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Research Article

On Complete Convergence of Moving Average Process for AANA Sequence

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We investigate the moving average process such that $X_n = \sum_{i=1}^{\infty} a_i Y_{i+n}$, $n \ge 1$, where $\sum_{i=1}^{\infty} |a_i| < \infty$ and $\{Y_i, 1 \le i < \infty\}$ is a sequence of asymptotically almost negatively associated (AANA) random variables. The complete convergence, complete moment convergence, and the existence of the moment of supermum of normed partial sums are presented for this moving average process.

1. Introduction

We assume that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed random variables with $E|Y_1| < \infty$. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, \quad n \ge 1 \tag{1.1}$$

be the *moving average process* based on the sequence $\{Y_i, -\infty < i < \infty\}$. As usual, $S_n = \sum_{k=1}^{n} X_k$, $n \ge 1$ denotes the sequence of partial sums.

Under the assumption that $\{Y_i, -\infty < i < \infty\}$ is a sequence of independent identically distributed random variables, various results of the moving average process $\{X_n, n \ge 1\}$ have been obtained. For example, Ibragimov [1] established the central limit theorem, Burton and Dehling [2] obtained a large deviation principle, and Li et al. [3] gave the complete convergence result for $\{X_n, n \ge 1\}$.

Many authors extended the complete convergence of moving average process to the case of dependent sequences, for example, Zhang [4] for φ-mixing sequence, Li and Zhang [5] for NA sequence. The following Theorems A and B are due to Zhang [4] and Kim et al. [6], respectively.

Theorem A. *Suppose that* $\{Y_i,\ -\infty < i < \infty\}$ *is a sequence of identically distributed* φ*-mixing random variables with* $\sum_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$ *and* $\{X_n,\ n \geq 1\}$ *is as in* (1.1). *Let* $h(x) > 0\ (x > 0)$ *be a slowly varying function and* $1 \leq p < 2$, $r \geq 1$. *If* $Y_1 = 0$ *and* $E|Y_1|^{rp} h(|Y_1|^p) < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\left(|S_n| \geq \varepsilon n^{1/p}\right) < \infty, \quad \forall \varepsilon > 0. \tag{1.2}$$

Theorem B. *Suppose that* $\{Y_i,\ -\infty < i < \infty\}$ *is a sequence of identically distributed* φ*-mixing random variables with* $EY_1 = 0$, $EY_1^2 < \infty$ *and* $\sum_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$ *and* $\{X_n,\ n \geq 1\}$ *is as in* (1.1). *Let* $h(x) > 0\ (x > 0)$ *be a slowly varying function and* $1 \leq p < 2$, $r > 1$. *If* $E|Y_1|^{rp} h(|Y_1|^p) < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{r-2-1/p} h(n) E\left(|S_n| - \varepsilon n^{1/p}\right)^{+} < \infty, \quad \forall \varepsilon > 0, \tag{1.3}$$

where $x^{+} = \max\{x, 0\}$.

Chen and Gan [7] investigated the moments of maximum of normed partial sums of ρ-mixing random variables and gave the following result.

Theorem C. *Let* $0 < r < 2$ *and* $p > 0$. *Assume that* $\{X_n,\ n \geq 1\}$ *is a mean zero sequence of identically distributed* ρ*-mixing random variables with the maximal correlation coefficient rate* $\sum_{n=1}^{\infty} \rho^{2/s}(2^n) < \infty$, *where* $s = 2$ *if* $p < 2$ *and* $s > p$ *if* $p \geq 2$. *Denote* $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. *Then*

$$\begin{cases} E|X_1|^r < \infty, & if\ p < r, \\ E\left[|X_1|^r \log(1 + |X_1|)\right] < \infty, & if\ p = r, \\ E|X_1|^p < \infty, & if\ p > r, \end{cases}$$
$$E\left(\sup_{n \geq 1} \left|\frac{X_n}{n^{1/r}}\right|^p\right) < \infty, \tag{1.4}$$
$$E\left(\sup_{n \geq 1} \left|\frac{S_n}{n^{1/r}}\right|^p\right) < \infty$$

are all equivalent.

Chen et al. [8] and Zhou [9] also studied limit behavior of moving average process under φ-mixing assumption. For more related details of complete convergence, one can refer to Hsu and Robbins [10], Chow [11], Shao [12], Li et al. [3], Zhang [4], Li and Zhang [5], Chen and Gan [7], Kim et al. [6], Sung [13–15], Chen and Li [16], Zhou and Lin [17], and so forth.

Inspired by Zhang [4], Kim et al. [6], Chen and Gan [7], Sung [13–15], and other papers above, we investigate the limit behavior of moving average process under AANA sequence, which is weaker than NA and obtain some similar results of Theorems A, B, and C. The main results can be seen in Section 2 and their proofs are given in Section 3.

