-1+sc)^{n-1}, \qquad x_{6n-7} = b(-1+br)^{n-1}, \qquad x_{6n-6} = a(-1+aq)^{n-1}, \\
y_{6n-11} = \frac{s}{(-1+sc)^{n-1}}, \qquad y_{6n-10} = \frac{r}{(-1+br)^{n-1}}, \qquad y_{6n-9} = \frac{q}{(-1+aq)^{n-1}}, \\
y_{6n-8} = p(-1+pf)^{n-1}, \qquad y_{6n-7} = h(-1+eh)^{n-1}, \qquad y_{6n-6} = g(-1+dg)^{n-1}.
\end{gathered} \tag{3.3}$$

Now, it follows from (3.1) that

$$\begin{aligned}
x_{6n-5} &= \frac{x_{6n-11}}{-1 + y_{6n-8}x_{6n-11}} = \frac{f/(-1+pf)^{n-1}}{\left(-1 + p(-1+pf)^{n-1} \times \left(f/(-1+pf)^{n-1}\right)\right)} \\
&= \frac{f/(-1+pf)^{n-1}}{(-1+pf)} = \frac{f}{(-1+pf)^n}, \\
y_{6n-5} &= \frac{y_{6n-11}}{-1 + x_{6n-8}y_{6n-11}} = \frac{\left(s/(-1+sc)^{n-1}\right)}{\left(-1 + c(-1+sc)^{n-1}\left(s/(-1+sc)^{n-1}\right)\right)} \\
&= \frac{s/(-1+sc)^{n-1}}{(-1+cs)} = \frac{s}{(-1+sc)^n}.
\end{aligned} \tag{3.4}$$

Also, we see from (3.1) that

$$\begin{aligned} x_{6n-2} &= \frac{x_{6n-8}}{-1+y_{6n-5}x_{6n-8}} = \frac{c(-1+sc)^{n-1}}{\left(-1+\left(s/(-1+sc)^{n}\right)c(-1+sc)^{n-1}\right)} \\ &= \frac{c(-1+sc)^{n-1}}{(-1+sc/(-1+sc))}\left(\frac{(-1+sc)}{(-1+sc)}\right) = \frac{c(-1+sc)^{n}}{1-sc+sc} = c(-1+sc)^{n}, \\ y_{6n-2} &= \frac{y_{6n-8}}{-1+x_{6n-5}y_{6n-8}} = \frac{p(-1+pf)^{n-1}}{\left(-1+p(-1+pf)^{n-1}(f/(-1+pf)^{n})\right)} \\ &= \frac{p(-1+pf)^{n-1}}{(-1+pf/(-1+pf))}\left(\frac{-1+pf}{-1+pf}\right) = \frac{p(-1+pf)^{n}}{(1-pf+pf)} = p(-1+pf)^{n}. \end{aligned} \tag{3.5}$$

Similarly, we can prove the other relations. □

Lemma 3.2. *The solutions of system* (3.1) *has unboundedness solutions except in the following case.*

Theorem 3.3. *System* (3.1) *has a periodic solution of period six if and only if* $pf = eh = dg = br = aq = sc = 2$ *and it will take the form* $\{x_n\} = \{f, e, d, c, b, a, f, e, \ldots\}$, $\{y_n\} = \{s, r, q, p, h, g, s, r, \ldots\}$.

Proof. First suppose that there exists a prime period-six solution

$$\{x_n\} = \{f, e, d, c, b, a, f, e, \ldots\}, \qquad \{y_n\} = \{s, r, q, p, h, g, s, r, \ldots\}, \tag{3.6}$$

of system (3.1). We see from the form of the solution of system (3.1) that

$$\begin{gathered} f = \frac{f}{(-1+pf)^{n}}, \qquad e = \frac{e}{(-1+eh)^{n}}, \qquad d = \frac{d}{(-1+dg)^{n}}, \\ c = c(-1+sc)^{n}, \qquad b = b(-1+br)^{n}, \qquad a = a(-1+aq)^{n}, \\ s = \frac{s}{(-1+sc)^{n}}, \qquad r = \frac{r}{(-1+br)^{n}}, \qquad q = \frac{q}{(-1+aq)^{n}}, \\ p = p(-1+pf)^{n}, \qquad h = h(-1+eh)^{n}, \qquad g = g(-1+dg)^{n}. \end{gathered} \tag{3.7}$$

