Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 735623, 10 pages
doi:10.1155/2012/735623

Research Article

Limit 2-Cycles for a Discrete-Time Bang-Bang Control Model

Chengmin Hou[1] and Sui Sun Cheng[2]

[1] *Department of Mathematics, Yanbian University, Yanji 133002, China*
[2] *Department of Mathematics, Tsing Hua University, Taiwan 30043, Taiwan*

Correspondence should be addressed to Sui Sun Cheng, sscheng@math.nthu.edu.tw

Received 3 August 2012; Revised 19 September 2012; Accepted 24 September 2012

Academic Editor: Raghib Abu-Saris

A discrete-time periodic model with bang-bang feedback control is investigated. It is shown that each solution tends to one of four different types of limit 2-cycles. Furthermore, the accompanying initial regions for each type of solutions can be determined. When a threshold parameter is introduced in the bang-bang function, our results form a complete bifurcation analysis of our control model. Hence, our model can be used in the design of a control system where the state variable fluctuates between two state values with decaying perturbation.

1. Introduction

Discrete-time control systems of the form

$$\mathbf{x}_n = A_n \mathbf{x}_{n-1} + G(n, \mathbf{u}_{n-1}), \tag{1.1}$$

with $\mathbf{x}_n \in \mathbf{R}^n$ and $\mathbf{u}_n \in \mathbf{R}^m$, are of great importance in engineering (see, e.g., any text books on discrete-time signals and systems).

Indeed, such a system consists of a linear part which is easily produced by design and a nonlinear part which allows nonlinear feedback controls of the form

$$\mathbf{u}_n = \mathbf{Q}(\mathbf{x}_n), \tag{1.2}$$

commonly seen in engineering designs.

In a commonly seen situation, $\mathbf{x}_n$ and $\mathbf{u}_n$ belong to $\mathbf{R}^1$, while $\mathbf{u}_n$ takes on two fixed values (on-off values) depending on whether the state variable is above or below a certain value (as commonly seen in thermostat control). In some cases, it is desirable to see that the state value x_n fluctuates between two fixed values with decaying perturbations as time goes by (an example will be provided at the end of this paper). Here, the important question is whether we can design such a control system that fulfils our objectives.

In this note, we will show that a very simple feedback system of the form

$$x_n = a_n x_{n-2} + b_n f_\lambda(x_{n-1}) + d_n, \quad n \in \mathbf{N} = \{0, 1, 2, \ldots\}, \tag{1.3}$$

can achieve such a goal provided that:
(i) we take $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{d_n\}_{n=0}^{\infty}$ to be 2-periodic sequences with a_0, $a_1 \in (0,1)$, $b_0, b_1 \in (0, +\infty)$, $d_0, d_1 \in \mathbf{R}$,
(ii) while the control function f_λ is taken to be the step (activation) or bang bang function [1] defined by

$$f_\lambda(u) = \begin{cases} 1, & \text{if } u \le \lambda, \\ -1, & \text{if } u > \lambda, \end{cases} \tag{1.4}$$

where λ may be regarded as a threshold parameter.
Remarks: (i) Note that in case $\lambda = 0$, our function f_0 is reduced to the well-known Heaviside function

$$H(u) = \begin{cases} 1, & \text{if } u \le 0, \\ -1, & \text{if } u > 0. \end{cases} \tag{1.5}$$

These bang bang controllers are indeed used in daily control mechanisms; for example, a water heater that maintains desired temperature by turning the applied power on and off based on temperature feedback is an example application.
(ii) As for the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$, we have assumed that they are periodic with a prime period ω. We could have considered more general periodic sequences since a large number of environmental parameters are generated in periodic manners, and such structural nature should be reflected in the choice of our sequences. However, in the early stage of our study, it is quite reasonable to assume that they have a common prime period 2 (instead of various prime periods).
(iii) Finally, we have selected $a_0, a_1 \in (0,1)$. A simple reason is that without the feedback control and forcing sequence $\{d_n\}$, our system is a stable one and which can easily be realized in practice.
(iv) Equation (1.3) has a second-order delay in the open loop part and a first-order delay in the control function. It may equally be well to choose a system that has a first-order delay in the open loop part and a second-order delay in the control function. Such a model will be handled in another paper.
(v) The simple prototype studied here is representative of a much wider class of discrete-time periodic systems with piecewise constant feedback controls [2–10], and hence we hope that our results will lead to much more general ones for complex systems involving such discontinuous controls.

Clearly, given any initial state value pair (x_{-2}, x_{-1}) in $\mathbf{R}^2$, we can generate through (1.3) a unique real sequence $\{x_n\}_{n=-2}^{\infty}$. Such a (state) sequence is called a solution of (1.3) originated from (x_{-2}, x_{-1}). What is interesting is that by elementary analysis, we can show that for any value of the threshold parameter λ, there are at most four possible types of limiting behaviors for solutions of (1.3), and we can determine exactly the range of the parameter values and the exact "initial region" from which each type of solutions originates from (see the concluding section for more details).