Recall that the sequence $\{X_n,\ n \ge 1\}$ is stochastically dominated by a nonnegative random variable X if

$$\sup_{n\ge1} P(|X_n| > t) \le CP(X > t) \quad \text{for some positive constant } C \text{ and } \forall t \ge 0. \tag{1.5}$$

Definition 1.1. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\left\{f(X_i : i \in A_1),\ g\left(X_j : j \in A_2\right)\right\} \le 0, \tag{1.6}$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists.

An infinite sequence $\{X_n,\ n \ge 1\}$ is NA if every finite subcollection is NA.

Definition 1.2. A sequence $\{X_n,\ n \ge 1\}$ of random variables is called asymptotically almost negatively associated (AANA) if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\operatorname{Cov}\left(f(X_n), g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})\right) \le q(n)\left[\operatorname{Var}\left(f(X_n)\right) \operatorname{Var}\left(g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})\right)\right]^{1/2}, \tag{1.7}$$

for all $n, k \ge 1$ and for all coordinate-wise nondecreasing continuous functions f and g whenever the variances exist.

The concept of NA sequence was introduced by Joag-Dev and Proschan [18]. For the basic properties and inequalities of NA sequence, one can refer to Joag-Dev and Proschan [18] and Matula [19]. The family of AANA sequence contains NA (in particular, independent) sequence (with $q(n) = 0,\ n \ge 1$) and some more sequences of random variables which are not much deviated from being negatively associated. An example of an AANA which is not NA was constructed by Chandra and Ghosal [20, 21]. For various results and applications of AANA random variables can be found in Chandra Ghosal [21], Wang et al. [22], Ko et al. [23], Yuan and An [24], and Wang et al. [25, 26] among others.

For simplicity, in this paper we consider the moving average process:

$$X_n = \sum_{i=1}^{\infty} a_i Y_{i+n}, \quad n \ge 1, \tag{1.8}$$

where $\sum_{i=1}^{\infty} |a_i| < \infty$ and $\{Y_i,\ 1 \le i < \infty\}$ is a mean zero sequence of AANA random variables.

The following lemmas are our basic techniques to prove our results.

Lemma 1.3. *Let $\{X_n,\ n \ge 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n),\ n \ge 1\}$. If $f_1, f_2, \ldots$ are all nondecreasing (or nonincreasing) continuous functions, then $\{f_n(X_n),\ n \ge 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{q(n),\ n \ge 1\}$.*

Remark 1.4. Lemma 1.3 comes from Lemma 2.1 of Yuan and An [24], but the functions of $f_1, f_2, \ldots$ in Lemma 2.1 of Yuan and An [24] are written to be all nondecreasing (or nonincreasing) functions. According to the definition of AANA, $f_1, f_2, \ldots$ should be all nondecreasing (or nonincreasing) continuous functions.

Lemma 1.5 (cf. Wang et al. [25, Lemma 1.4]). *Let $1 < p \le 2$ and $\{X_n,\ n \ge 1\}$ be a mean zero sequence of AANA random variables with mixing coefficients $\{q(n),\ n \ge 1\}$. If $\sum_{n=1}^{\infty} q^2(n) < \infty$, then there exists a positive constant C_p depending only on p such that*

$$E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \le C_p \sum_{i=1}^{n} E|X_i|^p, \tag{1.9}$$

for all $n \ge 1$, where $C_p = 2^p[2^{2-p} + (6p)^p(\sum_{n=1}^{\infty} q^2(n))^{p-1}]$.

Lemma 1.6 (cf. Wu [27, Lemma 4.1.6]). *Let $\{X_n,\ n \ge 1\}$ be a sequence of random variables, which is stochastically dominated by a nonnegative random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$\begin{gathered} E\left[|X_n|^\alpha I(|X_n| \le b)\right] \le C_1\{[EX^\alpha I(X \le b)] + b^\alpha P(X > b)\}, \\ E\left[|X_n|^\alpha I(|X_n| > b)\right] \le C_2 E[X^\alpha I(X > b)], \end{gathered} \tag{1.10}$$

where C_1 and C_2 are positive constants.

Throughout the paper, $I(A)$ is the indicator function of set A, $x^+ = \max\{x, 0\}$ and C, $C_1, C_2, \ldots$ denote some positive constants not depending on n, which may be different in various places.

