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Research Article

Dynamic Proportional Reinsurance and Approximations for Ruin Probabilities in the Two-Dimensional Compound Poisson Risk Model

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We consider the dynamic proportional reinsurance in a two-dimensional compound Poisson risk model. The optimization in the sense of minimizing the ruin probability which is defined by the sum of subportfolio is being ruined. Via the Hamilton-Jacobi-Bellman approach we find a candidate for the optimal value function and prove the verification theorem. In addition, we obtain the Lundberg bounds and the Cramér-Lundberg approximation for the ruin probability and show that as the capital tends to infinity, the optimal strategies converge to the asymptotically optimal constant strategies. The asymptotic value can be found by maximizing the adjustment coefficient.

1. Introduction

In an insurance business, a reinsurance arrangement is an agreement between an insurer and a reinsurer under which claims are split between them in an agreed manner. Thus, the insurer (cedent company) is insuring part of a risk with a reinsurer and pays premium to the reinsurer for this cover. Reinsurance can reduce the probability of suffering losses and diminish the impact of the large claims of the company. Proportional reinsurance is one of the reinsurance arrangement, which means the insurer pays a proportion, say a, when the claim occurs and the remaining proportion, $1-a$, is paid by the reinsurer. If the proportion a can be changed according to the risk position of the insurance company, this is the dynamic proportional reinsurance. Researches dealing with this problem in the one-dimensional risk model have been done by many authors. See for instance, Højgaard and Taksar [1, 2], Schmidli [3] considered the optimal proportional reinsurance policies for diffusion risk model and for compound Poisson risk model, respectively. Works combining proportional and other type of reinsurance polices for the diffusion model were presented in Zhang et al. [4]. If investment or dividend can be involved, this problem was discussed by Schmidli [5] and Azcue and Muler [6], respectively. References about dynamic reinsurance of large claim are Taksar and Markussen [7], Schmidli [8], and the references therein.

Although literatures on the optimal control are increasing rapidly, seemly that none of them consider this problem in the multidimensional risk model so far. This kind of model depicts that an unexpected claim event usually triggers several types of claims in an umbrella insurance policy, which means that a single event influences the risks of the entire portfolio. Such risk model has become more important for the insurance companies due to the fact that it is useful when the insurance companies handle dependent class of business. The previous work relating to multidimensional model without dynamic control mainly focuses on the ruin probability. See for example, Chan et al. [9] obtained the simple bounds for the ruin probabilities in two-dimensional case, and a partial integral-differential equation satisfied by the corresponding ruin probability. Yuen et al. [10] researched the finite-time survival probability of a two-dimensional compound Poisson model by the approximation of the so-called bivariate compound binomial model. Li et al. [11] studied the ruin probabilities of a two-dimensional perturbed insurance risk model and obtained a Lundberg-type upper bound for the infinite-time ruin probability. Dang et al. [12] obtained explicit expressions for recursively calculating the survival probability of the two-dimensional risk model by applying the partial integral-differential equation when claims are exponentially distributed. More literatures can be found in the references within the above papers.

In this paper, we will discuss the dynamic proportional reinsurance in a two-dimensional compound Poisson risk model. From the insurers point of view, we want to minimize the ruin probability or equivalently to maximize the survival probability.

We start with a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. $\mathcal{F}_t$ represents the information available at time t, and any decision is made upon it. Suppose that an insurance portfolio consists of two subportfolios $\{X_t^a\}$ and $\{Y_t^b\}$. $\{(U_n, V_n)\}$ is a sequence of *i.i.d* random vectors which denote the claim size for (X_t^a, Y_t^b). Let $G(u, v)$ denote their joint distribution function, and suppose $G(u, v)$ is continuous. At any time t the cedent may choose proportional reinsurance strategy (a_t, b_t). This implies that at time t the cedent company pays $(a_t U, b_t V)$. The reinsurance company pays the amount $((1 - a_t)U, (1 - b_t)V)$. $a = \{a_t\}$ and $b = \{b_t\}$ are admissible if they are adapted processes with value in $[0, 1]$. By $\mathcal{U}$ we denote the set of all admissible strategies. The model can be stated as

$$\begin{pmatrix} X_t^a \\ Y_t^b \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \int_0^t c_1(a_s)\mathrm{d}s \\ \int_0^t c_2(b_s)\mathrm{d}s \end{pmatrix} - \sum_{n=1}^{N_t} \begin{pmatrix} a_{\sigma_n -} U_n \\ b_{\sigma_n -} V_n \end{pmatrix} \tag{1.1}$$

u_1, u_2 are the initial capital of $\{X_t^a\}$ and $\{Y_t^b\}$, respectively. $c_1(a_t)$ and $c_2(b_t)$ denote the premium rates received by the insurance (cedent) company for the subportfolio $\{X_t^a\}$ and $\{Y_t^b\}$ at time t. Suppose $c_1(a)$ is continuous about a and $c_2(b)$ is continuous about b. Note that if full reinsurance, that is, $a = b = 0$ is chosen the premium rates, $c_1(0)$ and $c_2(0)$ are strictly negative. Otherwise, the insurer would reinsure the whole portfolio, then ruin would never occur for it. Let c_1, c_2 denote the premium if no reinsurance is chosen. Then $c_1(a_t) \leq c_1$, $c_2(b_t) \leq c_2$. For (U_n, V_n), their common arrival times constitute a counting process $\{N_t\}$, which is a Poisson process with rate λ and independent of (U_n, V_n). The net profit conditions are $c_1 > \lambda \mathrm{E} U_n$ and $c_2 > \lambda \mathrm{E} V_n$. $a_{\sigma_n -} U_n$ and $b_{\sigma_n -} V_n$ are the claim size that the cedent company pays at σ_n (time of the nth claim arrivals). This reinsurance form chosen prior to the claim prevents the insurer change the strategies to full reinsurance when the claim occurs and avoid the insurer owning all the premium while the reinsurer pays all the claims.

In realities, if the insurance company deals with multidimensional risk model, they may adjust the capital among every subportfolio. If the adjustment is reasonable, the company may run smoothly. So the actuaries care more about how the aggregate loss for the whole book of business effects the insurance company. Hence, in our problem we focus on the aggregate surplus:

$$R_t^{a,b} = X_t^a + Y_t^b = u + \int_0^t (c_1(a_s) + c_2(b_s))\mathrm{d}s - \sum_{n=1}^{N_t} (a_{\sigma_n -} U_n + b_{\sigma_n -} V_n), \tag{1.2}$$

where $u = u_1 + u_2$. Ruin time is defined by

$$\tau_{a,b} = \inf\left\{ t \geq 0; R_t^{a,b} < 0 \right\}, \tag{1.3}$$

which denotes the first time that the total of X_t^a and Y_t^b is negative. The ruin probability is

$$\psi_{a,b}(u) = \mathrm{P}\left(\tau_{a,b} < \infty \mid R_0^{a,b} = u \right). \tag{1.4}$$

The corresponding survival probability is

$$\delta_{a,b}(u) = \mathrm{P}\left(\tau_{a,b} = \infty \mid R_0^{a,b} = u \right). \tag{1.5}$$

Our optimization criterion is maximization of survival probability from the insurer (cedent company) point of view. So the objective is to find the optimal value function $\delta(u)$ which is defined by

$$\delta(u) = \sup_{(a,b) \in \mathcal{U}} \delta_{a,b}(u). \tag{1.6}$$

If the optimal strategy (a^*, b^*) exists, we try to determine it. Let $\{R_t\}$ denote the process under the optimal strategy (a^*, b^*) and τ^* the corresponding ruin time.

The paper is organized as follows. After the brief introduction of our model, in Section 2, we proof some useful properties of $\delta(u)$. The HJB equation satisfied by the optimal value function is presented in Section 3. Furthermore, we show that there exists a unique solution with certain boundary condition and give a proof of the verification theorem. Taking advantage of a very important technique of changing of measure, the Lundberg bounds for the controlled process are obtained in Section 4. In Section 5, we get the Cramér-Lundberg approximation for $\psi(u)$. The convergence of the optimal strategy is proved in Section 6. In the last section, we give a numerical example to illustrate how to get the upper bound of $\psi(u)$.

2. Some Properties of $\delta(u)$

We first give some useful properties of $\delta(u)$.

Lemma 2.1. *For any strategy (a,b), with probability 1, either ruin occurs or $R_t^{a,b} \to \infty$ as $t \to \infty$.*

Proof. Let (a,b) be a strategy. If the full reinsurance of each subportfolio is chosen, we denote $c_1^0 < 0$, $c_2^0 < 0$ be the premium left to the cedent insurance company. Let $B = \{(a,b) : c_1(a) + c_2(b) \geq (c_1^0 + c_2^0)/2\}$, let and $\overline{B}$ be its complementary set. Choose $\varepsilon < -(c_1^0 + c_2^0)/2$ and $\kappa = (-c_1^0 - c_2^0 - 2\varepsilon)/(2(c_1 + c_2) - c_1^0 - c_2^0)$. First, if $\int_t^{t+1} 1_{(a_s,b_s)\in B}\mathrm{d}s \leq \kappa$, then

$$\begin{aligned}
R_{t+1}^{a,b} &= R_t^{a,b} + \int_t^{t+1} (c_1(a) + c_2(b))\mathrm{d}s - \sum_{i=N(t)+1}^{N(t+1)} (a_{\sigma_{i-}} U_i + b_{\sigma_{i-}} V_i) \\
&\leq R_t^{a,b} + \int_t^{t+1} (c_1(a) + c_2(b))\mathrm{d}s \\
&= R_t^{a,b} + \int_t^{t+1} (c_1(a) + c_2(b))1_{(a,b)\in B}\mathrm{d}s + \int_t^{t+1} (c_1(a) + c_2(b))1_{(a,b)\in \overline{B}}\mathrm{d}s \\
&\leq R_t^{a,b} + (c_1 + c_2)\int_t^{t+1} 1_{(a,b)\in B}\mathrm{d}s + \frac{c_1^0 + c_2^0}{2}\int_t^{t+1} 1_{(a,b)\in \overline{B}}\mathrm{d}s \\
&\leq R_t^{a,b} + (c_1 + c_2)\kappa + (1-\kappa)\frac{c_1^0 + c_2^0}{2} \\
&= R_t^{a,b} - \varepsilon.
\end{aligned} \tag{2.1}$$

Otherwise, if $\int_t^{t+1} 1_{b_s\in B}\mathrm{d}s > \kappa$. Because $c_1(a)$, $c_2(b)$, au, and bv are continuous, we assume that ε is small enough such that

$$\mathrm{P}\left[\inf_{(a,b)\in B} aU + bV > \varepsilon\right] > 0. \tag{2.2}$$

Also

$$\mathrm{P}\left[\int_t^{t+1} 1_{(a_s,b_s)\in B}\mathrm{d}N_s \geq 1 + \frac{c_1 + c_2}{\varepsilon}\right] \geq \mathrm{P}\left[N_\kappa \geq 1 + \frac{c_1 + c_2}{\varepsilon}\right] > 0. \tag{2.3}$$