To this end, we first note that by the transformation $u_n = x_n - \lambda$, (1.3) is equivalent to

$$u_n = a_n u_{n-2} + b_n H(u_{n-1}) + c_n, \qquad n \in \mathbf{N}, \tag{1.6}$$

where $c_n = d_n + (a_n - 1)\lambda$. Next, by means of the identification $u_{2n} = y_n$ and $u_{2n+1} = z_n$ for $n \in \{-1, 0, \ldots\}$, we note further that (1.6) is equivalent to the following two-dimensional *autonomous* dynamical system:

$$y_n = a_0 y_{n-1} + b_0 H(z_{n-1}) + c_0, \qquad z_n = a_1 z_{n-1} + b_1 H(y_n) + c_1, \quad n \in \mathbf{N}, \tag{1.7}$$

which is a special case of the system (1.1). By such a transformation, we are then considering the subsequences $\{u_{2n}\}$ and $\{u_{2n+1}\}$ consisting of even and odd terms of the solution sequence $\{u_n\}$ of (1.6). Therefore, all the asymptotic properties of (1.6) can be obtained from those of (1.7).

To study the asymptotic properties of (1.7), we first note that its solution is of the form $\{(y_n, z_n)\}_{n=-1}^{\infty}$ where (y_{-1}, z_{-1}) is now a point in the real plane. By considering all possible initial data pairs $(y_{-1}, z_{-1}) \in \mathbf{R}^2$, we will be able to show that every solution of (1.7) tends to one of four vectors. To describe these four vectors, we set

$$\xi_i^{\pm} = \frac{c_i \pm b_i}{1 - a_i}, \quad i = 0, 1. \tag{1.8}$$

Then, the four vectors are

$$(\xi_0^-, \xi_1^-), (\xi_0^+, \xi_1^-), (\xi_0^+, \xi_1^+), (\xi_0^-, \xi_1^+), \tag{1.9}$$

and since $b_0, b_1 > 0$, we see that $\xi_0^- < \xi_0^+$ and $\xi_1^- < \xi_1^+$,and hence they form the corners of a rectangle.

Depending on the relative location of the origin $(0, 0)$ with respect to this rectangle, we may then distinguish eleven *exhaustive* (but *not* mutually distinct; see Section 3 in the following) cases:

(i) $0 > \max\{\xi_0^+, \xi_1^+\}$,

(ii) $0 < \min\{\xi_0^-, \xi_1^-\}$,

(iii) $\min\{\xi_0^+, \xi_1^+\} < 0 < \max\{\xi_0^+, \xi_1^+\}$,

(iv) $\min\{\xi_0^-, \xi_1^-\} < 0 < \max\{\xi_0^-, \xi_1^-\}$;

(v) $\max\{\xi_0^+, \xi_1^+\} = 0$,

(vi) $\min\{\xi_0^-, \xi_1^-\} = 0$;

(vii) $0 = \min\{\xi_0^+, \xi_1^+\} = \xi_0^+ < \xi_1^+$,

(viii) $0 = \min\{\xi_0^+, \xi_1^+\} = \xi_1^+ < \xi_0^+$;

(ix) $0 = \max\{\xi_0^-, \xi_1^-\} = \xi_1^- > \xi_0^-$,

(x) $0 = \max\{\xi_0^-, \xi_1^-\} = \xi_0^- > \xi_1^-$;

(xi) $\max\{\xi_0^-, \xi_1^-\} < 0 < \min\{\xi_0^+, \xi_1^+\}$.

For each case, we intend to show that solutions of (1.7) originated from different parts of the plane will tend to one of the four vectors in (1.9). To facilitate description of the various parts of the plane, we introduce the following notations:

$$A_{i,j}^{\pm} = -\frac{\left(1 - a_i^j\right)}{a_i^j}\xi_i^{\pm}, \quad j \in \mathbf{N}, \ i = 0, 1,$$
$$R^+ = (0, +\infty), \qquad R^- = (-\infty, 0]. \tag{1.10}$$

2. Main Results

Cases (i), (ii), (iii), and (iv)

First of all, the first four cases $0 > \max\{\xi_0^+, \xi_1^+\}$, $0 < \min\{\xi_0^-, \xi_1^-\}$, $\min\{\xi_0^+, \xi_1^+\} < 0 < \max\{\xi_0^+, \xi_1^+\}$, and $\min\{\xi_0^-, \xi_1^-\} < 0 < \max\{\xi_0^-, \xi_1^-\}$ are relatively easy. Indeed, suppose $0 > \max\{\xi_0^+, \xi_1^+\}$. Let $(y, z) = \{(y_n, z_n)\}_{n=-1}^{\infty}$ be a solution of (1.7). By (1.7), $y_n \le a_0 y_{n-1} + b_0 + c_0$, $z_n \le a_1 z_{n-1} + b_1 + c_1$. Then,