2. The Main Results

Theorem 2.1. *Let $r > 1$, $1 \le p < 2$ and $rp < 2$. Assume that $\{X_n,\ n \ge 1\}$ is a moving average process defined in (1.8), where $\{Y_i, 1 \le i < \infty\}$ is a mean zero sequence of AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$ and stochastically dominated by a nonnegative random variable Y. If $EY^{rp} < \infty$, then for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1\le k\le n}|S_k| > \varepsilon n^{1/p}\right) < \infty, \tag{2.1}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\sup_{k\ge n}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon\right) < \infty. \tag{2.2}$$

Theorem 2.2. *Let the conditions of Theorem 2.1 hold. Then for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\le k\le n}|S_k| - \varepsilon n^{1/p}\right)^+ < \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^{r-2} E\left(\sup_{k\ge n}\left|\frac{S_k}{k^{1/p}}\right| - \varepsilon\right)^+ < \infty. \tag{2.4}$$

Theorem 2.3. *Let $0 < r < 2$ and $0 < p < 2$. Assume that $\{X_n,\ n \ge 1\}$ is a moving average process defined in* (1.8), *where $\{Y_i,\ 1 \le i < \infty\}$ is a mean zero sequence of AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$ and stochastically dominated by a nonnegative random variable Y with $EY < \infty$. Suppose that*

$$\begin{cases} for\ p < r, & \begin{cases} E[Y\log(1+Y)] < \infty, & if\ r = 1, \\ EY^r < \infty, & if\ r > 1, \end{cases} \\ for\ p = r, & \begin{cases} E[Y\log(1+Y)] < \infty, & if\ 0 < r < 1, \\ E\left[Y\log^2(1+Y)\right] < \infty, & if\ r = 1, \\ E[Y^r\log(1+Y)] < \infty, & if\ r > 1, \end{cases} \\ for\ p > r, & \begin{cases} E[Y\log(1+Y)] < \infty, & if\ p = 1, \\ EY^p < \infty, & if p > 1. \end{cases} \end{cases} \tag{2.5}$$

Then

$$E\left(\sup_{n\ge 1}\left|\frac{S_n}{n^{1/r}}\right|^p\right) < \infty. \tag{2.6}$$

3. The Proofs of Main Results

Proof of Theorem 2.1. Firstly, we show that the moving average process (1.8) converges almost surely under the conditions of Theorem 2.1. Since $rp > 1$, it has $EY < \infty$, following from the condition $EY^{rp} < \infty$. On the other hand, by Lemma 1.6 with $\alpha = 1$ and $b = 1$, one has

$$E|Y_i| \le 1 + C_2 E[YI(Y > 1)] \le 1 + C_2 EY < \infty, \quad 1 \le i < \infty. \tag{3.1}$$

Consequently, by the condition $\sum_{i=1}^{\infty}|a_i| < \infty$, we have that

$$\sum_{i=1}^{\infty} E|a_i Y_{i+n}| \le C_3 \sum_{i=1}^{\infty}|a_i| < \infty, \tag{3.2}$$

which implies $\sum_{i=1}^{\infty} a_i Y_{i+n}$ converges almost surely.

Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n}\sum_{i=1}^{\infty} a_i Y_{i+k} = \sum_{i=1}^{\infty} a_i \sum_{k=i+1}^{i+n} Y_k, \quad n \ge 1. \tag{3.3}$$

Since $r > 1$ and $EY^{rp} < \infty$, one has $EY^p < \infty$. Combining $EY^p < \infty$ with $EY_j = 0$, $\sum_{i=1}^{\infty} |a_i| < \infty$ and Lemma 1.6, we can find that

$$\begin{aligned}
&n^{-1/p} \max_{1\le k\le n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} E\left[Y_j I\left(\left|Y_j\right| \le n^{1/p}\right)\right]\right| \\
&\quad = n^{-1/p} \max_{1\le k\le n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} E\left[Y_j I\left(\left|Y_j\right| > n^{1/p}\right)\right]\right| \\
&\quad \le n^{-1/p} \sum_{i=1}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E\left[\left|Y_j\right| I\left(\left|Y_j\right| > n^{1/p}\right)\right] \\
&\quad \le CE\left[\left(n^{1/p}\right)^{p-1} Y I\left(Y > n^{1/p}\right)\right] \le CE\left[Y^p I\left(Y > n^{1/p}\right)\right] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.4}$$

Meanwhile,

$$\begin{aligned}
&n^{-1/p} \max_{1\le k\le n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} \left(-n^{1/p}\right) E\left[I\left(Y_j < -n^{1/p}\right)\right]\right| \\
&\quad \le n^{-1/p} \sum_{i=1}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E\left[\left|Y_j\right| I\left(\left|Y_j\right| > n^{1/p}\right)\right] \le CE\left[Y^p I\left(Y > n^{1/p}\right)\right] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\
&n^{-1/p} \max_{1\le k\le n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} n^{1/p} E\left[I\left(Y_j > n^{1/p}\right)\right]\right| \\
&\quad \le n^{-1/p} \sum_{i=1}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E\left[\left|Y_j\right| I\left(\left|Y_j\right| > n^{1/p}\right)\right] \le CE\left[Y^p I\left(Y > n^{1/p}\right)\right] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.5}$$