While

$$\begin{aligned}
&\sum_{i=N_t+1}^{N_{t+1}} a_{\sigma_i -} U_i + b_{\sigma_i -} V_i \\
&\quad = \sum_{i=N_t+1}^{N_{t+1}} (a_{\sigma_i -} U_i + b_{\sigma_i -} V_i)1_{(a,b)\in B} + \sum_{i=N_t+1}^{N_{t+1}} (a_{\sigma_i -} U_i + b_{\sigma_i -} V_i)1_{(a,b)\in \overline{B}} \\
&\quad \geq \sum_{i=N_t+1}^{N_{t+1}} (a_{\sigma_i -} U_i + b_{\sigma_i -} V_i)1_{(a,b)\in B}.
\end{aligned} \tag{2.4}$$

Because $\mathrm{P}[\sum_{i=N_t+1}^{N_{t+1}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i)1_{(a,b)\in B} \geq (1+((c_1+c_2)/\varepsilon))\varepsilon = (c_1+c_2)+\varepsilon] > 0$, then

$$\mathrm{P}\left[\sum_{i=N_t+1}^{N_{t+1}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i) \geq (c_1+c_2)+\varepsilon\right] > 0. \tag{2.5}$$

We denote a lower bound by $\delta > 0$. Choose $M > 0$. Let $t_0 = 0$ and $t_{k+1} = \inf\{t \geq t_k + 1; R_t^{a,b} \leq M\}$. Here we define $t_{k+1} = \infty$ if $t_k = \infty$ or if $R_t^{a,b} > M$ for all $t \geq t_k + 1$. Because

$$\begin{aligned} M - R_{t_k+1}^{a,b} &\geq R_{t_k}^{a,b} - R_{t_k+1}^{a,b} \\ &= \sum_{i=N_{t_k}+1}^{N_{t_k+1}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i) - \int_{t_k}^{t_k+1}(c_1(a)+c_2(b))\mathrm{d}s \\ &\geq \sum_{i=N_{t_k}+1}^{N_{t_k+1}}(a_{\sigma_i-}U_i + b_{\sigma_i-}V_i) - (c_1+c_2). \end{aligned} \tag{2.6}$$

Then

$$\mathrm{P}\left[M - R_{t_k+1}^{a,b} \geq \varepsilon \mid \mathcal{F}_{t_k}\right] \geq \delta, \tag{2.7}$$

which can also be expressed by

$$\mathrm{P}\left[R_{t_k+1}^{a,b} \leq M - \varepsilon \mid \mathcal{F}_{t_k}\right] \geq \delta. \tag{2.8}$$

Let $W_k = 1_{t_k<\infty, R_{t_k+1}^{a,b}<M-\varepsilon}$, $Z_k = \delta 1_{t_k<\infty}$ and $S_n = \sum_{k=1}^{n}(W_k - Z_k)$. Because

$$\begin{aligned} \mathrm{E}|S_n| &= \mathrm{E}\left(\left|\sum_{k=1}^{n}1_{t_k<\infty, R_{t_k+1}^{a,b}<M-\varepsilon} - \sum_{k=1}^{n}\delta 1_{t_k<\infty}\right|\right) \leq 2n < \infty, \\ \mathrm{E}[S_{n+1} \mid \mathcal{F}_n] &= \mathrm{E}\left[\sum_{k=1}^{n+1}(W_k - Z_k) \mid \mathcal{F}_n\right] \\ &= \mathrm{E}\left[\sum_{k=1}^{n}(W_k - Z_k) + W_{n+1} - Z_{n+1} \mid \mathcal{F}_n\right] \\ &= S_n + \mathrm{E}[W_{n+1} - Z_{n+1} \mid \mathcal{F}_n] \\ &= S_n + \mathrm{E}\left[1_{t_{n+1}<\infty, R_{t_{n+1}+1}^{a,b}<M-\varepsilon} - \delta 1_{t_{n+1}<\infty} \mid \mathcal{F}_n\right] \\ &= S_n + \left(\mathrm{P}\left[R_{t_{n+1}+1}^{a,b} < M - \varepsilon \mid \mathcal{F}_{t_n}\right] - \delta\right)\mathrm{P}[t_{n+1} < \infty] \\ &\geq S_n. \end{aligned} \tag{2.9}$$

From above, we know that $\{S_n\}$ is a submartingale and $\{S_n\}$ satisfied the conditions of Lemma 1.15 in Schmidli [13]. So

$$P\left[\sum_{k=1}^{\infty} 1_{t_k<\infty, R^{a,b}_{t_k+1}<M-\varepsilon} < \infty, \sum_{k=1}^{\infty} \delta 1_{t_k<\infty} = \infty\right] = 0. \tag{2.10}$$

Thus $R^{a,b}_{t_k+1} < M - \varepsilon$ infinitely often. If $\liminf R^{a,b}_t \leq N$, then for $M = N + \varepsilon/2$, $t_n < \infty$ for all n. Then $R^{a,b}_{t_n+1} \leq N - \varepsilon/2$ infinitely often. In particular, $\liminf R^{a,b}_t \leq N - \varepsilon/2$. We can conclude that $\liminf R^{a,b}_t < \infty$ implies $\liminf R^{a,b}_t < -\varepsilon/2$. Therefore ruin occurs. While $\liminf R^{a,b}_t = \infty$ implies $R^{a,b}_t \to \infty$ as $t \to \infty$. □

Lemma 2.2. *The function $\delta(u)$ is strictly increasing.*

Proof. If $u < z$, we can use the same strategy (a, b) for initial capital u and z. Then we can conclude that $\delta_{a,b}(u) < \delta_{a,b}(z)$, so $\delta(u) = \sup_{(a,b)\in\mathcal{U}} \delta_{a,b}(u) \leq \sup_{(a,b)\in\mathcal{U}} \delta_{a,b}(z) = \delta(z)$. Suppose that $\delta(u) = \delta(z)$.

(a) From Lemma 2.1, we know that if $c_1(a) + c_2(b) \leq (c_1^0 + c_2^0)/2$ on the interval $[0, T_1)$, where $T_1 = [2(u+\kappa(c_1+c_2))]/(-c_1^0-c_2^0)+\kappa$ for all t except a set with measure κ, then

$$R_{T_1} \leq u + \kappa(c_1 + c_2) + \frac{2(u + \kappa(c_1 + c_2))}{-c_1^0 - c_2^0} \frac{c_1^0 + c_2^0}{2} \leq 0. \tag{2.11}$$

Then ruin occurs.

(b) Otherwise, let $T_2 = \inf\{t : \int_0^t 1_{(c_1(a)+c_2(b)\leq(c_1^0+c_2^0)/2)} ds > \kappa\}$. Similar to Lemma 2.1, we have

$$\begin{gathered} P\left[\inf_{(a,b)\in B} (aU + bV) > \varepsilon\right] > 0, \\ P\left[\int_0^{T_2} 1_{(a_s,b_s)\in B} dN_s \geq \frac{u + \kappa(c_1 + c_2)}{\varepsilon}\right] \geq P\left[N_\kappa \geq \frac{u + \kappa(c_1 + c_2)}{\varepsilon}\right] > 0. \end{gathered} \tag{2.12}$$

Thus

$$P\left[\sum_{i=1}^{N_{T_2}} (a_{\sigma_i-} U_i + b_{\sigma_i-} V_i) \geq u + \kappa(c_1 + c_2)\right] > 0. \tag{2.13}$$

This implies that ruin occurs with strictly positive probability.

From (a) and (b) above, we conclude that $\delta(u) < 1$.

The process $\{\delta_{a,b}(R^{a,b}_{\tau_{a,b}\wedge t})\}$ is a martingale, if we stop the the process starting in u at the first time T_z where $R^{a,b}_t = z$. Define $\overline{R}^{a,b}_t = R^{a,b}_t + z - u$ for $t \leq T_z$, and choose arbitrary strategy

$(\overline{a}, \overline{b})$ after time T_z. To the process $\{\overline{R}_t^{a,b}\}$, we define its corresponding characteristics by a bar sign. Then

$$\begin{aligned} \overline{\delta}_{a,b}(z) &= \mathrm{E}\left[\overline{\delta}_{a,b}\left(\overline{R}_{T_z \wedge \overline{\tau}_{a,b}}\right)\right] \\ &= \overline{\delta}_{a,b}(2z - u)\mathrm{P}[T_z < \overline{\tau}_{a,b}] \geq \overline{\delta}_{a,b}(2z - u)\mathrm{P}[T_z < \tau_{a,b}]. \end{aligned} \tag{2.14}$$

There exists a strategy such that $\mathrm{P}[T_z < \tau_{a,b}]$ is arbitrarily close to 1 due to $\delta_{a,b}(u) = \delta_{a,b}(z)\mathrm{P}[T_z < \tau_{a,b}]$. From the arbitrary property of $(\overline{a}, \overline{b})$, we have $\delta(2z - u) = \delta(z) = \delta(u)$. Thus, $\delta(z)$ would be a constant for all $z \geq u$. While $\delta(z) \to 1$ as $z \to \infty$, this is only possible if $\delta(u) = 1$. Then this is contract with $\delta(u) < 1$. From all above, we conclude that $\delta(u)$ is strictly increasing. □

3. HJB Equation and Verification of Optimality

In this section, we establish the Hamilton-Jacobi-Bellman (HJB for short) equation associated with our problem and give a proof of verification theorem.