$$\limsup_n y_n \le \frac{(b_0 + c_0)}{(1 - a_0)} = \xi_0^+ < 0, \tag{2.1}$$

$$\limsup_n z_n \le \frac{(b_1 + c_1)}{(1 - a_1)} = \xi_1^+ < 0. \tag{2.2}$$

Therefore, there exists an $m_0 \in \mathbf{N}$ such that $y_n, z_n \in R^-$ for all $n \ge m_0$. By (1.7) again, $y_n = a_0 y_{n-1} + b_0 + c_0$ and $z_n = a_1 z_{n-1} + b_1 + c_1$ for $n > m_0$. Then, $a_0, a_1 \in (0, 1)$ imply

$$\lim_n (y_n, z_n) = \left(\frac{c_0 + b_0}{1 - a_0}, \frac{c_1 + b_1}{1 - a_1}\right) = \left(\xi_0^+, \xi_1^+\right). \tag{2.3}$$

In summary, suppose $\max\{\xi_0^+, \xi_1^+\} < 0$ and suppose $(y_{-1}, z_{-1}) \in \mathbf{R}^2$, then the solution $\{(y_n, z_n)\}$ originated from (y_{-1}, z_{-1}) will tend to (ξ_0^+, ξ_1^+). We record this result as the first data row in Table 1.

By symmetric arguments, the second data row is also correct. To see the validity of the third data row, we first note that $\min\{\xi_0^+, \xi_1^+\} < 0 < \max\{\xi_0^+, \xi_1^+\}$ if and only if $\xi_0^+ < 0 < \xi_1^+$ or $\xi_1^+ < 0 < \xi_0^+$. If $\xi_0^+ < 0 < \xi_1^+$ holds, then by (1.7), $y_n \le a_0 y_{n-1} + b_0 + c_0$ for all $n \in \mathbf{N}$. Hence, $\limsup_n y_n \le (b_0 + c_0)/(1 - a_0) = \xi_0^+ < 0$. Therefore, there exists an $m_0 \in \mathbf{N}$ such that $y_n < 0$ for $n \ge m_0$. Thus, $z_n = a_1 z_{n-1} + b_1 + c_1$ for $n > m_0$. Then $\lim_n z_n = \xi_1^+ > 0$. Therefore, there exists an $m_1 \ge m_0$ such that $z_n > 0$ for all $n > m_1$. Then, by (1.7) again, $y_n = a_0 y_{n-1} - b_0 + c_0$ for all $n > m_1 + 1$, and hence $\lim_n y_n = \xi_0^-$. The case where $\xi_1^+ < 0 < \xi_0^+$ is similarly proved. Finally, the fourth data row is established by arguments symmetric to those for the third row.

Case (v)

Next, we assume that $0 = \max\{\xi_0^+, \xi_1^+\}$. Since $a_i \in (0,1), b_i \in (0,+\infty)$, and $c_i \in \mathbf{R}$ for $i = 0,1$, then $\xi_i^+ = (c_i + b_i)/(1-a_i) > (c_i - b_i)/(1-a_i) = \xi_i^-$ for $i = 0,1$. We see that $\max\{\xi_0^-, \xi_1^-\} < 0$, and $A_{i,0}^- = 0$, $\lim_j A_{i,j}^- = \lim_j(-(1-a_i^j)/(a_i^j))\xi_i^- = +\infty$ for $i = 0,1$. Therefore, $R^+ = \bigcup_{j=0}^{+\infty}(A_{i,j}^-, A_{i,j+1}^-]$ for $i = 0,1$. Furthermore, if $\xi_0^+ < \xi_1^+ = 0$, then $\lim_j A_{0,j}^+ = +\infty, R^+ = \bigcup_{j=0}^{+\infty}(A_{0,j}^+, A_{0,j+1}^+]$, and if $\xi_1^+ < \xi_0^+ = 0$, then $\lim_j A_{1,j}^+ = +\infty, R^+ = \bigcup_{j=0}^{+\infty}(A_{1,j}^+, A_{1,j+1}^+]$. We need to consider three cases: (i) $\xi_0^+ < \xi_1^+ = 0$, (ii) $\xi_1^+ < \xi_0^+ = 0$, and (iii) $\xi_0^+ = \xi_1^+ = 0$. By arguments similar to those used in the derivation of Table 1, we may derive Table 2.