Let

$$\begin{gathered}
Y_{nj} = -n^{1/p} I\left(Y_j < -n^{1/p}\right) + Y_j I\left(\left|Y_j\right| \le n^{1/p}\right) + n^{1/p} I\left(Y_j > n^{1/p}\right), \quad j \ge 1, \\
\widetilde{Y}_{nj} = Y_{nj} - EY_{nj}, \quad j \ge 1.
\end{gathered} \tag{3.6}$$

Hence, for for all $\varepsilon > 0$, there exists an n_0 such that

$$n^{-1/p}\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}EY_{nj}\right| < \frac{\varepsilon}{4}, \quad n \ge n_0. \tag{3.7}$$

Denote

$$Y_{nj}^{*} = n^{1/p}I\left(Y_j < -n^{1/p}\right) - n^{1/p}I\left(Y_j > n^{1/p}\right) + Y_jI\left(\left|Y_j\right| > n^{1/p}\right), \quad j \ge 1. \tag{3.8}$$

Noting that $Y_j = Y_{nj}^{*} + Y_{nj}$, $j \ge 1$, we can find

$$\begin{aligned}
\sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}|S_k| > \varepsilon n^{1/p}\right) &\le \sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{nj}^{*}\right| > \frac{\varepsilon n^{1/p}}{2}\right)\\
&\quad + \sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{nj}\right| > \frac{\varepsilon n^{1/p}}{2}\right)\\
&\le \sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{nj}^{*}\right| > \frac{\varepsilon n^{1/p}}{2}\right)\\
&\quad + \sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}\widetilde{Y}_{nj}\right| > \frac{\varepsilon n^{1/p}}{4}\right)\\
&\quad + C + \sum_{n=n_0}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}EY_{nj}\right| > \frac{\varepsilon n^{1/p}}{4}\right)\\
&\le C + \sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{nj}^{*}\right| > \frac{\varepsilon n^{1/p}}{2}\right)\\
&\quad + \sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}\widetilde{Y}_{nj}\right| > \frac{\varepsilon n^{1/p}}{4}\right)\\
&=: C + I + J.
\end{aligned} \tag{3.9}$$

For I, by Markov's inequality, $\sum_{i=1}^{\infty}|a_i| < \infty, |Y_{nj}^*| \le |Y_j| I(|Y_j| > n^{1/p})$, Lemma 1.6 and $EY^{rp} < \infty$, it has

$$
\begin{aligned}
I &\le \frac{2}{\varepsilon}\sum_{n=1}^{\infty} n^{r-2} n^{-1/p} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^*\right|\right) \\
&\le C_1 \sum_{n=1}^{\infty} n^{r-2} n^{-1/p} \sum_{i=1}^{\infty} |a_i| E\left(\max_{1\le k\le n}\sum_{j=i+1}^{i+k}|Y_j| I\left(|Y_j| > n^{1/p}\right)\right) \\
&\le C_2 \sum_{n=1}^{\infty} n^{r-1-1/p} E\left[Y I\left(Y > n^{1/p}\right)\right] \\
&= C_2 \sum_{n=1}^{\infty} n^{r-1-1/p} \sum_{m=n}^{\infty} E[Y I(m < Y^p \le m+1)] \\
&= C_2 \sum_{m=1}^{\infty} E[Y I(m < Y^p \le m+1)] \sum_{n=1}^{m} n^{r-1-1/p} \\
&\le C_3 \sum_{m=1}^{\infty} m^{r-1/p} E[Y I(m < Y^p \le m+1)] \le C E Y^{rp} < \infty.
\end{aligned}
\tag{3.10}
$$

Since $f_j(x) = -n^{1/p} I(x < -n^{1/p}) + x I(|x| \le n^{1/p}) + n^{1/p} I(x > n^{1/p})$ is a nondecreasing continuous function of x, we can find by using Lemma 1.3 that $\{\widetilde{Y}_{nj}, 1 \le j < \infty\}$ is a mean zero AANA sequence and $E\widetilde{Y}_{nj}^2 \le EY_{nj}^2$, $Y_{nj}^2 = Y_j^2 I(|Y_j| \le n^{1/p}) + n^{2/p} I(|Y_j| > n^{1/p})$, $j \ge 1$. Consequently, by the property of AANA, Markov's inequality, Hölder's inequality, Lemma 1.5, C_r inequality, and Lemma 1.6, we can check that