Then, we get

$$-1+pf = -1+eh = -1+dg = -1+sc = -1+br = -1+aq = 1. \tag{3.8}$$

Thus,

$$pf = eh = dg = br = aq = sc = 2. \tag{3.9}$$

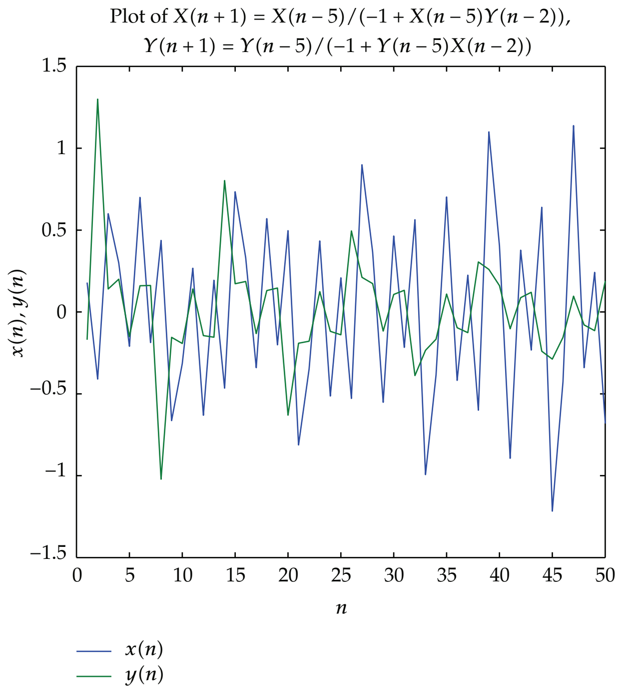

Figure 2

Second, assume that $pf = eh = dg = br = aq = sc = 2$. Then, we see from the form of the solution of system (3.1) that

$$\begin{gathered}
x_{6n-5} = f, \quad x_{6n-4} = e, \quad x_{6n-3} = d, \\
x_{6n-2} = c, \quad x_{6n-1} = b, \quad x_{6n} = a, \\
y_{6n-5} = s, \quad y_{6n-4} = r, \quad y_{6n-3} = q, \\
y_{6n-2} = p, \quad y_{6n-1} = h, \quad y_{6n} = g.
\end{gathered} \tag{3.10}$$

Thus, we have a periodic solution of period six and the proof is complete. □

Example 3.4. Figure 2 shows the behavior of the solution of the difference system (3.1) with the initial conditions, where $x_{-5} = 0.18$, $x_{-4} = -0.41$, $x_{-3} = .6$, $x_{-2} = .3$, $x_{-1} = -0.21$, $x_0 = .7$, $y_{-5} = -0.17$, $y_{-4} = 1.3$, $y_{-3} = .14$, $y_{-2} = 0.2$, $y_{-1} = -.15$, and $y_0 = 0.16$.

Example 3.5. If we consider the difference equation system (3.1) with the initial conditions, where $x_{-5} = 5$, $x_{-4} = -2$, $x_{-3} = .1$, $x_{-2} = -7$, $x_{-1} = 4$, $x_0 = -3$, $y_{-5} = -2/7$, $y_{-4} = 0.5$, $y_{-3} = -2/3$, $y_{-2} = 0.4$, $y_{-1} = -1$, and $y_0 = 20$, then we get the shape of Figure 3.

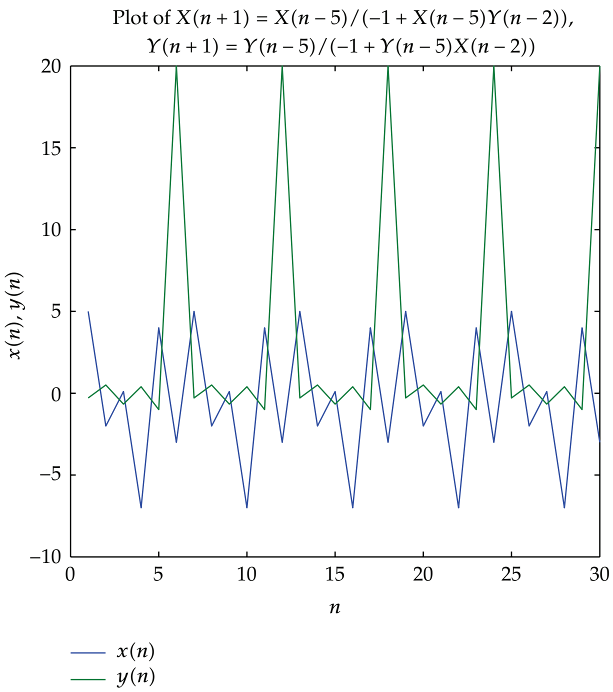

Figure 3

4. The Third System: $x_{n+1} = x_{n-5}/(-1 + y_{n-2}x_{n-5})$, $y_{n+1} = y_{n-5}/(1 - x_{n-2}y_{n-5})$

In this section, we obtain the form of the solution of the system of two difference equations as