For instance, suppose $\xi_0^+ < \xi_1^+ = 0$. Let $(y_{-1}, z_{-1}) \in R^- \times R^-$. Then, by (1.7), we have $y_0 = a_0 y_{-1} + b_0 + c_0 < a_0 y_{-1} < 0, z_0 = a_1 z_{-1} + b_1 + c_1 = a_1 z_{-1} < 0$, and by induction, we may easily see that $y_n, z_n \in R^-$ for all $n \in \mathbf{N}$. Thus, $y_n = a_0 y_{n-1} + b_0 + c_0$, $z_n = a_1 z_{n-1} + b_1 + c_1$, and hence $\lim_n(y_n, z_n) = (\xi_0^+, \xi_1^+)$. As another example, let $(y_{-1}, z_{-1}) \in R^+ \times R^-$, then $y_{-1} \in (A_{0,k}^+, A_{0,k+1}^+]$ for some $k \in \mathbf{N}$. By (1.7) and induction, we may easily see that $(y_k, z_k) \in R^- \times R^-$. Our conclusion comes from the previous case. As a further example, let $(y_{-1}, z_{-1}) \in (A_{0,k}^-, A_{0,k+1}^-] \times (A_{1,s}^-, A_{1,s+1}^-] \subset R^+ \times R^+$, where $0 \le k \le s$, then by (1.7) and induction, we may easily see that $(y_k, z_k) \in R^- \times R^+$. Our conclusion now follows from the fourth data row.

Case (vi)

This case is a dual of the Case (v). Indeed, assume that $0 = \min\{\xi_0^-, \xi_1^-\}$. Then, $\min\{\xi_0^+, \xi_1^+\} > 0$ and $A_{i,0}^+ = 0$, $\lim_j A_{i,j}^+ = \lim_j(-(1-a_i^j)/(a_i^j))\xi_0^+ = -\infty$ for $i = 0,1$. Thus, $R^- = \bigcup_{j=0}^{+\infty}(A_{i,j+1}^+, A_{i,j}^+]$ for $i = 0,1$. Furthermore, if $0 = \xi_1^- < \xi_0^-$, then $\lim_j A_{0,j}^- = -\infty, R^- = \bigcup_{j=0}^{+\infty}(A_{0,j+1}^-, A_{0,j}^-]$, and if $0 = \xi_0^- < \xi_1^-$, then $\lim_j A_{1,j}^- = -\infty, R^- = \bigcup_{j=0}^{+\infty}(A_{1,j+1}^-, A_{1,j}^-]$. We need to consider three cases: (i) $0 = \xi_1^- < \xi_0^-$, (ii) $0 = \xi_0^- < \xi_1^-$, and (iii) $\xi_0^- = \xi_1^- = 0$. By arguments similar to those in the previous case, we may obtain the asymptotic behaviors of (1.7) summarized in Table 3.

Cases (vii) and (viii)

By arguments similar to those described previously, the corresponding asymptotic behaviors of (1.7) can be summarized in Tables 4 and 5.

Case (ix) and (x)

By arguments similar to those described previously, the corresponding asymptotic behaviors of (1.7) can be summarized in Tables 6 and 7.

Case (xi)

By arguments similar to those described previously, the corresponding asymptotic behaviors of (1.7) can be summarized in Table 8.

3. Remarks

We remark that the different Cases (i)–(xi) discussed above may not be mutually distinct. For instance, the conditions $\min\{\xi_0^+, \xi_1^+\} < 0 < \max\{\xi_0^+, \xi_1^+\}$ and $\min\{\xi_0^-, \xi_1^-\} < 0 < \max\{\xi_0^-, \xi_1^-\}$ are

Table 1

Case	y_{-1}	z_{-1}	Condition	$\lim_n(y_n, z_n)$
$0 > \max\{\xi_0^+, \xi_1^+\}$	$\in \mathbf{R}$	$\in \mathbf{R}$		(ξ_0^+, ξ_1^+)
$0 < \min\{\xi_0^-, \xi_1^-\}$	$\in \mathbf{R}$	$\in \mathbf{R}$		(ξ_0^-, ξ_1^-)
$\min\{\xi_0^+, \xi_1^+\} < 0 < \max\{\xi_0^+, \xi_1^+\}$	$\in \mathbf{R}$	$\in \mathbf{R}$	$\xi_0^+ < 0 < \xi_1^+$	(ξ_0^-, ξ_1^+)
			$\xi_1^+ < 0 < \xi_0^+$	(ξ_0^+, ξ_1^-)
$\min\{\xi_0^-, \xi_1^-\} < 0 < \max\{\xi_0^-, \xi_1^-\}$	$\in \mathbf{R}$	$\in \mathbf{R}$	$\xi_0^- < 0 < \xi_1^-$	(ξ_0^-, ξ_1^+)
			$\xi_1^- < 0 < \xi_0^-$	(ξ_0^+, ξ_1^-)

Table 2: $\max\{\xi_0^+, \xi_1^+\} = 0$.