We first derive the HJB equation. Let $(a, b) \in [0, 1]$ be two arbitrary constants and $\varepsilon > 0$. If the initial capital $u = 0$, we assume that $c_1(a) + c_2(b) \geq 0$ in order to avoid immediate ruin. If $u > 0$, assume that $h > 0$ is small enough such that $u + (c_1(a) + c_2(b))h > 0$. Define

$$\left(u_t^1, u_t^2\right) = \begin{cases} (a, b), & \text{for } t \leq \sigma_1 \wedge h, \\ \left(a^{\varepsilon}_{t-(\sigma_1 \wedge h)}, b^{\varepsilon}_{t-(\sigma_1 \wedge h)}\right), & \text{for } t > \sigma_1 \wedge h, \end{cases} \tag{3.1}$$

where $(a_t^{\varepsilon}, b_t^{\varepsilon})$ are strategies satisfying $\delta_{a_t^{\varepsilon}, b_t^{\varepsilon}}(x) > \delta(x) - \varepsilon$. The first claim happens with density $\lambda e^{-\lambda t}$ and $\mathrm{P}(\sigma_1 > h) = e^{-\lambda h}$. This yields by conditioning on $\mathcal{F}_{\sigma_1 \wedge h}$

$$\begin{aligned} \delta(u) \geq \delta_{u^1, u^2}(u) &= e^{-\lambda h}\delta_{a^{\varepsilon}, b^{\varepsilon}}(u + (c_1(a) + c_2(b))h) \\ &\quad + \int_0^h \int_0^{(u+(c_1(a)+c_2(b))t)/a} \int_0^{(u+(c_1(a)+c_2(b))t-ax)/b} \delta_{a^{\varepsilon}, b^{\varepsilon}}(u + (c_1(a) + c_2(b))t - au - bv) \\ &\qquad \times \mathrm{d}G(u, v)\lambda e^{-\lambda t}\mathrm{d}t \\ &\geq e^{-\lambda h}\delta(u + (c_1(a) + c_2(b))h) \\ &\quad + \int_0^h \int_0^{(u+(c_1(a)+c_2(b))t)/a} \int_0^{(u+(c_1(a)+c_2(b))t-ax)/b} \delta(u + (c_1(a) + c_2(b))t - au - bv) \\ &\qquad \times \mathrm{d}G(u, v)\lambda e^{-\lambda t}\mathrm{d}t - \varepsilon. \end{aligned} \tag{3.2}$$

Because ε is arbitrary, let $\varepsilon = 0$. The above expression can be expressed as

$$\frac{\delta(u+(c_1(a)+c_2(b))h)-\delta(u)}{h}-\frac{1-e^{-\lambda h}}{h}\delta(u+(c_1(a)+c_2(b))h) + \frac{1}{h}\int_0^h\int_0^{u/a}\int_0^{(u-ax)/b}\delta(u+(c_1(a)+c_2(b))t-au-bv)\mathrm{d}G(u,v)\lambda e^{-\lambda t}\mathrm{d}t \le 0. \tag{3.3}$$

If we assume that $\delta(u)$ is differentiable and $h \to 0$, yields

$$[c_1(a)+c_2(b)]\delta'(u)+\lambda\int_0^{u/a}\int_0^{(u-ax)/b}\delta(u-ax-by)\mathrm{d}G(x,y)-\lambda\delta(u)\le 0. \tag{3.4}$$

For all $(a,b)\in\mathcal{U}$, (3.4) is true. We first consider such a HJB equation

$$\sup_{(a,b)\in[0,1]\times[0,1]}[c_1(a)+c_2(b)]f'(u)+\lambda\int_0^{\infty}\int_0^{\infty}f(u-ax-by)\mathrm{d}G(x,y)-\lambda f(u)=0. \tag{3.5}$$

For the moment, we are not sure whether $\delta(u)$ fulfills the HJB equation and just conjecture that $\delta(u)$ is one of the solutions, so we replace $\delta(u)$ by $f(u)$. Because $\delta(u)$ is a survival function, we are interested in a function $f(x)$ which is strictly increasing, $f(x)=0$ for $x<0$ and $f(0)>0$. Because the function for which the supremum is taken is continuous in a, b, and $[0,1]\times[0,1]$ is compact, for $u\ge 0$, there are values $a(u)$, $b(u)$ for which the supremum is attained. In (3.5), we also need $c_1(a)+c_2(b)\ge 0$. Otherwise, (3.5) will never be true. Furthermore, $\mathrm{P}(aU_n+bV_n>0)>0$, so $c_1(a)+c_2(b)>0$. We rewrite (3.5) by

$$\sup_{(a,b)\in\widetilde{U}}[c_1(a)+c_2(b)]f'(u)+\lambda\int_0^{u/a}\int_0^{(u-ax)/b}f(u-ax-by)\mathrm{d}G(x,y)-\lambda f(u)=0, \tag{3.6}$$

where $\widetilde{U}=\{(a,b)\in[0,1]\times[0,1]:c_1(a)+c_2(b)>0\}$ and $u\ge 0$. Define that $u/0=\infty$.

From (3.6), we have

$$f'(u)\le\frac{\lambda}{c_1(a)+c_2(b)}\left[f(u)-\int_0^{u/a}\int_0^{(u-ax)/b}f(u-ax-by)\mathrm{d}G(x,y)\right]. \tag{3.7}$$

When $(a,b)=(a^*,b^*)$, equality holds. Then $f(u)$ also satisfies the following equivalent equation:

$$f'(u)=\inf_{(a,b)\in\widetilde{U}}\frac{\lambda}{c_1(a)+c_2(b)}\left[f(u)-\int_0^{u/a}\int_0^{(u-ax)/b}f(u-ax-by)\mathrm{d}G(x,y)\right]. \tag{3.8}$$

Equations (3.4) and (3.8) are equivalent for strictly increasing functions. Solutions solved from (3.8) are only up to a constant, and we can choose $f(0)=1$.

In the next theorem we prove the existence of a solution of HJB equation and also give the properties of the solution.

Theorem 3.1. *There is a unique solution to the HJB equation* (3.8) *with* $f(0) = 1$. *The solution is bounded, strictly increasing, and continuously differentiable.*

Proof. Reformulate the expression by integrating by part,

$$\begin{aligned} f(u) &- \int_0^{u/a}\int_0^{(u-ax)/b} f(u-ax-by)\,\mathrm{d}G(x,y) \\ &= f(u) - \int_0^u f(u-x)\,\mathrm{d}G_{aU+bV}(x) \\ &= f(u) - \int_0^u \left(\int_0^{u-x} f'(y)\,\mathrm{d}y - 1\right)\mathrm{d}G_{aU+bV}(x) \\ &= \int_0^u f'(y)\left(1-G_{aU+bV}(u-y)\right)\mathrm{d}y + 1 - G_{aU+bV}(u). \end{aligned} \tag{3.9}$$

Let $\mho$ be an operator, and let g be a positive function, define

$$\mho g(u) = \inf_{(a,b)\in\overline{U}} \frac{\lambda}{c_1(a)+c_2(b)}\left[\int_0^u g(y)\left(1-G_{aU+bV}(u-y)\right)\mathrm{d}y + 1 - G_{aU+bV}(u)\right]. \tag{3.10}$$

First we will show the existence of a solution. If no reinsurance is taken to every subportfolio, the survival probability $\delta_1(u)$ satisfied the equation (See Rolski et al. [14]) as follows:

$$\begin{aligned} \delta_1'(u) &= \frac{\lambda}{c_1+c_2}\left[\delta_1(u) - \int_0^u \delta_1(u-x)\,\mathrm{d}G_{U+V}(x)\right] \\ &= \frac{\lambda}{c_1+c_2}\left[\int_0^u \delta_1'(y)\left(1-G_{U+V}(u-y)\right)\mathrm{d}y + 1 - G_{U+V}(u)\right]. \end{aligned} \tag{3.11}$$

Let

$$g_0(u) = \frac{(c_1+c_2)\delta_1'(u)}{\lambda(\mathrm{E}(U+V))} = \frac{\delta_1'(u)}{\delta_1(0)}, \tag{3.12}$$

where $\delta_1(0) = (\lambda \mathrm{E}(U+V))/(c_1+c_2)$ (this result can be found in Schmidli [13] Appendix D.1.) Next we define recursively $g_n(u) = \mho g_{n-1}(u)$. Because

$$\begin{aligned} g_1(u) &= \mho g_0(u) \\ &= \inf_{(a,b)\in U} \frac{\lambda}{c_1(a)+c_2(b)}\left[\int_0^u g_0(y)\left(1-G_{aU+bV}(u-y)\right)\mathrm{d}y + 1 - G_{aU+bV}(u)\right] \\ &\le \frac{\lambda}{c_1+c_2}\left[\int_0^u g_0(y)\left(1-G_{U+V}(u-y)\right)\mathrm{d}y + 1 - G_{U+V}(u)\right] \\ &= \frac{\lambda}{c_1+c_2}\left[\int_0^u \frac{\delta_1'(y)}{\delta_1(0)}\left(1-G_{U+V}(u-y)\right)\mathrm{d}y + 1 - G_{U+V}(u)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\delta_1(0)}\frac{\lambda}{c_1+c_2}\left[\int_0^u \delta_1'(y)(1-G_{U+V}(u-y))\mathrm{d}y + (1-G_{U+V}(u))\delta_1(0)\right]\\
&\le \frac{1}{\delta_1(0)}\frac{\lambda}{c_1+c_2}\left[\int_0^u \delta_1'(y)(1-G_{U+V}(u-y))\mathrm{d}y + (1-G_{U+V}(u))\right]\\
&= \frac{\delta_1'(u)}{\delta_1(0)} = g_0(u).
\end{aligned}
\tag{3.13}
$$

Then $g_1(u) \le g_0(u)$. We conclude that $g_n(u)$ is decreasing in n. Indeed, suppose that $g_{n-1}(u) \ge g_n(u)$. Let (a_n, b_n) be the points where $\mho g_{n-1}(u)$ attains the minimum. Such a pair of points exist because the right side of (3.8) is continuous in both a and b, the set $\{(a,b) : c_1(a)+c_2(b) \ge 0\}$ is compact, and the right side of (3.8) converges to infinity as (a,b) approach the point (a_0, b_0) where $c_1(a_0) = 0$, $c_2(b_0) = 0$. Then

$$
\begin{aligned}
g_n(u) - g_{n+1}(u) &= \mho g_{n-1}(u) - \mho g_n(u)\\
&\ge \frac{\lambda}{c_1(a_n)+c_2(b_n)}\left[\int_0^u (g_{n-1}(y) - g_n(y))(1-G_{a_nU+b_nV}(u-y))\mathrm{d}y\right] \ge 0.
\end{aligned}
\tag{3.14}
$$

So $g_n(u) \ge g_{n+1}(u) > 0$, and we have $g(u) = \lim_{n\to\infty} g_n(u)$ exists point wise. By the bounded convergence, for each u, a, and b

$$
\lim_{n\to\infty}\int_0^u g_n(y)(1-G_{aU+bV}(u-y))\mathrm{d}y = \int_0^u g(y)(1-G_{aU+bV}(u-y))\mathrm{d}y. \tag{3.15}
$$

Let a, b be points which $\mho g(u)$ attains its minimum. For

$$
\begin{aligned}
g_n(u) &= \frac{\lambda}{c_1(a_n)+c_2(b_n)}\left[1-G_{a_nU+b_nV}(u) + \int_0^u g_{n-1}(y)(1-G_{a_nU+b_nV}(u-y))\mathrm{d}y\right]\\
&\le \frac{\lambda}{c_1(a)+c_2(b)}\left[1-G_{aU+bV}(u) + \int_0^u g_{n-1}(y)(1-G_{aU+bV}(u-y))\mathrm{d}y\right].
\end{aligned}
\tag{3.16}
$$

So $g(u) \le \mho g(u)$ by letting $n \to \infty$. On the other hand, $g_n(z)$ is decreasing, then

$$
\begin{aligned}
g_n(u) &= \frac{\lambda}{c_1(a_n)+c_2(b_n)}\left[1-G_{a_nU+b_nV}(u) + \int_0^u g_{n-1}(y)(1-G_{a_nU+b_nV}(u-y))\mathrm{d}y\right]\\
&\ge \frac{\lambda}{c_1(a_n)+c_2(b_n)}\left[1-G_{a_nU+b_nV}(u) + \int_0^u g(y)(1-G_{a_nU+b_nV}(u-y))\mathrm{d}y\right]\\
&\ge \frac{\lambda}{c_1(a)+c_2(b)}\left[1-G_{aU+bV}(u) + \int_0^u g(y)(1-G_{aU+bV}(u-y))\mathrm{d}y\right].
\end{aligned}
\tag{3.17}
$$

So $g(u) = \mho g(u)$. Define $f(u) = 1 + \int_0^u g(x)\mathrm{d}x$. By the bounded convergence, $f(u)$ fulfills (3.8). Then $f(u)$ is increasing, continuously differentiable and bounded by $(c_1 + c_2)/(\lambda \mathrm{E}(U+V))$. From (3.8), $f'(0) > 0$. Let $x_0 = \inf\{z : f'(z) = 0\}$. Because $f(u)$ is strictly increasing in $[0, x_0]$, we must have $G_{aU+bV}(x_0) = 1$ and $ax + by = 0$ for all points of increase of $G_{aU+bV}(z)$. But this would be $a = b = 0$, which is impossible. Thus $f(u)$ is strictly increasing.