$$x_{n+1} = \frac{x_{n-5}}{-1 + y_{n-2}x_{n-5}}, \qquad y_{n+1} = \frac{y_{n-5}}{1 - x_{n-2}y_{n-5}}, \tag{4.1}$$

where the initial conditions are arbitrary real numbers such that $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0 \neq \pm 1$, and $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0y_{-3} \neq 1/2, 1$.

Theorem 4.1. *Suppose that* $\{x_n, y_n\}$ *are solutions of system* (4.1). *Then*

$$x_{12n-5} = \frac{(-1)^n f}{(-1+pf)^n(1+pf)^n}, \qquad x_{12n-4} = \frac{(-1)^n e}{(-1+eh)^n(1+eh)^n},$$

$$x_{12n-3} = \frac{(-1)^n d}{(-1+dg)^n(1+dg)^n}, \qquad x_{12n-2} = \frac{(-1)^n c(1-sc)^{2n}}{(-1+2sc)^n},$$

$$x_{12n-1}=\frac{(-1)^n b(1-br)^{2n}}{(-1+2br)^n},\qquad x_{12n}=\frac{(-1)^n a(1-aq)^{2n}}{(-1+2aq)^n},$$
$$x_{12n+1}=\frac{(-1)^n f}{(-1+pf)^{n+1}(1+pf)^n},\qquad x_{12n+2}=\frac{(-1)^n e}{(-1+eh)^{n+1}(1+eh)^n},$$
$$x_{12n+3}=\frac{(-1)^n d}{(-1+dg)^{n+1}(1+dg)^n},\qquad x_{12n+4}=\frac{(-1)^n c(1-sc)^{2n+1}}{(-1+2sc)^{n+1}},$$
$$x_{12n+5}=\frac{(-1)^n b(1-br)^{2n+1}}{(-1+2br)^{n+1}},\qquad x_{12n+6}=\frac{(-1)^n a(1-aq)^{2n+1}}{(-1+2aq)^{n+1}},$$
$$y_{12n-5}=\frac{(-1)^n s(-1+2sc)^n}{(1-sc)^{2n}},\qquad y_{12n-4}=\frac{(-1)^n r(-1+2br)^n}{(1-br)^{2n}},$$
$$y_{12n-3}=\frac{(-1)^n q(-1+2aq)^n}{(1-aq)^{2n}},\qquad y_{12n-2}=(-1)^n p(-1+pf)^n(1+pf)^n,$$
$$y_{12n-1}=(-1)^n h(-1+eh)^n(1+eh)^n,\qquad y_{12n}=(-1)^n g(-1+dg)^n(1+dg)^n,$$
$$y_{12n+1}=\frac{(-1)^n s(-1+2sc)^n}{(1-sc)^{2n+1}},\qquad y_{12n+2}=\frac{(-1)^n r(-1+2br)^n}{(1-br)^{2n+1}},$$
$$y_{12n+3}=\frac{(-1)^n q(-1+2aq)^n}{(1-aq)^{2n+1}},\qquad y_{12n+4}=(-1)^{n+1} p(-1+pf)^{n+1}(1+pf)^n,$$
$$y_{12n+5}=(-1)^{n+1} h(-1+eh)^{n+1}(1+eh)^n,\qquad y_{12n+6}=(-1)^{n+1} g(-1+dg)^{n+1}(1+dg)^n,\tag{4.2}$$

where $x_{-5}=f$, $x_{-4}=e$, $x_{-3}=d$, $x_{-2}=c$, $x_{-1}=b$, $x_0=a$, $y_{-5}=s$, $y_{-4}=r$, $y_{-3}=q$, $y_{-2}=p$, $y_{-1}=h$, $y_0=g$.

Proof. As the proof of Theorem 2.1, and so it will be omitted. □

Example 4.2. Figure 4 shows the behavior of the solutions of the system (4.1) with the initial conditions $x_{-5}=0.05$, $x_{-4}=-.42$, $x_{-3}=.11$, $x_{-2}=0.07$, $x_{-1}=-0.4$, $x_0=-3$, $y_{-5}=-1.7$, $y_{-4}=0.12$, $y_{-3}=-1.2$, $y_{-2}=0.2$, $y_{-1}=-1$, and $y_0=0.13$.