y_{-1}	z_{-1}	Condition	Condition	$\lim_n(y_n, z_n)$
$\in R^-$	$\in R^-$		$\xi_0^+ < \xi_1^+ = 0$	(ξ_0^+, ξ_1^+)
			$\xi_1^+ < \xi_0^+ = 0$	(ξ_0^+, ξ_1^+)
			$\xi_1^+ = \xi_0^+ = 0$	(ξ_0^+, ξ_1^+)
$\in R^-$	$\in R^+$		$\xi_0^+ < \xi_1^+ = 0$	(ξ_0^-, ξ_1^+)
			$\xi_1^+ < \xi_0^+ = 0$	(ξ_0^+, ξ_1^+)
			$\xi_1^+ = \xi_0^+ = 0$	(ξ_0^-, ξ_1^+)
$\in R^+$	$\in R^-$		$\xi_0^+ < \xi_1^+ = 0$	(ξ_0^+, ξ_1^+)
			$\xi_1^+ < \xi_0^+ = 0$	(ξ_0^+, ξ_1^-)
			$\xi_1^+ = \xi_0^+ = 0$	(ξ_0^+, ξ_1^-)
$\in (A_{0,k}^-, A_{0,k+1}^-] \subset R^+$	$\in (A_{1,s}^-, A_{1,s+1}^-] \subset R^+$	$0 \le k \le s$	$\xi_0^+ < \xi_1^+ = 0$	(ξ_0^-, ξ_1^+)
			$\xi_1^+ < \xi_0^+ = 0$	(ξ_0^+, ξ_1^+)
			$\xi_1^+ = \xi_0^+ = 0$	(ξ_0^-, ξ_1^+)
$\in (A_{0,k}^-, A_{0,k+1}^-] \subset R^+$	$\in (A_{1,s}^-, A_{1,s+1}^-] \subset R^+$	$0 \le s < k$	$\xi_0^+ < \xi_1^+ = 0$	(ξ_0^+, ξ_1^+)
			$\xi_1^+ < \xi_0^+ = 0$	(ξ_0^+, ξ_1^-)
			$\xi_1^+ = \xi_0^+ = 0$	(ξ_0^+, ξ_1^-)

Table 3: $\min\{\xi_0^-, \xi_1^-\} = 0$.

y_{-1}	z_{-1}	Condition	Condition	$\lim_n(y_n, z_n)$
$\in R^+$	$\in R^+$		$0 = \xi_1^- < \xi_0^-$	(ξ_0^-, ξ_1^-)
			$0 = \xi_0^- < \xi_1^-$	(ξ_0^-, ξ_1^-)
			$0 = \xi_1^- = \xi_0^-$	(ξ_0^-, ξ_1^-)
$\in R^+$	$\in R^-$		$0 = \xi_1^- < \xi_0^-$	(ξ_0^+, ξ_1^-)
			$0 = \xi_0^- < \xi_1^-$	(ξ_0^-, ξ_1^-)
			$0 = \xi_1^- = \xi_0^-$	(ξ_0^+, ξ_1^-)
$\in R^-$	$\in R^+$		$0 = \xi_1^- < \xi_0^-$	(ξ_0^-, ξ_1^-)
			$0 = \xi_0^- < \xi_1^-$	(ξ_0^-, ξ_1^+)
			$0 = \xi_1^- = \xi_0^-$	(ξ_0^-, ξ_1^+)
$\in (A_{0,k+1}^+, A_{0,k}^+] \subset R^-$	$\in (A_{1,s+1}^+, A_{1,s}^+] \subset R^-$	$0 \le k \le s$	$0 = \xi_1^- < \xi_0^-$	(ξ_0^+, ξ_1^-)
			$0 = \xi_0^- < \xi_1^-$	(ξ_0^-, ξ_1^-)
			$0 = \xi_1^- = \xi_0^-$	(ξ_0^+, ξ_1^-)
$\in (A_{0,k+1}^-, A_{0,k}^-] \subset R^-$	$\in (A_{1,s+1}^-, A_{1,s}^-] \subset R^-$	$0 \le s < k$	$0 = \xi_1^- < \xi_0^-$	(ξ_0^-, ξ_1^-)
			$0 = \xi_0^- < \xi_1^-$	(ξ_0^-, ξ_1^+)
			$0 = \xi_1^- = \xi_0^-$	(ξ_0^-, ξ_1^+)

Table 4: $0 = \min\{\xi_0^+, \xi_1^+\} = \xi_0^+ < \xi_1^+$.

y_{-1}	z_{-1}	Condition	Condition	$\lim_n(y_n, z_n)$
$\in R^-$	$\in \mathbf{R}$			(ξ_0^-, ξ_1^+)
$\in R^+$	$\in R^-$		$\xi_1^- \leq 0$	(ξ_0^+, ξ_1^-)
			$\xi_1^- > 0$	(ξ_0^-, ξ_1^+)
$\in R^+$	$\in R^+$		$\xi_1^- \geq 0$	(ξ_0^-, ξ_1^+)
$\in(A_{0,k}^-, A_{0,k+1}^-] \subset R^+$	$\in(A_{1,s}^-, A_{1,s+1}^-] \subset R^+$	$0 \leq k \leq s$	$\xi_1^- < 0$	(ξ_0^-, ξ_1^+)
$\in(A_{0,k}^-, A_{0,k+1}^-] \subset R^+$	$\in(A_{1,s}^-, A_{1,s+1}^-] \subset R^+$	$0 \leq s < k$	$\xi_1^- < 0$	(ξ_0^+, ξ_1^-)