Next we want to show the uniqueness of the solution. Suppose that $f_1(u)$ and $f_2(u)$ are the solutions to (3.8) with $f_1(0) = f_2(0) = 1$. Define $g_i(u) = f_i'(u)$, and (a_i, b_i) is the value which minimize (3.8). To a constant $\overline{x} > 0$, because the right hand of (3.8) is continuous both in a and b and tends to infinity as $c_1(a) + c_2(b)$ approach 0, the $c_1(a) + c_2(b)$ is bounded away from 0 on $(0, \overline{x}]$. Let $x_1 = \inf\{\min_i c_1(a_i(x)) + c_2(b_i(x)) : 0 \le u \le \overline{x}\}/(2\lambda)$ and $x_n = nx_1 \wedge \overline{x}$. Suppose we have proved that $f_1(u) = f_2(u)$ on $[0, x_n]$. For $n = 0$, it is obviously true. Then for $u \in [x_n, x_{n+1}]$, with $m = \sup_{x_n \le u \le x_{n+1}} |g_1(u) - g_2(u)|$

$$\begin{aligned} g_1(u) - g_2(u) &= \mho g_1(u) - \mho g_2(u) \\ &\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} \left[\int_0^u (g_1(y) - g_2(y))(1 - G_{a_2U+b_2V}(u-y))\mathrm{d}y \right] \\ &= \frac{\lambda}{c_1(a_2) + c_2(b_2)} \left[\int_{x_n}^u (g_1(y) - g_2(y))(1 - G_{a_2U+b_2V}(u-y))\mathrm{d}y \right] \\ &\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} m(u - x_n) \\ &\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} m(x_{n+1} - x_n) \\ &\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} m x_1 \\ &\le \frac{\lambda}{c_1(a_2) + c_2(b_2)} m \frac{c_1(a_2) + c_2(b_2)}{2\lambda} = \frac{m}{2}. \end{aligned} \tag{3.18}$$

Once revers the role of $g_1(u)$ and $g_2(u)$, then $|g_1(u) - g_2(u)| \le m/2$. This is impossible for all $u \in [x_n, x_{n+1}]$ if $m \ne 0$. This shows that $f_1(u) = f_2(u)$ on $[0, x_{n+1}]$. So $f_1(u) = f_2(u)$ on $[0, \overline{x}]$. The uniqueness is true from the arbitrary of $\overline{x}$. □

Denoted by $a^*(u), b^*(u)$ the value of a and b maximize (3.6).

From the next theorem, so-called verification theorem, we conclude that a solution to the HJB equation which satisfies some conditions really is the desired value function.

Theorem 3.2. *Let $f(u)$ be the unique solution to the HJB equation* (3.8) *with $f(0) = 1$. Then $f(u) = \delta(u)/\delta(0)$. An optimal strategy is given by (a_t^*, b_t^*), which minimize* (3.8)*, and $\{R_t\}$ is the process under the optimal strategy.*

Proof. Let (a, b) be an arbitrary strategy with the risk processes $\{R_t^{a,b}\}$. Since $f(u)$ is bounded, then for each $t \ge 0$,

$$\mathrm{E}\left(\sum_{n:\sigma_n \le t} \left| f\left(R^{a,b}(\sigma_n)\right) - f\left(R^{a,b}(\sigma_n -)\right) \right| \right) < \infty. \tag{3.19}$$

Let $\mathscr{A}$ denotes the generator of $\{R_t^{a,b}\}$. From Theorem 11.2.2 in Rolski et al. [14], we know that $f \in \mathfrak{D}(\mathscr{A})$, where $\mathfrak{D}(\mathscr{A})$ is the domain of $\mathscr{A}$. Then

$$f\left(R^{a,b}_{\tau_{a,b}\wedge t}\right) - \int_0^{\tau_{a,b}\wedge t}\left[(c_1(a)+c_2(b))f'\left(R_s^{a,b}\right)\right.$$
$$\left. +\lambda\left(\int_0^{R_t^{a,b}/a}\int_0^{(R_t^{a,b}-ax)/b} f\left(R_s^{a,b}-ax-by\right)\mathrm{d}G(x,y) - f\left(R_s^{a,b}\right)\right)\right]\mathrm{d}s \tag{3.20}$$

is a martingale. From (3.6) we know that $\{f(R_t^{a,b})1_{\tau_{a,b}>t}\}$ is a supermartingale, then

$$\mathrm{E}\left(f\left(R_t^{a,b}\right)1_{\tau_{a,b}>t}\right) = \mathrm{E}\left(f\left(R^{a,b}_{\tau_{a,b}\wedge t}\right)\right) \le f(u). \tag{3.21}$$

If $(a,b)=(a^*,b^*)$, then $\{f(R_{\tau^*\wedge t})\}$ is a martingale. So $\mathrm{E}(f(R_t)1_{\tau^*>t}) = f(u)$. Let $t \to \infty$, from the bounded property of $f(u)$, we have

$$f(\infty)\delta_{a,b}(u) = f(\infty)\mathrm{P}[\tau_{a,b}=\infty] \le f(u) = f(\infty)\delta_{a^*,b^*}(u) = \delta(u)f(\infty). \tag{3.22}$$

For $u=0$, we obtain that $f(\infty)=1/\delta(0)$. Then $\delta(u)=f(u)/f(\infty)=f(u)\delta(0)$. Furthermore, the associated policy with (a^*,b^*) is indeed an optimal strategy. □

4. Lundberg Bounds and the Change of Measure Formula

In Section 3, we have seen when considering the dynamic reinsurance police the explicit expression of ruin probability is not easy to derive. Therefore the asymptotic optimal strategies are very important. In the classical risk theory, we have Lundberg bounds and Cramér-Lundberg approximation for the ruin probability. The former gives the upper and lower bounds for ruin probability, and the latter gives the asymptotic behavior of ruin probability as the capital tends to infinity. They both provide the useful information in getting the nature of underlying risks. In researching the two-dimensional risk model controlled by reinsurance strategy, we can also discuss the analogous problems. References are Schmidli [15, 16], Hipp and Schmidli [17], and so forth. The key in researching the asymptotic behavior is adjustment coefficient. Next we will discuss it in detail.

Assume that $\mathrm{E}e^{r(U+V)} < \infty$ for $r > 0$. To the *fixed* (a,b), let $R(a,b)$ be adjustment coefficient satisfied

$$\theta(r;a,b) := \lambda\left(\mathrm{E}e^{r(aU+bV)} - 1\right) - r(c_1(a)+c_2(b)) = 0. \tag{4.1}$$

We focus on $R = \sup_{(a,b)\in[0,1]\times[0,1]} R(a,b)$, which is the adjustment coefficient for our problem. By the assumption that $c_1(a)$ and $c_2(b)$ are continuous, then $\theta(r;a,b)$ is continuous both in a and b. Moreover

$$\frac{\partial^2\theta(r;a,b)}{\partial r^2} = \lambda \mathrm{E}(aU + bV)^2 e^{r(aU+bV)} > 0,$$
$$\theta(0;a,b) = 0, \qquad \theta(R(a,b);a,b) = 0. \tag{4.2}$$

We can get that $\theta(r;a,b)$ is strictly convex in r and $\theta(R;a,b) > 0$. If $r < R$, then there are a and b such that $R(a,b) > r$ and $\theta(r;a,b) < 0$. Because $\theta(R;a,b)$ is continuous in a and b, also $[0,1]\times[0,1]$ is compact, there exist $\tilde{a}$ and $\tilde{b}$ for which $\theta(R;\tilde{a},\tilde{b}) = 0$.

Lemma 4.1. *Suppose that $M(r,a,b)$, $c_1(a)$, and $c_2(b)$ are all twice differentiable (with respect to r, a, and b). Moreover that*

$$c_1''(a) \le 0, \qquad c_2''(b) \le 0, \tag{4.3}$$

then there is a unique maximum of $R(a,b)$.

Proof. $R(a,b)$ satisfies (4.1):

$$\lambda\left(\mathrm{E}e^{R(a,b)(aU+bV)} - 1\right) - (c_1(a) + c_2(b))R(a,b) = 0. \tag{4.4}$$

Let $M(r,a,b) = \mathrm{E}e^{r(aU+bV)}$, and $M_r(r,a,b)$, $M_a(r,a,b)$, $M_b(r,a,b)$, R_a, and R_b denote the partial derivatives.

Taking partial derivative of (4.4) with respect to a,

$$\lambda M_r R_a(a,b) + \lambda M_a - c_1'(a)R(a,b) - (c_1(a) + c_2(b))R_a(a,b) = 0. \tag{4.5}$$

Because the left-side hand of (4.4) is a convex function in r, we have $\lambda M_r - (c_1(a)+c_2(b)) > 0$. So

$$R_a(a,b) = -\frac{\lambda M_a - c_1'(a)R(a,b)}{\lambda M_r - (c_1(a) + c_2(b))}. \tag{4.6}$$

Similarly

$$R_b(a,b) = -\frac{\lambda M_b - c_2'(b)R(a,b)}{\lambda M_r - (c_1(a) + c_2(b))}. \tag{4.7}$$

Let $(\tilde{a},\tilde{b})$ be the point such that $R_a(\tilde{a},\tilde{b}) = R_b(\tilde{a},\tilde{b}) = 0$. Then

$$\begin{aligned}
R_{a,a}\left(\tilde{a},\tilde{b}\right) &= -\frac{\lambda M_{a,a}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - c_1''(a)R\left(\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)} \\
&= -\frac{\lambda \mathrm{E}\left(R\left(\tilde{a},\tilde{b}\right)U\right)^2 e^{R(\tilde{a},\tilde{b})(\tilde{a}U+\tilde{b}V)} - c_1''(\tilde{a})R\left(\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)} < 0, \\
R_{b,b}\left(\tilde{a},\tilde{b}\right) &= -\frac{\lambda M_{b,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - c_2''(b)R\left(\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)} \\
&= -\frac{\lambda \mathrm{E}\left(R\left(\tilde{a},\tilde{b}\right)V\right)^2 e^{R(\tilde{a},\tilde{b})(\tilde{a}U+\tilde{b}V)} - c_2''\left(\tilde{b}\right)R\left(\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)} < 0, \\
R_{a,b}\left(\tilde{a},\tilde{b}\right) &= -\frac{\lambda M_{a,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)}{\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)}.
\end{aligned} \tag{4.8}$$