$$
\begin{aligned}
J &\le \left(\frac{4}{\varepsilon}\right)^2 \sum_{n=1}^{\infty} n^{r-2} n^{-2/p} E\left\{\max_{1\le k\le n}\left(\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} \widetilde{Y}_{nj}\right)^2\right\} \\
&\le \left(\frac{4}{\varepsilon}\right)^2 \sum_{n=1}^{\infty} n^{r-2} n^{-2/p} E\left\{\sum_{i=1}^{\infty}\left(|a_i|^{1/2}\right)\left(|a_i|^{1/2}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\widetilde{Y}_{nj}\right|\right)\right\}^2 \\
&\le \left(\frac{4}{\varepsilon}\right)^2 \sum_{n=1}^{\infty} n^{r-2} n^{-2/p}\left(\sum_{i=1}^{\infty}|a_i|\right)^2 \sup_{i\ge 1} E\left\{\max_{1\le k\le n}\left(\sum_{j=i+1}^{i+k}\widetilde{Y}_{nj}\right)^2\right\} \\
&\le C_1 \sum_{n=1}^{\infty} n^{r-2} n^{-2/p} \sup_{i\ge 1}\sum_{j=i+1}^{i+n} E\widetilde{Y}_{nj}^2 \\
&\le C_2 \sum_{n=1}^{\infty} n^{r-2} n^{-2/p} \sup_{i\ge 1}\sum_{j=i+1}^{i+n}\left\{E\left[Y_j^2 I\left(|Y_j| \le n^{1/p}\right)\right] + n^{2/p} E\left[I\left(|Y_j| > n^{1/p}\right)\right]\right\}
\end{aligned}
$$

$$\begin{aligned}
&\leq C_3\sum_{n=1}^{\infty}n^{r-1-2/p}E\left[Y^2I\left(Y\leq n^{1/p}\right)\right]+C_4\sum_{n=1}^{\infty}n^{r-1}P\left(Y>n^{1/p}\right)\\
&\leq C_3\sum_{n=1}^{\infty}n^{r-1-2/p}E\left[Y^2I\left(Y\leq n^{1/p}\right)\right]+C_4\sum_{n=1}^{\infty}n^{r-1-1/p}E\left[YI\left(Y>n^{1/p}\right)\right]\\
&=: C_3J_1+C_4J_2.
\end{aligned}\tag{3.11}$$

Since $rp<2$, it can be seen by $EY^{rp}<\infty$ that

$$\begin{aligned}
J_1&=\sum_{n=1}^{\infty}n^{r-1-2/p}\sum_{i=1}^{n}E\left[Y^2I\left((i-1)^{1/p}<Y\leq i^{1/p}\right)\right]\\
&=\sum_{i=1}^{\infty}E\left[Y^2I\left((i-1)^{1/p}<Y\leq i^{1/p}\right)\right]\sum_{n=i}^{\infty}n^{r-1-2/p}\\
&\leq C_1\sum_{i=1}^{\infty}E\left[Y^{rp}Y^{2-rp}I\left((i-1)^{1/p}<Y\leq i^{1/p}\right)\right]i^{r-2/p}\leq C_2EY^{rp}<\infty.
\end{aligned}\tag{3.12}$$

By the proof of (3.10), we have $J_2\leq CEY^{rp}<\infty$. Therefore, (2.1) follows from (3.9), (3.10), (3.11), and (3.12).

Inspired by the proof of Theorem 12.1 of Gut [28], it can be checked that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{r-2}P\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right|>2^{2/p}\varepsilon\right)\\
&\quad=\sum_{m=1}^{\infty}\sum_{n=2^{m-1}}^{2^m-1}n^{r-2}P\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right|>2^{2/p}\varepsilon\right)\\
&\quad\leq 2^{2-r}\sum_{m=1}^{\infty}P\left(\sup_{k\geq 2^{m-1}}\left|\frac{S_k}{k^{1/p}}\right|>2^{2/p}\varepsilon\right)\sum_{n=2^{m-1}}^{2^m-1}2^{m(r-2)}\\
&\quad\leq 2^{2-r}\sum_{m=1}^{\infty}2^{m(r-1)}P\left(\sup_{k\geq 2^{m-1}}\left|\frac{S_k}{k^{1/p}}\right|>2^{2/p}\varepsilon\right)\\
&\quad=2^{2-r}\sum_{m=1}^{\infty}2^{m(r-1)}P\left(\sup_{l\geq m}\max_{2^{l-1}\leq k<2^l}\left|\frac{S_k}{k^{1/p}}\right|>2^{2/p}\varepsilon\right)\\
&\quad\leq 2^{2-r}\sum_{m=1}^{\infty}2^{m(r-1)}\sum_{l=m}^{\infty}P\left(\max_{1\leq k\leq 2^l}|S_k|>\varepsilon 2^{(l+1)/p}\right)\\
&\quad=2^{2-r}\sum_{l=1}^{\infty}P\left(\max_{1\leq k\leq 2^l}|S_k|>\varepsilon 2^{(l+1)/p}\right)\sum_{m=1}^{l}2^{m(r-1)}
\end{aligned}$$

$$\begin{aligned}
&\le C_1\sum_{l=1}^{\infty}2^{l(r-1)}P\left(\max_{1\le k\le 2^l}|S_k|>\varepsilon 2^{(l+1)/p}\right)\\
&=2^{2-r}C_1\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{l+1}-1}2^{(l+1)(r-2)}P\left(\max_{1\le k\le 2^l}|S_k|>\varepsilon 2^{(l+1)/p}\right)\\
&\le 2^{2-r}C_1\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{l+1}-1}n^{r-2}P\left(\max_{1\le k\le n}|S_k|>\varepsilon n^{1/p}\right)\quad(\text{since } r<2)\\
&\le 2^{2-r}C_1\sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}|S_k|>\varepsilon n^{1/p}\right).
\end{aligned}\tag{3.13}$$