5. The Fourth System: $x_{n+1}=x_{n-5}/(-1+y_{n-2}x_{n-5})$, $y_{n+1}=y_{n-5}/(-1-x_{n-2}y_{n-5})$

We get, in this section, the solution of the following system of the difference equations:

$$x_{n+1}=\frac{x_{n-5}}{-1+y_{n-2}x_{n-5}},\qquad y_{n+1}=\frac{y_{n-5}}{-1-x_{n-2}y_{n-5}},\tag{5.1}$$

where $n\in\mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

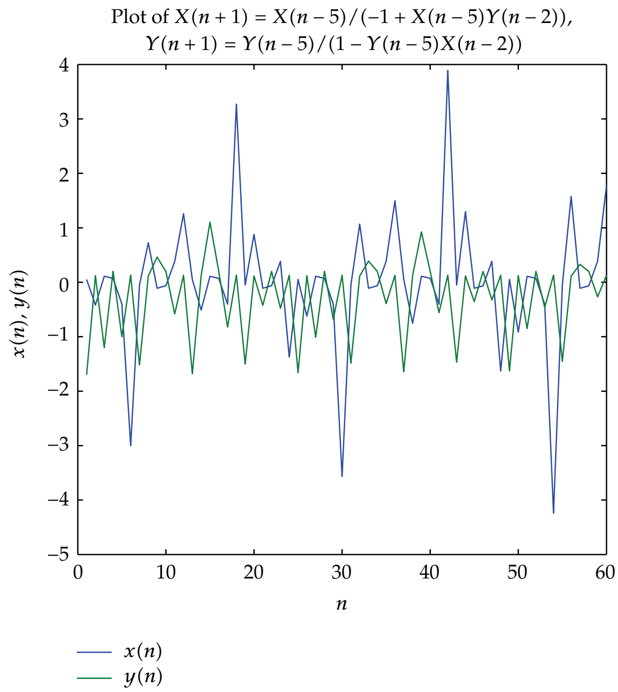

Figure 4

Theorem 5.1. *Let* $\{x_n, y_n\}_{n=-5}^{+\infty}$ *be solutions of system* (5.1)*. Then for* $n = 0, 1, 2, \ldots$ *one has*

$$
\begin{gathered}
x_{6n-5} = f\prod_{i=0}^{n-1}\frac{(1-(2i)pf)}{(-1+(2i+1)pf)}, \qquad x_{6n-4} = e\prod_{i=0}^{n-1}\frac{(1-(2i)eh)}{(-1+(2i+1)eh)}, \\
x_{6n-3} = d\prod_{i=0}^{n-1}\frac{(1-(2i)dg)}{(-1+(2i+1)dg)}, \qquad x_{6n-2} = c\prod_{i=0}^{n-1}\frac{(-1-(2i+1)sc)}{(1+(2i+2)sc)}, \\
x_{6n-1} = b\prod_{i=0}^{n-1}\frac{(-1-(2i+1)br)}{(1+(2i+2)br)}, \qquad x_{6n} = a\prod_{i=0}^{n-1}\frac{(-1-(2i+1)aq)}{(1+(2i+2)aq)}, \\
y_{6n-5} = s\prod_{i=0}^{n-1}\frac{(1+(2i)sc)}{(-1-(2i+1)sc)}, \qquad y_{6n-4} = r\prod_{i=0}^{n-1}\frac{(1+(2i)br)}{(-1-(2i+1)br)}, \\
y_{6n-3} = q\prod_{i=0}^{n-1}\frac{(1+(2i)aq)}{(-1-(2i+1)aq)}, \qquad y_{6n-2} = p\prod_{i=0}^{n-1}\frac{(-1+(2i+1)pf)}{(1-(2i+2)pf)}, \\
y_{6n-1} = h\prod_{i=0}^{n-1}\frac{(-1+(2i+1)eh)}{(1-(2i+2)eh)}, \qquad y_{6n} = g\prod_{i=0}^{n-1}\frac{(-1+(2i+1)dg)}{(1-(2i+2)dg)},
\end{gathered}
\tag{5.2}
$$

where $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$, $y_{-5} = s$, $y_{-4} = r$, $y_{-3} = q$, $y_{-2} = p$, $y_{-1} = h$, $y_0 = g$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$, that is,