While

$$\begin{aligned}
&R_{a,a}\left(\tilde{a},\tilde{b}\right)R_{b,b}\left(\tilde{a},\tilde{b}\right) - R_{a,b}^2\left(\tilde{a},\tilde{b}\right) \\
&\quad= \frac{\left[\lambda M_{a,a}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - c_1''(a)R\left(\tilde{a},\tilde{b}\right)\right]\left[\lambda M_{b,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - c_2''(b)R\left(\tilde{a},\tilde{b}\right)\right]}{\left[\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)\right]^2} \\
&\qquad - \frac{\lambda^2 M_{a,b}^2\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)}{\left[\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)\right]^2} \\
&\quad= \frac{\left[M_{a,a}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)M_{b,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - M_{a,b}^2\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)\right]\lambda^2}{\left[\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)\right]^2} \\
&\qquad + \frac{c_1''(a)c_2''(b)R^2 + \left[M_{b,b}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)c_1''(a) + M_{a,a}\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right)c_2''(b)R\left(\tilde{a},\tilde{b}\right)\right]\lambda R}{\left[\lambda M_r\left(R\left(\tilde{a},\tilde{b}\right),\tilde{a},\tilde{b}\right) - \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)\right]^2}.
\end{aligned} \tag{4.9}$$

From Hölder inequality, we have that the first term of above expression is positive. Owning to the conditions given by the lemma, we also find that the second term of above is positive. Therefore, $R(\tilde{a},\tilde{b})$ is a maximum value. □

We now let $\psi(u)$ be the ruin probability under the optimal strategy. First we give a Lundberg upper bound of $\psi(u)$.

Theorem 4.2. *The minimal ruin probability $\psi(u)$ is bounded by e^{-Ru}, that is, $\psi(u) < e^{-Ru}$.*

Proof. To the fixed proportional reinsurance $(\tilde{a},\tilde{b})$, $\psi_{a,b}(u)$ can be calculated by the result on ruin probability of the classical risk model. We have the following expression of $\psi_{\tilde{a},\tilde{b}}(u)$:

$$\begin{aligned}\psi_{\tilde{a},\tilde{b}}(u) &= \mathrm{P}\left(\tau_{\tilde{a},\tilde{b}} < \infty\right)\\ &= \mathrm{E}^{(R)}\left[\exp\left\{RR^{\tilde{a},\tilde{b}}_{\tau_{\tilde{a},\tilde{b}}}\right\}\right]e^{-Ru}\\ &< e^{-Ru}.\end{aligned} \tag{4.10}$$

So the minimal ruin probability is bounded by $\psi(u) \le \psi_{\tilde{a},\tilde{b}}(u) < e^{-Ru}$. □

From Theorem 4.2, the adjustment coefficient R can be looked upon as a risk measure to estimate the optimal ruin probability.

For the considerations below we define the strategy: if $u < 0$, we let $a^*(u) = b^*(u) = 1$. In order to obtain the lower bound, we start by defining a process M_t as follows:

$$\begin{aligned}M_t &= \exp\left\{-R(R_t - u) - \int_0^t \theta(R; a^*(R_s), b^*(R_s))\mathrm{d}s\right\}\\ &= \exp\left\{\sum_{n=1}^{N_t} R(a^*(R_{\sigma_n-})U_n + b^*(R_{\sigma_n-})V_n) - \int_0^t \lambda\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)} - 1\right)\mathrm{d}s\right\}.\end{aligned} \tag{4.11}$$

Lemma 4.3. *The process M_t is a strictly positive martingale with mean value 1.*

Proof. First we will show that $\{M_{\sigma_n \wedge t}\}$ is a martingale. Indeed, $\mathrm{E}M_{\sigma_0\wedge t} = \mathrm{E}M_{\sigma_0} = 1$, and we suppose that $\mathrm{E}M_{\sigma_{n-1}\wedge t} = 1$. Given $\mathcal{F}_{\sigma_{n-1}}$, the progress $\{(X_t, Y_t)\}$ is deterministic on $[\sigma_{n-1}, \sigma_n)$. We split into the event $\{\sigma_n > t\}$ and $\{\sigma_n \le t\}$. From the Markov property of M_t and for $\sigma_{n-1} < t$, we have

$$\begin{aligned}\mathrm{E}M_{\sigma_n\wedge t} &= \mathrm{E}\{\mathrm{E}[M_{\sigma_n\wedge t} \mid \mathcal{F}_{\sigma_{n-1}}]\}\\ &= \mathrm{E}\{\mathrm{E}[M_{\sigma_n\wedge t} \mid M_{\sigma_{n-1}}]\}\\ &= \mathrm{E}\{\mathrm{E}[1_{\sigma_n>t}M_t \mid M_{\sigma_{n-1}}]\} + \mathrm{E}\{\mathrm{E}[1_{\sigma_n\le t}M_{\sigma_n} \mid M_{\sigma_{n-1}}]\}.\end{aligned} \tag{4.12}$$

For convenience, let $A = \mathrm{E}\{\mathrm{E}[1_{\sigma_n>t}M_t \mid M_{\sigma_{n-1}}]\}$ and $B = \mathrm{E}\{\mathrm{E}[1_{\sigma_n\le t}M_{\sigma_n} \mid M_{\sigma_{n-1}}]\}$. Next we calculate A and B, respectively,

$$\begin{aligned}A &= \mathrm{E}\{\mathrm{E}[1_{\sigma_n>t}M_t \mid M_{\sigma_{n-1}}]\}\\ &= \mathrm{E}\left\{1_{\sigma_n>t>\sigma_{n-1}}\mathrm{E}\left[M_{\sigma_n-1}\exp\left\{-\lambda\int_{\sigma_{n-1}}^t \left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)} - 1\right)\mathrm{d}s\right\} \mid M_{\sigma_{n-1}}\right]\right\}\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}M_{\sigma_{n-1}}\mathrm{P}(\sigma_{n-1} < t < \sigma_n)\exp\left\{-\lambda\int_{\sigma_{n-1}}^{t}\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)}-1\right)\mathrm{d}s\right\}\\
&= e^{-\lambda(t-\sigma_{n-1})}\exp\left\{-\lambda\int_{\sigma_{n-1}}^{t}\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)}-1\right)\mathrm{d}s\right\}\\
&= \exp\left\{-\lambda\int_{\sigma_{n-1}}^{t}\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)}\mathrm{d}s\right\},\\
B &= \mathrm{E}\{\mathrm{E}[1_{\sigma_n\le t}M_n \mid M_{\sigma_{n-1}}]\}\\
&= \mathrm{E}\left\{1_{\sigma_n\le t}\mathrm{E}\left[M_{\sigma_n-1}\exp\left\{R(a^*(R_{\sigma_n-})U+b^*(R_{\sigma_n-})V)\right.\right.\right.\\
&\qquad\left.\left.\left.-\lambda\int_{\sigma_{n-1}}^{\sigma_n}\left(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)}-1\right)\mathrm{d}s\right\} \mid M_{\sigma_{n-1}}\right]\right\}\\
&= \int_{\sigma_{n-1}}^{t}\mathrm{E}\exp\left\{R(a^*(R_{s-})U+b^*(R_{s-})V)-\lambda\int_{\sigma_{n-1}}^{s}\left(\mathrm{E}e^{R(a^*(R_w)U+b^*(R_w)V)}-1\right)\mathrm{d}w\right\}\\
&\quad\times\lambda e^{-\lambda(s-\sigma_{n-1})}\mathrm{d}s\\
&= \int_{\sigma_{n-1}}^{t}\lambda\mathrm{E}e^{R(a^*(R_{s-})U+b^*(R_{s-})V)}\exp\left\{-\lambda\int_{\sigma_{n-1}}^{s}\mathrm{E}e^{R(a^*(R_w)U+b^*(R_w)V)}\mathrm{d}w\right\}\mathrm{d}s.
\end{aligned}
\tag{4.13}
$$

Let $f(s) = \lambda\mathrm{E}e^{R(a^*(R_{s-})U+b^*(R_{s-})V)}$, then

$$
\mathrm{E}M_{\sigma_n\wedge t} = e^{-\int_{\sigma_{n-1}}^{t}f(s)\mathrm{d}s} + \int_{\sigma_{n-1}}^{t}f(s)e^{-\int_{\sigma_{n-1}}^{s}f(w)\mathrm{d}w}\mathrm{d}s. \tag{4.14}
$$

Because $(e^{-\int_{\sigma_{n-1}}^{t}f(s)\mathrm{d}s})' = f(t)e^{-\int_{\sigma_{n-1}}^{t}f(w)\mathrm{d}w}$, using the integration by part, we have $\mathrm{E}M_{\sigma_n\wedge t} = 1$. From above we know that $\mathrm{E}[M_{\sigma_n\wedge t} \mid \mathcal{F}_{\sigma_{n-1}}] = M_{\sigma_{n-1}\wedge t}$. Furthermore, following the assumption that $\mathrm{E}e^{R(U+V)} < \infty$, then

$$
\exp\left\{\sum_{n=1}^{N_t}R(a^*(R_{\sigma_n-})U_n + b^*(R_{\sigma_n-})V_n)\right\} \le \exp\left\{\sum_{n=1}^{N_t}R(U_n+V_n)\right\} < \infty. \tag{4.15}
$$

So for each t, $\{M_{\sigma_n\wedge t}\}$ is uniform integrable. This finishes the proof of Lemma 4.3. □

Based on the martingale $\{M(t), t \ge 0\}$ given above, we consider a family of new measure $\mathrm{P}_t^*[A] = \mathrm{E}[M_t; A]$, $A \in \mathcal{F}_t$. From the Kolmogorov's extension theorem, there exists a probability measure P^* such that the restriction of P^* to $\mathcal{F}_t$ is P_t^*. Moreover, if T is an $\mathcal{F}_t$-stopping time and $A \subset \{T < \infty\}$ such that $A \in \mathcal{F}_T$, then $\mathrm{P}^*[A] = \mathrm{E}[M_T; A]$. The change of measure technique is a powerful tool in investigating ruin probability. The following theorem gives us the feature of R_t under the new measure.

Theorem 4.4. *Under the new measure P^*, the process $\{R_t\}$ is a piecewise deterministic Markov process (PDMP for short) with jump intensity $\lambda^*(x) = \lambda E e^{R(a^*(x)U+b^*(x)V)}$ and claim size distribution*

$$G_x^*(u,v) = \frac{1}{Ee^{R(a^*(x)U+b^*(x)V)}} \int_0^u \int_0^v e^{R(a^*(x)r+b^*(x)s)} dG(r,s). \tag{4.16}$$

The premium rates for each subportfolios are $c_1(a^*(x))$ and $c_2(b^*(x))$, respectively.