Combining (2.1) with the inequality above, we obtain (2.2) immediately. □

Proof of Theorem 2.2. For all $\varepsilon>0$, it has

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{r-2-1/p}E\left(\max_{1\le k\le n}|S_k|-\varepsilon n^{1/p}\right)^{+}\\
&\quad=\sum_{n=1}^{\infty}n^{r-2-1/p}\int_0^{\infty}P\left(\max_{1\le k\le n}|S_k|-\varepsilon n^{1/p}>t\right)dt\\
&\quad=\sum_{n=1}^{\infty}n^{r-2-1/p}\int_0^{n^{1/p}}P\left(\max_{1\le k\le n}|S_k|-\varepsilon n^{1/p}>t\right)dt\\
&\qquad+\sum_{n=1}^{\infty}n^{r-2-1/p}\int_{n^{1/p}}^{\infty}P\left(\max_{1\le k\le n}|S_k|-\varepsilon n^{1/p}>t\right)dt\\
&\quad\le\sum_{n=1}^{\infty}n^{r-2}P\left(\max_{1\le k\le n}|S_k|>\varepsilon n^{1/p}\right)+\sum_{n=1}^{\infty}n^{r-2-1/p}\int_{n^{1/p}}^{\infty}P\left(\max_{1\le k\le n}|S_k|>t\right)dt.
\end{aligned}\tag{3.14}$$

By Theorem 2.1, in order to prove (2.3), we only have to prove that

$$\sum_{n=1}^{\infty}n^{r-2-1/p}\int_{n^{1/p}}^{\infty}P\left(\max_{1\le k\le n}|S_k|>t\right)dt<\infty.\tag{3.15}$$

For $t>0$, let

$$\begin{aligned}
Y_{tj}&=-tI(Y_j<-t)+Y_jI(|Y_j|\le t)+tI(Y_j>t),\quad j\ge 1,\\
\widetilde{Y}_{tj}&=Y_{tj}-EY_{tj},\quad j\ge 1,\\
Y_{tj}^{*}&=tI(Y_j<-t)-tI(Y_j>t)+Y_jI(|Y_j|>t),\quad j\ge 1.
\end{aligned}\tag{3.16}$$

Since $Y_j = Y_{tj}^* + Y_{tj}$, $j \geq 1$, it is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} P\left(\max_{1\leq k\leq n}|S_k| > t\right)dt \\
&\quad\leq \sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{tj}^*\right| > \frac{t}{2}\right)dt \\
&\qquad+ \sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} \widetilde{Y}_{tj}\right| > \frac{t}{4}\right)dt \\
&\qquad+ \sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} EY_{tj}\right| > \frac{t}{4}\right)dt \\
&\quad=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.17}$$

For I_1, by Markov's inequality, $|Y_{tj}^*| \leq |Y_j| I(|Y_j| > t)$, Lemma 1.6 and $EY^{rp} < \infty$, we get that

$$\begin{aligned}
I_1 &\leq 2\sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} t^{-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{tj}^*\right|\right)dt \\
&\leq 2\sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} t^{-1} \sum_{i=1}^{\infty} |a_i| E\left(\max_{1\leq k\leq n} \sum_{j=i+1}^{i+k} |Y_j| I(|Y_j| > t)\right)dt \\
&\leq C_1 \sum_{n=1}^{\infty} n^{r-1-1/p} \int_{n^{1/p}}^{\infty} t^{-1} E[Y I(Y > t)]dt \\
&= C_1 \sum_{n=1}^{\infty} n^{r-1-1/p} \sum_{m=n}^{\infty} \int_{m^{1/p}}^{(m+1)^{1/p}} t^{-1} E\left[Y I\left(Y > m^{1/p}\right)\right]dt \\
&\leq C_2 \sum_{n=1}^{\infty} n^{r-1-1/p} \sum_{m=n}^{\infty} m^{1/p-1-1/p} E\left[Y I\left(Y > m^{1/p}\right)\right] \\
&= C_2 \sum_{m=1}^{\infty} m^{-1} E\left[Y I\left(Y > m^{1/p}\right)\right] \sum_{n=1}^{m} n^{r-1-1/p} \\
&\leq C_3 \sum_{m=1}^{\infty} m^{r-1-1/p} E\left[Y I\left(Y > m^{1/p}\right)\right] \leq C_4 EY^{rp} < \infty.
\end{aligned} \tag{3.18}$$