Table 5: $0 = \min\{\xi_0^+, \xi_1^+\} = \xi_1^+ < \xi_0^+$.

y_{-1}	z_{-1}	Condition	Condition	$\lim_n(y_n, z_n)$
$\in \mathbf{R}$	$\in R^-$			(ξ_0^+, ξ_1^-)
$\in R^-$	$\in R^+$		$\xi_0^- \leq 0$	(ξ_0^-, ξ_1^+)
			$\xi_0^- > 0$	(ξ_0^+, ξ_1^-)
$\in R^+$	$\in R^+$		$\xi_0^- \geq 0$	(ξ_0^+, ξ_1^-)
$\in(A_{0,k}^-, A_{0,k+1}^-] \subset R^+$	$\in(A_{1,s}^-, A_{1,s+1}^-] \subset R^+$	$0 \leq k \leq s$	$\xi_0^- < 0$	(ξ_0^-, ξ_1^+)
$\in (A_{0,k}^-, A_{0,k+1}^-] \subset R^+$	$\in (A_{1,s}^-, A_{1,s+1}^-] \subset R^+$	$0 \leq s < k$	$\xi_0^- < 0$	(ξ_0^+, ξ_1^-)

Table 6: $0 = \max\{\xi_0^-, \xi_1^-\} = \xi_1^- > \xi_0^-$.

y_{-1}	z_{-1}	Condition	Condition	$\lim_n(y_n, z_n)$
$\in \mathbf{R}$	$\in R^+$			(ξ_0^-, ξ_1^+)
$\in R^+$	$\in R^-$		$\xi_0^+ \geq 0$	(ξ_0^+, ξ_1^-)
			$\xi_0^+ < 0$	(ξ_0^-, ξ_1^+)
$\in R^-$	$\in R^-$		$\xi_0^+ \leq 0$	(ξ_0^-, ξ_1^+)
$\in(A_{0,k+1}^+, A_{0,k}^+] \subset R^-$	$\in(A_{1,s+1}^+, A_{1,s}^+] \subset R^-$	$0 \leq k \leq s$	$\xi_0^+ > 0$	(ξ_0^+, ξ_1^-)
$\in(A_{0,k+1}^+, A_{0,k}^+] \subset R^-$	$\in(A_{1,s+1}^+, A_{1,s}^+] \subset R^-$	$0 \leq s < k$	$\xi_0^+ > 0$	(ξ_0^-, ξ_1^+)

Table 7: $0 = \max\{\xi_0^-, \xi_1^-\} = \xi_0^- > \xi_1^-$.

y_{-1}	z_{-1}	Condition	Condition	$\lim_n(y_n, z_n)$
$\in R^+$	$\in \mathbf{R}$			(ξ_0^+, ξ_1^-)
$\in R^-$	$\in R^+$		$\xi_1^+ \geq 0$	(ξ_0^-, ξ_1^+)
			$\xi_1^+ < 0$	(ξ_0^+, ξ_1^-)
$\in R^-$	$\in R^-$		$\xi_1^+ \leq 0$	(ξ_0^+, ξ_1^-)
$\in(A_{0,k+1}^+, A_{0,k}^+] \subset R^-$	$\in(A_{1,s+1}^+, A_{1,s}^+] \subset R^-$	$0 \leq k \leq s$	$\xi_1^+ > 0$	(ξ_0^+, ξ_1^-)
$\in(A_{0,k+1}^+, A_{0,k}^+] \subset R^-$	$\in(A_{1,s+1}^+, A_{1,s}^+] \subset R^-$	$0 \leq s < k$	$\xi_1^+ > 0$	(ξ_0^-, ξ_1^+)

Table 8: $\max\{\xi_0^-, \xi_1^-\} < 0 < \min\{\xi_1^+, \xi_0^+\}$.

y_{-1}	z_{-1}	Condition	$\lim_n(y_n, z_n)$
$\in R^-$	$\in R^+$		(ξ_0^-, ξ_1^+)
$\in R^+$	$\in R^-$		(ξ_0^+, ξ_1^-)
$\in(A_{0,k}^-, A_{0,k+1}^-] \subset R^+$	$\in(A_{1,s}^-, A_{1,s+1}^-] \subset R^+$	$0 \leq k \leq s$	(ξ_0^-, ξ_1^+)
$\in(A_{0,k}^-, A_{0,k+1}^-] \subset R^+$	$\in(A_{1,s}^-, A_{1,s+1}^-] \subset R^+$	$0 \leq s < k$	(ξ_0^+, ξ_1^-)
$\in(A_{0,k+1}^+, A_{0,k}^+] \subset R^-$	$\in(A_{1,s+1}^+, A_{1,s}^+] \subset R^-$	$0 \leq k \leq s$	(ξ_0^+, ξ_1^-)
$\in(A_{0,k+1}^+, A_{0,k}^+] \subset R^-$	$\in(A_{1,s+1}^+, A_{1,s}^+] \subset R^-$	$0 \leq s < k$	(ξ_0^-, ξ_1^+)

not mutually exclusive. However, the corresponding conclusions in Table 1 show that they are compatible (and hence should not cause any problem).