Proof. Let B be a Borel set. Refer to Lemma C.1 in Schmidli [13], we have

$$\begin{aligned} \mathrm{P}^*[R_{t+s} \in B \mid \mathcal{F}_t] &= \mathrm{E}\Big[M_t^{-1} M_{t+s}; R_{t+s} \in B \mid \mathcal{F}_t\Big] \\ &= \mathrm{E}\Big[M_t^{-1} M_{t+s}; R_{t+s} \in B \mid \mathrm{X}_t\Big]. \end{aligned} \tag{4.17}$$

This means that under the new measure P^*, $\{R_t\}$ is still a Markov process. On the other hand, the path between jumps is deterministic. So $\{R_t\}$ is a PDMP under P^*. Next we will calculate the distribution of σ_1 (the time of the first claim happens), U, and V. Let r_s denote the deterministic path on $[0, \sigma_1)$. The distribution of σ_1 can be obtained by

$$\begin{aligned} \mathrm{P}^*[\sigma_1 > t] &= \mathrm{E}[M_t; \sigma_1 > t] \\ &= e^{-\lambda t} \exp\left\{ -\int_0^t \lambda \Big(\mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)} - 1 \Big) \mathrm{d}s \right\} \\ &= \exp\left\{ -\int_0^t \lambda \mathrm{E}e^{R(a^*(R_s)U+b^*(R_s)V)} \mathrm{d}s \right\}. \end{aligned} \tag{4.18}$$

So $\lambda^*(x) = \lambda \mathrm{E}e^{R(a^*(x)U+b^*(x)V)}$.

Next we consider the first claim size (U_1, V_1). Let B_1, B_2 be two Borel sets.

$$\begin{aligned} &\mathrm{P}^*[\sigma_1 \le t,\ U_1 \in B_1, V_1 \in B_2] \\ &\quad= \mathrm{E}[M_t; \sigma_1 \le t,\ U_1 \in B_1, V_1 \in B_2] \\ &\quad= \mathrm{E}[\mathrm{E}[M_t; \sigma_1 \le t,\ U_1 \in B_1, V_1 \in B_2 \mid \mathcal{F}_{\sigma_1}]] \\ &\quad= \mathrm{E}[M_{\sigma_1}; \sigma_1 \le t,\ U_1 \in B_1, V_1 \in B_2] \\ &\quad= \int_0^t \int_0^\infty \int_0^\infty \exp\Big\{ R(a^*(r_s)u + b^*(r_s)v) \\ &\qquad\qquad -\lambda \int_0^s \Big(\mathrm{E}e^{R(a^*(r_w)U_1+b^*(r_w)V_1)} - 1 \Big) \mathrm{d}w \Big\} 1_{B_1\times B_2}(u,v) \mathrm{d}G(u,v) \lambda e^{-\lambda s} \mathrm{d}s \\ &\quad= \int_0^t \int_0^\infty \int_0^\infty 1_{B_1\times B_2}(u,v) \mathrm{d}G^*(u,v) \lambda^*(r_s) e^{-\int_0^s \lambda^*(r_w)\mathrm{d}w} \mathrm{d}s. \end{aligned} \tag{4.19}$$

At last, since the set of trajectories of R_t is same under P and P*, it is clear that the deterministic premium rates remain $c_1(a^*)$ and $c_2(b^*)$. □

If we consider the drift of R_t under the new measure P*, then

$$\begin{aligned}
&c_1(a^*(x)) + c_2(b^*(x)) - \lambda^*(x)\int_0^\infty\int_0^\infty (a^*(x)u + b^*(x)v)\mathrm{d}G_x^*(u,v) \\
&\quad = c_1(a^*(x)) + c_2(b^*(x)) - \lambda \mathrm{E}e^{R(a^*(x)U+b^*(x)V)} \\
&\qquad \times \int_0^\infty\int_0^\infty (a^*(x)u + b^*(x)v)\frac{1}{\mathrm{E}e^{R(a^*(x)U+b^*(x)V)}}e^{R(a^*(x)u+b^*(x)v)}\mathrm{d}G(u,v) \\
&\quad = c_1(a^*(x)) + c_2(b^*(x)) - \lambda \mathrm{E}R(a^*(x)U + b^*(x)V)e^{R(a^*(x)U+b^*(x)V)} \\
&\quad = -\frac{\partial\theta}{\partial R}(R; a^*(x), b^*(x)).
\end{aligned} \tag{4.20}$$

From the convexity property of $\theta(r; a^*(x), b^*(x))$ about r, we know that $\theta_r'(R; a^*(x), b^*(x)) > 0$. This implies that $\mathrm{P}^*[\tau^* < \infty] = 1$, and

$$\psi(u) = \mathrm{E}^*\left[e^{RR_{\tau^*}+\int_0^{\tau^*}\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s}\right]e^{-Ru}. \tag{4.21}$$

The following theorem gives a lower bound for $\psi(u)$.

Theorem 4.5. *Let*

$$C_- = \inf_z \frac{1}{E\left[e^{R(U+V-z)} \mid U+V>z\right]}, \tag{4.22}$$

where z is taken over the set $\{z : P[U+V>z]>0\}$. Then $\psi(u) \geq C_- e^{-Ru}$.

Proof. Suppose that $R_{\tau^*-} = z$, then

$$\begin{aligned}
&\mathrm{E}^*\left[\exp\{RR_{\tau^*}\} \mid R_{\tau^*-} = z\right] \\
&\quad = \mathrm{E}^*\left[\exp\{R(z - a^*(z)U - b^*(z)V)\} \mid a^*(z)U + b^*(z)V > z\right] \\
&\quad = \frac{1}{\mathrm{E}\left[\exp\{R((a^*(z)U + b^*(z)V) - z)\} \mid a^*(z)U + b^*(z)V > z\right]} \\
&\quad = \frac{1}{\mathrm{E}\left[\exp\left\{Ra^*(z)b^*(z)\left[\dfrac{U}{b^*(z)} + \dfrac{V}{a^*(z)} - \dfrac{z}{a^*(z)b^*(z)}\right]\right\} \mid \dfrac{U}{b^*(z)} + \dfrac{V}{a^*(z)} > \dfrac{z}{a^*(z)b^*(z)}\right]} \\
&\quad \geq \inf_{a,b}\frac{1}{\mathrm{E}\left[\exp\left\{Rab\left[\dfrac{U}{b^*(z)} + \dfrac{V}{a^*(z)} - \dfrac{z}{a^*(z)b^*(z)}\right]\right\} \mid \dfrac{U}{b^*(z)} + \dfrac{V}{a^*(z)} > \dfrac{z}{a^*(z)b^*(z)}\right]}.
\end{aligned} \tag{4.23}$$

Thus,

$$
\begin{aligned}
&\mathrm{E}^*\left[\exp\{RR_{\tau^*}\}\right] \\
&\quad\geq \inf_{a,b,z}\frac{1}{\mathrm{E}\left[\exp\left\{R\left[\dfrac{ab}{b^*(z)}U+\dfrac{ab}{a^*(z)}V-\dfrac{ab}{a^*(z)b^*(z)}z\right]\right\}\mid \dfrac{ab}{b^*(z)}U+\dfrac{ab}{a^*(z)}V>\dfrac{ab}{a^*(z)b^*(z)}z\right]} \\
&\quad\geq \inf_{a,b,z}\frac{1}{\mathrm{E}\left[\exp\{R[abU+abV-abz]\}\mid abU+abV>abz\right]} \\
&\quad\geq \inf_{z}\frac{1}{\mathrm{E}\left[\exp\{R[U+V-z]\}\mid U+V>z\right]}=C_-.
\end{aligned}
\tag{4.24}
$$

Then $\psi(u)\geq \mathrm{E}^*[\exp\{RR_{\tau^*}\}]e^{-Ru}\geq C_-e^{-Ru}$. □

5. The Cramér-Lundberg Approximation

In this section we will consider the asymptotic behavior of $\psi(x)e^{Rx}$, called Cramér-Lundberg approximation. First from the Fubini's theorem, we transform the expression below:

$$
\begin{aligned}
&\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)}\psi(x-a^*(x)u-b^*(x)v)\mathrm{d}G(u,v) \\
&\quad=\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)}\left(\int_0^{x-a^*(x)u-b^*(x)v}\psi'(z)\mathrm{d}z+\psi(0)\right)\mathrm{d}G(u,v) \\
&\quad=\psi(0)\mathrm{P}[a^*(x)U+b^*(x)V<x]+\int_0^x\int_0^{x-r}\psi'(z)\mathrm{d}z\mathrm{d}G_{a^*(x)U+b^*(x)V}(r) \\
&\quad=\psi(0)\mathrm{P}[a^*(x)U+b^*(x)V<x]+\int_0^x\int_0^{x-z}\mathrm{d}G_{a^*(x)U+b^*(x)V}(r)\psi'(z)\mathrm{d}z \\
&\quad=\psi(0)\mathrm{P}[a^*(x)U+b^*(x)V<x]+\int_0^x\mathrm{P}(a^*(x)U+b^*(x)V<x-z)\psi'(z)\mathrm{d}z \\
&\quad=\psi(0)\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)}\mathrm{d}G(u,v) \\
&\qquad+\int_0^x\left(\int_0^{(x-z)/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)}\mathrm{d}G(u,v)\right)\psi'(z)\mathrm{d}z \\
&\quad=\psi(0)\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)}\mathrm{d}G(u,v)+\int_0^x\psi'(z)G_{a^*(x)U+b^*(x)V}(x-z)\mathrm{d}z.
\end{aligned}
\tag{5.1}
$$

Because $\psi(x) = 1 - \delta(x)$, then the HJB equation can be changed into

$$\begin{aligned}
&(c_1(a^*(x)) + c_2(b^*(x)))\psi'(x) \\
&\quad + \lambda\Big[1 - G_{a^*(x)U+b^*(x)V}(x) - \psi(0)\big(1 - G_{a^*(x)U+b^*(x)V}(x)\big) \\
&\quad - \int_0^x \psi'(z)\big(1 - G_{a^*(x)U+b^*(x)V}(x-z)\big)\mathrm{d}z\Big] = 0.
\end{aligned} \tag{5.2}$$

Let $f(x) = \psi(x)e^{Rx}$, then $\psi'(x)e^{Rx} = f'(x) - Rf(x)$, and

$$\begin{aligned}
&[c_1(a^*(x)) + c_2(b^*(x))](f'(x) - Rf(x)) \\
&\quad + \lambda\Big[\delta(0)\big(1 - G_{a^*(x)U+b^*(x)V}(x)\big)e^{Rx} \\
&\quad + \int_0^x (Rf(z) - f'(z))\big(1 - G_{a^*(x)U+b^*(x)V}(x-z)\big)e^{R(x-z)}\mathrm{d}z\Big] = 0.
\end{aligned} \tag{5.3}$$