$$
\begin{aligned}
x_{6n-11} &= f\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(-1+(2i+1)pf)}, & x_{6n-10} &= e\prod_{i=0}^{n-2}\frac{(1-(2i)eh)}{(-1+(2i+1)eh)},\\
x_{6n-9} &= d\prod_{i=0}^{n-2}\frac{(1-(2i)dg)}{(-1+(2i+1)dg)}, & x_{6n-8} &= c\prod_{i=0}^{n-2}\frac{(-1-(2i+1)sc)}{(1+(2i+2)sc)},\\
x_{6n-7} &= b\prod_{i=0}^{n-2}\frac{(-1-(2i+1)br)}{(1+(2i+2)br)}, & x_{6n-6} &= a\prod_{i=0}^{n-2}\frac{(-1-(2i+1)aq)}{(1+(2i+2)aq)},\\
y_{6n-11} &= s\prod_{i=0}^{n-2}\frac{(1+(2i)sc)}{(-1-(2i+1)sc)}, & y_{6n-10} &= r\prod_{i=0}^{n-2}\frac{(1+(2i)br)}{(-1-(2i+1)br)},\\
y_{6n-9} &= q\prod_{i=0}^{n-2}\frac{(1+(2i)aq)}{(-1-(2i+1)aq)}, & y_{6n-8} &= p\prod_{i=0}^{n-2}\frac{(-1+(2i+1)pf)}{(1-(2i+2)pf)},\\
y_{6n-7} &= h\prod_{i=0}^{n-2}\frac{(-1+(2i+1)eh)}{(1-(2i+2)eh)}, & y_{6n-6} &= g\prod_{i=0}^{n-2}\frac{(-1+(2i+1)dg)}{(1-(2i+2)dg)}.
\end{aligned}
\tag{5.3}
$$

It follows from (3.1) that

$$
\begin{aligned}
&x_{6n-5}\\
&= \frac{x_{6n-11}}{-1+y_{6n-8}x_{6n-11}}\\
&= \frac{f\prod_{i=0}^{n-2}((1-(2i)pf)/(-1+(2i+1)pf))}{\left(-1+p\prod_{i=0}^{n-2}((-1+(2i+1)pf)/(1-(2i+2)pf))f\prod_{i=0}^{n-2}((1-(2i)pf)/(-1+(2i+1)pf))\right)}\\
&= \frac{f\prod_{i=0}^{n-2}((1-(2i)pf)/(-1+(2i+1)pf))}{\left(-1+pf\prod_{i=0}^{n-2}((1-(2i)pf)/(1-(2i+2)pf))\right)} = \frac{f\prod_{i=0}^{n-2}((1-(2i)pf)/(-1+(2i+1)pf))}{(-1+pf/(1-(2n-2)pf))}\\
&= f\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(-1+(2i+1)pf)}\frac{(1-(2n-2)pf)}{(-1+(2n-2)pf+pf)} = f\prod_{i=0}^{n-1}\frac{(1-(2i)pf)}{(-1+(2i+1)pf)},
\end{aligned}
$$

$$
\begin{aligned}
y_{6n-5} &= \frac{y_{6n-11}}{-1 - x_{6n-8}y_{6n-11}} \\
&= \frac{\prod_{i=0}^{n-2}((1+(2i)sc)/(-1-(2i+1)sc))}{\left(-1 - c\prod_{i=0}^{n-2}((-1-(2i+1)sc)/(1+(2i+2)sc))s\prod_{i=0}^{n-2}((1+(2i)sc)/(-1-(2i+1)sc))\right)} \\
&= \frac{\prod_{i=0}^{n-2}((1+(2i)sc)/(-1-(2i+1)sc))}{\left(-1 - sc\prod_{i=0}^{n-2}((1+(2i)sc)/(1+(2i+2)sc))\right)} = \frac{\prod_{i=0}^{n-2}((1+(2i)sc)/(-1-(2i+1)sc))}{(-1-sc/(1+(2n-2)sc))} \\
&= \prod_{i=0}^{n-2}\frac{(1+(2i)sc)}{(-1-(2i+1)sc)}\frac{(1+(2n-2)sc)}{(-1-(2n-2)sc-sc)}.
\end{aligned}
\tag{5.4}
$$

Then, we see that

$$
y_{6n-5} = s\prod_{i=0}^{n-1}\frac{(1+(2i)sc)}{(-1-(2i+1)sc)}. \tag{5.5}
$$

Similarly, we can prove the other relations. This completes the proof. □

Lemma 5.2. *If $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}$, and y_0 are arbitrary real numbers and $\{x_n, y_n\}$ are solutions of system* (5.1)*, then the following statements are true.*