Next, we turn our attention to our original equation (1.3). By $c_i = d_i + (a_i - 1)\lambda$, we may see that

$$\xi_i^{\pm} = \frac{c_i \pm b_i}{1 - a_i} = -\lambda + \frac{d_i \pm b_i}{1 - a_i}, \quad i = 0, 1. \tag{3.1}$$

Therefore, the results in the previous section for the system (1.7) can easily be translated into results for (1.3). For instance, by Table 1, we may see that when $0 > \max\{\xi_0^+, \xi_1^+\}$, that is, $\lambda > \max\{(b_0 + d_0)/(1 - a_0), (b_1 + d_1)/(1 - a_1)\}$, a solution $\{x_n\}_{n=-2}^{\infty}$ with $(x_{-2}, x_{-1}) \in \mathbf{R}^2$ will satisfy

$$\lim_n x_{2n} = \frac{b_0 + d_0}{1 - a_0}, \qquad \lim_n x_{2n+1} = \frac{b_1 + d_1}{1 - a_1}. \tag{3.2}$$

As another example, the condition $\xi_0^+ < \xi_1^+ = 0$ is equivalent to $(b_0 + d_0)/(1 - a_0) < (b_1 + d_1)/(1 - a_1) = \lambda$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of (1.3) with $(x_{-2}, x_{-1}) \in R^- \times R^+$. Then, by Table 2, we may see that

$$\lim_n x_{2n} = \frac{-b_0 + d_0}{1 - a_0}, \qquad \lim_n x_{2n+1} = \frac{b_1 + d_1}{1 - a_1}. \tag{3.3}$$

By arguments similar to those just described, the corresponding asymptotic behaviors of solutions $\{x_n\}$ of (1.3) can be summarized as follow:

(i) if $\lambda < \min\{(d_0 - b_0)/(1 - a_0), (d_1 - b_1)/(1 - a_1)\}$, then

$$\{(x_{2n}, x_{2n+1})\} \longrightarrow \left(\frac{d_0 - b_0}{1 - a_0}, \frac{d_1 - b_1}{1 - a_1}\right), \tag{3.4}$$

(ii) if $\lambda = \min\{(d_0 - b_0)/(1 - a_0), (d_1 - b_1)/(1 - a_1)\}$, then

$$\{(x_{2n}, x_{2n+1})\} \longrightarrow \left(\frac{d_0 - b_0}{1 - a_0}, \frac{d_1 - b_1}{1 - a_1}\right), \left(\frac{d_0 + b_0}{1 - a_0}, \frac{d_1 - b_1}{1 - a_1}\right) \text{ or } \left(\frac{d_0 - b_0}{1 - a_0}, \frac{d_1 + b_1}{1 - a_1}\right), \tag{3.5}$$

(iii) if $\min\{(d_0 - b_0)/(1 - a_0), (d_1 - b_1)/(1 - a_1)\} < \lambda < \max\{(d_0 + b_0)/(1 - a_0), (d_1 + b_1)/(1 - a_1)\}$, then

$$\{(x_{2n}, x_{2n+1})\} \longrightarrow \left(\frac{d_0 - b_0}{1 - a_0}, \frac{d_1 + b_1}{1 - a_1}\right) \text{ or } \left(\frac{d_0 + b_0}{1 - a_0}, \frac{d_1 - b_1}{1 - a_1}\right), \tag{3.6}$$

(iv) if $\lambda = \max\{(d_0 + b_0)/(1 - a_0), (d_1 + b_1)/(1 - a_1)\}$, then

$$\{(x_{2n}, x_{2n+1})\} \longrightarrow \left(\frac{d_0 + b_0}{1 - a_0}, \frac{d_1 + b_1}{1 - a_1}\right), \left(\frac{d_0 + b_0}{1 - a_0}, \frac{d_1 - b_1}{1 - a_1}\right) \text{ or } \left(\frac{d_0 - b_0}{1 - a_0}, \frac{d_1 + b_1}{1 - a_1}\right), \tag{3.7}$$

(v) if $\lambda > \max\{(d_0 + b_0)/(1 - a_0), (d_1 + b_1)/(1 - a_1)\}$, then

$$\{(x_{2n}, x_{2n+1})\} \longrightarrow \left(\frac{d_0 + b_0}{1 - a_0}, \frac{d_1 + b_1}{1 - a_1}\right). \tag{3.8}$$

We remark that the precise initial regions of each type of solutions in the above statements can be inferred from our previous tables. Such repetitions, however, need not to be spelled out in detail for obvious reasons. Instead, based on the statements made above, it is more important to point out that our original motivation can be fulfilled.