Because $\psi(x)$ is strictly decreasing, then $\psi'(x)e^{Rx} < 0$. So $f'(x) < Rf(x)$. Thus $f'(x)$ is bounded from above. Let $g(x) = Rf(x) - f'(x)$, we get

$$\begin{aligned}
&\lambda\Big[\delta(0)\big(1 - G_{a^*(x)U+b^*(x)V}(x)\big)e^{Rx} + \int_0^x g(z)\big(1 - G_{a^*(x)U+b^*(x)V}(x-z)\big)e^{R(x-z)}\mathrm{d}z\Big] \\
&\quad - g(x)[c_1(a^*(x)) + c_2(b^*(x))] = 0.
\end{aligned} \tag{5.4}$$

Changing the order of the integral, we have

$$\begin{aligned}
&\lambda\Big[\delta(0)\big(1 - G_{a^*(x)U+b^*(x)V}(x)\big)e^{Rx} + \int_0^x g(x-y)\big(1 - G_{a^*(x)U+b^*(x)V}(y)\big)e^{Ry}\mathrm{d}y\Big] \\
&\quad - g(x)[c_1(a^*(x)) + c_2(b^*(x))] = 0.
\end{aligned} \tag{5.5}$$

If we replace $a^*(x)$, $b^*(x)$ by $\tilde{a}$, $\tilde{b}$, we will obtain the inequality

$$\begin{aligned}
&\lambda\Big[\delta(0)\big(1 - G_{\tilde{a}U+\tilde{b}V}(x)\big)e^{Rx} + \int_0^x g(x-y)\big(1 - G_{\tilde{a}U+\tilde{b}V}(y)\big)e^{Ry}\mathrm{d}y\Big] \\
&\quad - g(x)\Big[c_1(\tilde{a}) + c_2\big(\tilde{b}\big)\Big] \geq 0.
\end{aligned} \tag{5.6}$$

Note that

$$\mathrm{E}e^{R(\tilde{a}U+\tilde{b}V)} = \int_0^\infty \int_0^\infty e^{R(\tilde{a}u+\tilde{b}v)}\mathrm{d}G(u,v) = \int_0^\infty e^{Rx}\mathrm{d}G_{\tilde{a}U+\tilde{b}V}(x) = 1 + R\int_0^\infty e^{Ry}(1 - G_{\tilde{a}U+\tilde{b}V}(y))\mathrm{d}y. \quad (5.7)$$

From the definition of $\tilde{a}$ and $\tilde{b}$,

$$\lambda\left[\mathrm{E}e^{R(\tilde{a}U+\tilde{b}V)} - 1\right] = \left(c_1(\tilde{a}) + c_2\left(\tilde{b}\right)\right)R. \quad (5.8)$$

Thus $c_1(\tilde{a}) + c_2(\tilde{b}) = \lambda \int_0^\infty e^{Ry}(1 - G_{\tilde{a}U+\tilde{b}V}(y))\mathrm{d}y$. Take the expression of $c_1(\tilde{a}) + c_2(\tilde{b})$ into the above inequality, and obtain

$$\lambda\left[\delta(0)(1 - G_{\tilde{a}U+\tilde{b}V}(x))e^{Rx} + \int_0^x g(x-y)(1 - G_{\tilde{a}U+\tilde{b}V}(y))e^{Ry}\mathrm{d}y\right] - g(x)\lambda\int_0^\infty e^{Ry}(1 - G_{\tilde{a}U+\tilde{b}V}(y))\mathrm{d}y \ge 0. \quad (5.9)$$

After transforming

$$\int_0^x [g(x-y) - g(x)]\left(1 - G_{\tilde{a}U+\tilde{b}V}(y)\right)e^{Ry}\mathrm{d}y \ge \int_x^\infty (1 - G_{\tilde{a}U+\tilde{b}V}(y))e^{Ry}\mathrm{d}y \cdot g(x) - \delta(0)(1 - G_{\tilde{a}U+\tilde{b}V}(x))e^{Rx}. \quad (5.10)$$

From Lemma A.12 in Schmidli [13], we know $\lim_{x\to\infty}(1 - G_{\tilde{a}U+\tilde{b}V}(x))e^{Rx} = 0$. First we consider two functions $f(x) = \varphi(x)e^{Rx}$ and $g(x) = Rf(x) - f'(x)$, which are important in investigating the Cramér-Lundberg approximation. Repeating the proof of Lemma 4.10 in Schmidli [13] (note (5.10) will be uesd in the proof) gives the analogous results.

Lemma 5.1.

(a) $g(x)$ is bounded. In particular, $f'(x)$ is bounded.

(b) Let $\xi = \limsup_{x\to\infty} g(x)/R$, then $\limsup_{x\to\infty} f(x) = \xi$. In particular, $\xi > 0$ if $C_- > 0$.

(c) For any $\beta > 0$, $x_0 > 0$, and $\varepsilon > 0$, there is an $x \ge x_0$ such that $f(y) > \xi - \varepsilon$ for $y \in [x-\beta, x]$.

The main result of this section is as follows.

Theorem 5.2. *Suppose that $C_- > 0$. Then $\lim_{u\to\infty}\psi(u)e^{Ru} = \xi > 0$, where ξ is defined in Lemma 5.1.*

Proof. Choose $\beta > 0$, $\varepsilon > 0$. There exists $x_0 \geq \beta$ such that $f(x) > \xi - \varepsilon$ for $x \in [x_0 - \beta, x_0]$. If $x \geq 2x_0$ and define $T = \inf\{t > 0, R_t < x_0\}$, then

$$\begin{aligned} f(x) &= \mathrm{E}^*\left[e^{RR_{\tau^*}+\int_0^{\tau^*}\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s}\right] \\ &= \mathrm{E}^*\left[\mathrm{E}^*\left[e^{RR_{\tau^*}+\int_0^{\tau^*}\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s} \mid \mathcal{F}_T\right]\right] \\ &= \mathrm{E}^*\left[\mathrm{E}^*\left[e^{RR_{\tau^*}+\int_T^{\tau^*}\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s} \mid R_T\right]e^{\int_0^T\theta(R;a^*(R_s),b^*(R_s))\mathrm{d}s}\right] \\ &\geq \mathrm{E}^*\left[f(R(T))\right] \geq \mathrm{E}^*\left[f(R(T))1_{x_0-\beta\leq R_T\leq x_0}\right] \\ &> (\xi-\varepsilon)\mathrm{P}^*\left[x_0 - R_T \leq \beta\right]. \end{aligned} \tag{5.11}$$

By choosing β appropriately, we can get $\mathrm{P}^*[x_0 - R_T \leq \beta] > 1 - \varepsilon$. Then $f(x) > (\xi - \varepsilon)(1 - \varepsilon)$. Thus

$$\liminf_{x\to\infty} f(x) \geq \xi = \limsup_{x\to\infty} f(x). \tag{5.12}$$

By Lemma 5.1, this theorem can be proved. □

6. Convergence of the Strategies

After discussing the asymptotic behavior of $\psi(u)e^{Ru}$, in this section we will study the behavior of the optimal strategies (a^*, b^*) when the capital is large enough. If the optimal strategies converge, then using the convergent limit value we can obtain the asymptotic behavior of the optimal ruin probability. The following theorem indicates the convergence of (a^*, b^*).

Theorem 6.1. *Suppose that $C_- > 0$. Then $\lim_{x\to\infty} f'(x) = 0$. Moreover if $(\tilde{a}, \tilde{b})$ is unique, then $\lim_{x\to\infty} a^*(x) = \tilde{a}$, $\lim_{x\to\infty} b^*(x) = \tilde{b}$.*

Proof. First we replace $\psi(x)$ by $f(x)e^{-Rx}$ in the HJB equation to get

$$\begin{aligned} &[c_1(a^*(x)) + c_2(b^*(x))]\left[f'(x) - Rf(x)\right] \\ &\quad - \lambda\left[G_{a^*(x)U+b^*(x)V}(x)e^{R(x)} - \int_0^x f(x-y)e^{Ry}\mathrm{d}G_{a^*(x)U+b^*(x)V}(y) - e^{Rx} + f(x)\right] = 0. \end{aligned} \tag{6.1}$$

That is

$$\begin{aligned} &\lambda\left[\int_0^x f(x-y)e^{Ry}\mathrm{d}G_{a^*(x)U+b^*(x)V}(y) - f(x)\right] + \lambda e^{Rx}\left(1 - G_{a^*(x)U+b^*(x)V}(x)\right) \\ &\quad + [c_1(a^*(x)) + c_2(b^*(x))]f'(x) - [c_1(a^*(x)) + c_2(b^*(x))]Rf(x) = 0. \end{aligned} \tag{6.2}$$

Then

$$\begin{aligned}
&[c_1(a^*(x)) + c_2(b^*(x))]f'(x)\\
&\quad = -\lambda\left[\int_0^x f(x-y)e^{Ry}\mathrm{d}G_{a^*(x)U+b^*(x)V}(y) - f(x) - \xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right)\right]\\
&\qquad - \lambda\xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right) + [c_1(a^*(x)) + c_2(b^*(x))]Rf(x)\\
&\qquad - [c_1(a^*(x)) + c_2(b^*(x))]R\xi + [c_1(a^*(x))c_2(b^*(x))]R\xi - \lambda e^{Rx}\left(1 - G_{a^*(x)U+b^*(x)V}(x)\right)\\
&\quad = -\lambda\left[\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)} f(x - a^*(x)u - b^*(x)v)e^{-R(a^*(x)u+b^*(x)v)}\mathrm{d}G(u,v) - f(x)\right.\\
&\qquad\qquad \left. -\xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right)\right] - \xi\theta(R;a^*(x),b^*(x))\\
&\qquad + [c_1(a^*(x)) + c_2(b^*(x))]R\left(f(x) - \xi\right) - \lambda e^{Rx}\left(1 - G_{a^*(x)U+b^*(x)V}(x)\right)\\
&\quad < \lambda\left|\left[\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)} f(x - a^*(x)u - b^*(x)v)e^{-R(a^*(x)u+b^*(x)v)}\mathrm{d}G(u,v) - f(x)\right.\right.\\
&\qquad\qquad \left.\left. -\xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right)\right]\right|\\
&\qquad + \left|[c_1(a^*(x)) + c_2(b^*(x))]R\left(f(x) - \xi\right)\right| - \xi\theta(R;a^*(x),b^*(x)).
\end{aligned} \tag{6.3}$$

Note that when $x \to \infty$

$$\begin{gathered}
\left|\left[\int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)} f(x - a^*(x)u - b^*(x)v)e^{-R(a^*(x)u+b^*(x)v)}\mathrm{d}G(u,v) - f(x)\right.\right.\\
\left.\left. -\xi\left(\mathrm{E}e^{R(a^*(x)U+b^*(x)V)} - 1\right)\right]\right| < \frac{\varepsilon}{4},\\
\left|[c_1(a^*(x)) + c_2(b^*(x))]R\left(f(x) - \xi\right)\right| < \frac{\varepsilon}{4}.
\end{gathered} \tag{6.4}$$

Then

$$[c_1(a^*(x)) + c_2(b^*(x))]f'(x) < -\xi\theta(R;a^*(x),b^*(x)) + \frac{\varepsilon}{2}. \tag{6.5}$$

If for each $\varepsilon > 0$, there exists x_0 such that $c_1(a^*(x_0)) + c_2(b^*(x_0)) < \varepsilon$. Because $\theta(R;a^*(x_0), b^*(x_0)) > 0$, and $f'(x)$ is bounded, under this case we cannot get (6.5). So $c_1(a^*(x)) + c_2(b^*(x))$ cannot be arbitrary small. That means $c_1(a^*(x)) + c_2(b^*(x))$ are bounded away from 0. Therefore $\limsup_{x\to\infty} f'(x) \le 0$. Clearly, we have $\liminf_{x\to\infty} f'(x) = R\xi - \limsup_{x\to\infty} g(x) = 0$.