From the fact that $\{\tilde{Y}_{tj},\ 1 \le j < \infty\}$ is a mean zero AANA sequence and $E\tilde{Y}_{tj}^2 \le EY_{tj}$, $Y_{tj}^2 = Y_j^2 I(|Y_j| \le t) + t^2 I(|Y_j| > t),\ j \ge 1$, similar to the proof of (3.11), we have

$$\begin{aligned}
I_2 &\le 4^2 \sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} t^{-2} E\left(\max_{1\le k\le n} \left(\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+k} \tilde{Y}_{tj} \right)^2 \right) dt \\
&\le 4^2 \sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} t^{-2} \left(\sum_{i=1}^{\infty} |a_i| \right)^2 \sup_{i\ge 1} E\left\{ \max_{1\le k\le n} \left(\sum_{j=i+1}^{i+k} \tilde{Y}_{tj} \right)^2 \right\} dt \\
&\le C_1 \sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} t^{-2} \sup_{i\ge 1} \sum_{j=i+1}^{i+n} E\tilde{Y}_{tj}^2 dt \\
&\le C_2 \sum_{n=1}^{\infty} n^{r-2-1/p} \int_{n^{1/p}}^{\infty} t^{-2} \sup_{i\ge 1} \sum_{j=i+1}^{i+n} \left\{ E\left[Y_j^2 I(|Y_j| \le t) \right] + t^2 E\left[I(|Y_j| > t) \right] \right\} dt \\
&\le C_3 \sum_{n=1}^{\infty} n^{r-1-1/p} \int_{n^{1/p}}^{\infty} t^{-2} E\left[Y^2 I(Y \le t) \right] dt + C_4 \sum_{n=1}^{\infty} n^{r-1-1/p} \int_{n^{1/p}}^{\infty} P(Y > t) dt \\
&=: C_3 I_{21} + C_4 I_{22}.
\end{aligned} \tag{3.19}$$

It follows from $rp < 2$ and $EY^{rp} < \infty$ that

$$\begin{aligned}
I_{21} &= \sum_{n=1}^{\infty} n^{r-1-1/p} \sum_{m=n}^{\infty} \int_{m^{1/p}}^{(m+1)^{1/p}} t^{-2} E\left[Y^2 I(Y \le t) \right] dt \\
&\le C_1 \sum_{n=1}^{\infty} n^{r-1-1/p} \sum_{m=n}^{\infty} m^{1/p-1-2/p} E\left[Y^2 I\left(Y \le (m+1)^{1/p} \right) \right] \\
&= C_1 \sum_{m=1}^{\infty} m^{-1/p-1} E\left[Y^2 I\left(Y \le (m+1)^{1/p} \right) \right] \sum_{n=1}^{m} n^{r-1-1/p} \\
&\le C_2 \sum_{m=1}^{\infty} m^{r-1-2/p} E\left[Y^2 I\left(Y \le (m+1)^{1/p} \right) \right] \\
&= C_2 \sum_{m=1}^{\infty} m^{r-1-2/p} \sum_{i=1}^{m+1} E\left[Y^2 I\left((i-1)^{1/p} < Y \le i^{1/p} \right) \right] \\
&= C_2 \sum_{m=1}^{\infty} m^{r-1-2/p} E\left[Y^2 I\left(m^{1/p} < Y \le (m+1)^{1/p} \right) \right] \\
&\quad + C_2 \sum_{m=1}^{\infty} m^{r-1-2/p} \sum_{i=1}^{m} E\left[Y^2 I\left((i-1)^{1/p} < Y \le i^{1/p} \right) \right] \\
&= C_2 \sum_{m=1}^{\infty} m^{r-1-2/p} E\left[Y^{rp} Y^{2-rp} I\left(m^{1/p} < Y \le (m+1)^{1/p} \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&+C_2\sum_{i=1}^{\infty}E\Big[Y^2I\Big((i-1)^{1/p}<Y\le i^{1/p}\Big)\Big]\sum_{m=i}^{\infty}m^{r-1-2/p}\\
&\le 2^{(2-rp)/p}C_2\sum_{m=1}^{\infty}m^{-1}E\Big[Y^{rp}I\Big(m^{1/p}<Y\le(m+1)^{1/p}\Big)\Big]\\
&\quad+C_3\sum_{i=1}^{\infty}E\Big[Y^{rp}Y^{2-rp}I\Big((i-1)^{1/p}<Y\le i^{1/p}\Big)\Big]i^{r-2/p}\le C_4EY^{rp}<\infty.
\end{aligned}
\tag{3.20}
$$