(i) Equation (1.3) possesses exactly four 2-periodic solutions $\{\xi_0^{\pm}, \xi_1^{\pm}\}$ with $\xi_i^{\pm} = -\lambda + (d_i \pm b_i)/(1 - a_i))$. Every other solution tends to one of these four solutions "according to the information given in the previous section."

As an example, consider a plant which is supposed to produce a type of products with capacity x_n, where n now denotes economic stages. Suppose that the stages reflect booms and busts experienced by an economy characterized by alternating periods of economic growth and contraction. Then, during busts, the plant should be managed in a fashion so as to produce at low capacity and during booms at high capacity. Suppose that it is estimated that $\xi_0^- = 1$ unit capacity is demanded during busts and $\xi_1^- = 10$ unit capacity during booms. Then, an automated plant of the form (1.7) may be built to fit the estimated demands:

$$y_n = \frac{1}{2}y_{n-1} + \frac{3}{2}H(z_{n-1}) + 2, \qquad z_n = \frac{1}{3}z_{n-1} + \frac{2}{3}H(y_n) + \frac{22}{3}, \quad n \in \mathbf{N}, \tag{3.9}$$

where the "structural" parameters $a_0 = 1/2$, $a_1 = 1/3$, $b_0 = 3/2, b_1 = 2/3$, $c_0 = 2$, and $c_1 = 22/3$ are chosen since they, as may be checked easily, guarantee that the capacities y_n and z_n will tend to 1 and 10, respectively. In fact, for all $(y_{-1}, z_{-1}) \in \mathbf{R}^2$, by (1.7), we have $y_n = (1/2)y_{n-1} + (3/2)H(z_{n-1}) + 2 \geq (1/2)y_{n-1} - (3/2) + 2 = (1/2)y_{n-1} + 1/2$, and $z_n = (1/3)z_{n-1} + (2/3)H(y_n) + (22/3) \geq (1/3)z_{n-1} - (2/3) + (22/3) = (1/3)z_{n-1} + (20/3)$ for $n \in \mathbf{N}$. Thus, $\liminf_n y_n \geq 1$ and $\liminf_n z_n \geq 10$. Therefore, there is $n' \in \mathbf{N}$ such that $y_n, z_n \in R^+$ for $n \geq n'$. Then,

$$y_n = \frac{1}{2}y_{n-1} + \frac{1}{2}, \qquad z_n = \frac{1}{3}z_{n-1} + \frac{20}{3}, \qquad n > n'. \tag{3.10}$$

We get $\lim_n y_n = 1 = (c_0 - b_0)/(1 - a_0) = \xi_0^-$ and $\lim_n z_n = 10 = (c_1 - b_1)/(1 - a_1) = \xi_1^-$.

Acknowledgment

Project supported by the National Natural Science Foundation of China (11161049). Mathematics Subject Classifications: 39A11, 39A23, 92B20.

References

[1] Z. Artstein, "Discrete and continuous bang-bang and facial spaces, or: look for the extreme points," *SIAM Review*, vol. 22, no. 2, pp. 172–185, 1980.

[2] Q. Ge, C. M. Hou, and S. S. Cheng, “Complete asymptotic analysis of a nonlinear recurrence relation with threshold control,” *Advances in Difference Equations*, vol. 2010, Article ID 143849, 19 pages, 2010.

[3] Y. Chen, “All solutions of a class of difference equations are truncated periodic,” *Applied Mathematics Letters*, vol. 15, no. 8, pp. 975–979, 2002.

[4] H. Y. Zhu and L. H. Huang, “Asymptotic behavior of solutions for a class of delay difference equation,” *Annals of Differential Equations*, vol. 21, no. 1, pp. 99–105, 2005.

[5] Z. H. Yuan, L. H. Huang, and Y. M. Chen, “Convergence and periodicity of solutions for a discrete-time network model of two neurons,” *Mathematical and Computer Modelling*, vol. 35, no. 9-10, pp. 941–950, 2002.

[6] H. Sedaghat, *Nonlinear Difference Equations Mathematical Modelling: Theory and Applications Theory with Applications to Social Science Models*, Kluwer Academic Publishers, Dordrecht, The Netherland, 1st edition, 2003.

[7] M. di Bernardo, C. J. Budd, A. R. Champneys, and P. Kowalczyk, *Piecewise-Smooth Dynamical Systems*, vol. 163 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1st edition, 2008.

[8] C. M. Hou and S. S. Cheng, “Eventually periodic solutions for difference equations with periodic coeffcients and nonlinear control functions,” *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 179589, 21 pages, 2008.

[9] C. M. Hou, C. L. Wang, and S. S. Cheng, “Bifurcation analysis for a nonlinear recurrence relation with threshold control and periodic coeffcients,” *International Journal of Bifurcation and Chaos*, vol. 22, no. 3, Article ID 1250055, 12 pages, 2012.

[10] C. M. Hou and S. S. Cheng, “Complete asymptotic analysis of a two-nation arms race model with piecewise constant nonlinearities,” *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 745697, 17 pages, 2012.