Thus $\lim_{x\to\infty} f'(x) = 0$. If x is large enough, we get $\lambda e^{Rx}(1 - \int_0^{x/a^*(x)}\int_0^{(x-a^*(x)u)/b^*(x)} dG(u,v)) < \varepsilon/4$ and $[c_1(a^*(x)) + c_2(b^*(x))]|f'(x)| < \varepsilon/4$. Thus we have

$$-\frac{\varepsilon}{4} < [c_1(a^*(x)) + c_2(b^*(x))]f'(x) < -\xi\theta(R; a^*(x), b^*(x)) + \frac{3\varepsilon}{4}, \tag{6.6}$$

which is equal to

$$0 \le \xi\theta(R; a^*(x), b^*(x)) < \varepsilon. \tag{6.7}$$

This proves that $\lim_{x\to\infty}\theta(R; a^*(x), b^*(x)) = 0$. If $(\tilde{a}, \tilde{b})$ is unique, this is only possible if $\lim_{x\to\infty} a^*(x) = \tilde{a}$, $\lim_{x\to\infty} b^*(x) = \tilde{b}$. □

7. Example

To the multidimensional risk model, it seems impossible to get a closed form solution for the optimal ruin probability $\psi(u)$. In this section, from a numerical example, we will give an explicit procedure to obtain an exponential upper bound of $\psi(u)$ and the asymptotic optimal reinsurance strategies.

Example 7.1. Suppose that U_n and V_n are independent. The distribution function of them are given by $F_U = 1 - e^{-2x}$ and $F_V = 1 - e^{-x}$, respectively. So the joint distribution function of (U_n, V_n) is $G(x, y) = (1 - e^{-2x})(1 - e^{-y})$, and the joint density function is $p(x, y) = 2e^{-2x-y}$. Then $\mu_1 = \mathrm{E}U_n = 1/2$ and $\mu_2 = \mathrm{E}V_n = 1$. Let $\lambda = 1$. The expected value principle is used for our premium. Suppose that the relative safety loading for each subportfolios from the insurer point of view $\theta_1 = \theta_2 = 0.5$, and from the reinsurer $\eta_1 = \eta_2 = 0.7$. So $c_1(a) = (1.7a - 0.2)/2$, $c_2(b) = 1.7b - 0.2$.

Theorem 4.2 shows us that e^{-Ru} is an exponential type upper bound for $\psi(u)$. R can be get from $R = \sup_{(a,b)\in U} R(a, b)$, where $R(a, b)$ satisfied (4.1), that is, $\lambda(\mathrm{E}e^{R(a,b)(aU+bV)} - 1) = (c_1(a) + c_2(b))R(a, b)$. We can easily get when $\tilde{a} = 0.77$ and $\tilde{b} = 0.38$, $R(a, b)$ solved from previous equation reaches the maximum $R = 0.4194$. So $\psi(u) \le e^{-0.4194u}$. Moreover, $\tilde{a}$, $\tilde{b}$ work well as the optimal reinsurance constant strategies for "large" capital according to Theorem 6.1. So the asymptotic optimal constant strategies are $(\tilde{a}, \tilde{b}) = (0.77, 0.38)$.

Remark 7.2. When considering the two-dimensional risk model without dynamic control, the problem of the sum of two subportfolio indeed can be convert back to the one-dimensional case (e.g., Yuen et al. [18]). If we consider the dynamic proportional reinsurance in the two-dimensional compound Poisson risk model again from the this point, then the aggregate process $\tilde{R}_t$ is as follows:

$$\tilde{R}_t = X_t + Y_t = u + (c_1 + c_2)t - \sum_{n=1}^{N_t}(U_n + V_n). \tag{7.1}$$

We consider the dynamic proportional reinsurance strategy $\{\alpha_t\}$ on the one-dimensional risk model $\widetilde{R}_t$:

$$\widetilde{R}_t^{\alpha} = u + \int_0^t (c_1(\alpha_s) + c_2(\alpha_s))\mathrm{d}s - \sum_{n=1}^{N_t} \alpha_{\sigma_n-}(U_n + V_n). \tag{7.2}$$

The optimal reinsurance in one-dimensional case had been discussed in Schmidli [5]. So from the equation $\lambda(\mathrm{E}e^{R_1(\alpha)\alpha(U+V)} - 1) = (c_1(\alpha) + c_2(\alpha))R_1(\alpha)$ for some fixed α, we can calculate the maximal adjustment coefficient $R_1 = 0.40412$.

Obviously, $R_1 < R$. From the point of comparing the upper bound of $\psi(u)$, this tells us that from the two-dimensional point of view to consider the dynamic proportional reinsurance strategies for each subportfolio is better than considering the strategy just for the aggregate portfolio.

Remark 7.3. Another approach based on one-dimensional risk model to deal with our problem is that we may view each subportfolio as a one-dimensional case and discuss them, respectively. So we can handle the example as follows. First, we consider the subportfolio $\{X_t^a\}$. Similar to Schmidli [5], from $\lambda(\mathrm{E}e^{R(a)aU} - 1) = c_1(a)R(a)$ we derive the asymptotic optimal constant reinsurance strategy $\widetilde{a} = 0.504847$ for $\{X_t^a\}$. Meanwhile using the same way to $\{Y_t^b\}$, we can get the asymptotic optimal reinsurance strategy $\widetilde{b} = 0.504847$ for $\{Y_t^b\}$. Till now, we have get a constant strategy $(\widetilde{a}, \widetilde{b}) = (0.504847, 0.504847)$. Next we think over the sum of two subportfolio, that is, $R_t^{\widetilde{a},\widetilde{b}} = X_t^{\widetilde{a}} + Y_t^{\widetilde{b}}$. From $\lambda(\mathrm{E}e^{R_2(\widetilde{a},\widetilde{b})(\widetilde{a}U+\widetilde{b}V)} - 1) = (c_1(\widetilde{a}) + c_2(\widetilde{b}))R_2(\widetilde{a}, \widetilde{b})$, the adjustment coefficient $R_2 = 0.40408$ for $\{R_t^{\widetilde{a},\widetilde{b}}\}$ can be obtained.

We find that $R_2 < R$. This implies that even though $\widetilde{a}$ and $\widetilde{b}$ are the asymptotic optimal constant strategy for $\{X_t^a\}$ and $\{Y_t^b\}$, respectively, under $(\widetilde{a}, \widetilde{b})$ the upper bound ruin probability of $\{R_t^{\widetilde{a},\widetilde{b}}\}$ is not optimized at the same time.

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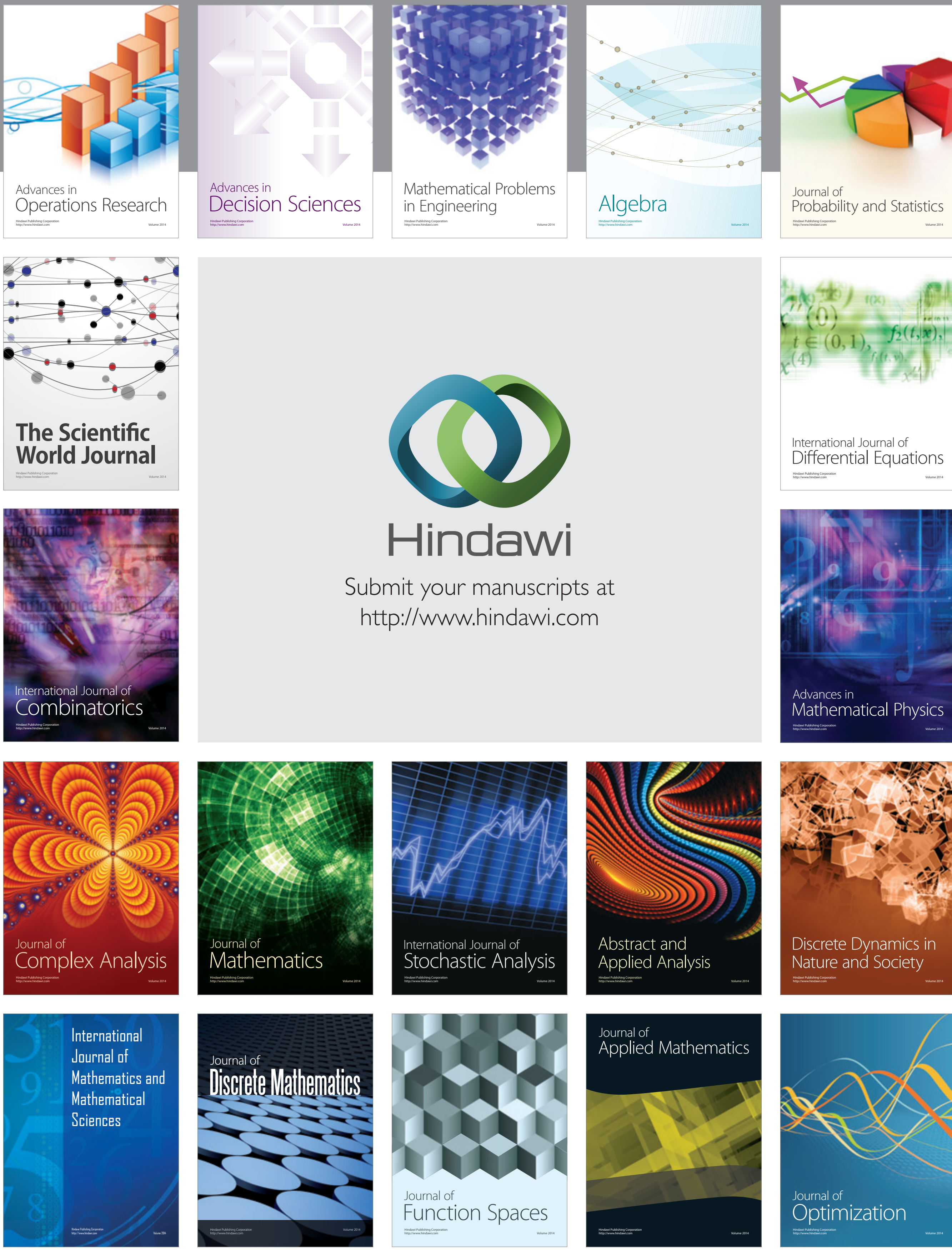
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