(i) *If $x_{-5} = 0$, $y_{-2} \neq 0$, then we have $x_{6n-5} = 0$ and $y_{6n-2} = (-1)^n y_{-2}$.*

(ii) *If $x_{-4} = 0$, $y_{-1} \neq 0$, then we have $x_{6n-4} = 0$ and $y_{6n-1} = (-1)^n y_{-1}$.*

(iii) *If $x_{-3} = 0$, $y_0 \neq 0$, then we have $x_{6n-3} = 0$ and $y_{6n} = (-1)^n y_0$.*

(iv) *If $x_{-2} = 0$, $y_{-5} \neq 0$, then we have $x_{6n-2} = 0$ and $y_{6n-5} = (-1)^n y_{-5}$.*

(v) *If $x_{-1} = 0$, $y_{-4} \neq 0$, then we have $x_{6n-1} = 0$ and $y_{6n-4} = (-1)^n y_{-4}$.*

(vi) *If $x_0 = 0$, $y_{-3} \neq 0$, then we have $x_{6n} = 0$ and $y_{6n-3} = (-1)^n y_{-3}$.*

(vii) *If $y_{-5} = 0$, $x_{-2} \neq 0$, then we have $y_{6n-5} = 0$ and $x_{6n-2} = (-1)^n x_{-2}$.*

(viii) *If $y_{-4} = 0$, $x_{-1} \neq 0$, then we have $y_{6n-4} = 0$ and $x_{6n-1} = (-1)^n x_{-1}$.*

(ix) *If $y_{-3} = 0$, $x_0 \neq 0$, then we have $y_{6n-3} = 0$ and $x_{6n} = (-1)^n x_0$.*

(x) *If $y_{-2} = 0$, $x_{-5} \neq 0$, then we have $y_{6n-2} = 0$ and $x_{6n-5} = (-1)^n x_{-5}$.*

(xi) *If $y_{-1} = 0$, $x_{-4} \neq 0$, then we have $y_{6n-1} = 0$ and $x_{6n-4} = (-1)^n x_{-4}$.*

(xii) *If $y_0 = 0$, $x_{-3} \neq 0$, then we have $y_{6n} = 0$ and $x_{6n-3} = (-1)^n x_{-3}$.*

Proof. The proof follows from the form of the solution of system (5.1). □

Example 5.3. If we take the system of difference equations (5.1) with the initial conditions $x_{-5} = 0.05$, $x_{-4} = -0.42$, $x_{-3} = .101$, $x_{-2} = 0.07$, $x_{-1} = -0.4$, $x_0 = -3$, $y_{-5} = -0.7$, $y_{-4} = .12$, $y_{-3} = -1.2$, $y_{-2} = 0.2$, $y_{-1} = -0.11$, and $y_0 = 0.13$, we get the following shape of the solution, see Figure 5.

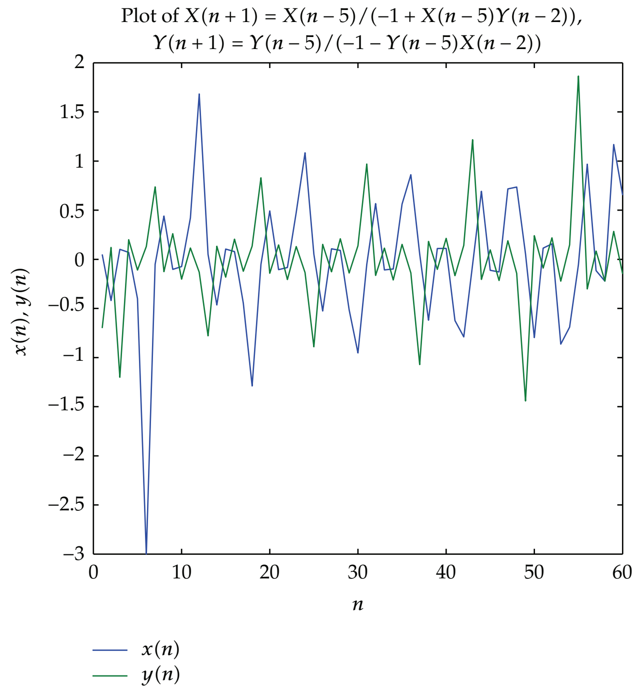

Figure 5

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