By the proof of (3.18), one has

$$
I_{22}\le\sum_{n=1}^{\infty}n^{r-1-1/p}\int_{n^{1/p}}^{\infty}t^{-1}E[YI(Y>t)]dt\le CEY^{rp}<\infty. \tag{3.21}
$$

On the other hand, by the property $EY_j=0$, we have

$$
\begin{aligned}
&\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}EY_{tj}\right|\\
&=\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}\{E[Y_jI(|Y_j|\le t)]-tE[I(Y_j<-t)]+tE[I(Y_j>t)]\}\right|\\
&=\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}\{E[Y_jI(|Y_j|>t)]+tE[I(Y_j<-t)]-tE[I(Y_j>t)]\}\right|\\
&\le 2\sum_{i=1}^{\infty}|a_i|\sum_{j=i+1}^{i+n}E[|Y_j|I(|Y_j|>t)].
\end{aligned}
\tag{3.22}
$$

Thus, by Lemma 1.6 and the proof of (3.18), it can be seen that

$$
\begin{aligned}
I_3&\le 4\sum_{n=1}^{\infty}n^{r-2-1/p}\int_{n^{1/p}}^{\infty}t^{-1}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{\infty}a_i\sum_{j=i+1}^{i+k}EY_{tj}\right|\right)dt\\
&\le 8\sum_{n=1}^{\infty}n^{r-2-1/p}\int_{n^{1/p}}^{\infty}t^{-1}\sum_{i=1}^{\infty}|a_i|\sum_{j=i+1}^{i+n}E[|Y_j|I(|Y_j|>t)]dt\\
&\le C_1\sum_{n=1}^{\infty}n^{r-1-1/p}\int_{n^{1/p}}^{\infty}t^{-1}E[YI(Y>t)]dt\le C_2EY^{rp}<\infty.
\end{aligned}
\tag{3.23}
$$

Consequently, by (3.14), (3.17), (3.18), (3.19), (3.20), (3.21), and (3.23) and Theorem 2.1, (2.3) holds true.

Next, we prove (2.4). It is easy to see that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} E\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right| - \varepsilon 2^{2/p}\right)^{+} \\
&\quad = \sum_{n=1}^{\infty} n^{r-2}\int_0^{\infty} P\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right)dt \\
&\quad = \sum_{m=1}^{\infty}\sum_{n=2^{m-1}}^{2^m-1} n^{r-2}\int_0^{\infty} P\left(\sup_{k\geq n}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right)dt \\
&\quad \leq 2^{2-r}\sum_{m=1}^{\infty}\int_0^{\infty} P\left(\sup_{k\geq 2^{m-1}}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right)dt \sum_{n=2^{m-1}}^{2^m-1} 2^{m(r-2)} \\
&\quad \leq 2^{2-r}\sum_{m=1}^{\infty} 2^{m(r-1)}\int_0^{\infty} P\left(\sup_{k\geq 2^{m-1}}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right)dt \\
&\quad = 2^{2-r}\sum_{m=1}^{\infty} 2^{m(r-1)}\int_0^{\infty} P\left(\sup_{l\geq m}\max_{2^{l-1}\leq k<2^l}\left|\frac{S_k}{k^{1/p}}\right| > \varepsilon 2^{2/p} + t\right)dt \\
&\quad \leq 2^{2-r}\sum_{m=1}^{\infty} 2^{m(r-1)}\sum_{l=m}^{\infty}\int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > \left(\varepsilon 2^{2/p} + t\right)2^{(l-1)/p}\right)dt \\
&\quad = 2^{2-r}\sum_{l=1}^{\infty}\int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > \left(\varepsilon 2^{2/p} + t\right)2^{(l-1)/p}\right)dt\sum_{m=1}^{l} 2^{m(r-1)} \\
&\quad \leq 2^{2-r}\sum_{l=1}^{\infty} 2^{l(r-1)}\int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > \left(\varepsilon 2^{2/p} + t\right)2^{(l-1)/p}\right)dt \quad \left(\text{let } s = 2^{(l-1)/p}t\right) \\
&\quad \leq C_1\sum_{l=1}^{\infty} 2^{l(r-1-1/p)}\int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > \varepsilon 2^{(l+1)/p} + s\right)ds \\
&\quad = 2^{(2+1/p-r)}C_1\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{l+1}-1} 2^{(l+1)(r-2-1/p)}\int_0^{\infty} P\left(\max_{1\leq k\leq 2^l}|S_k| > \varepsilon 2^{(l+1)/p} + s\right)ds \\
&\quad \leq 2^{(2+1/p-r)}C_1\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{l+1}-1} n^{r-2-1/p}\int_0^{\infty} P\left(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p} + s\right)ds \quad (\text{since } r<2) \\
&\quad \leq 2^{(2+1/p-r)}C_1\sum_{n=1}^{\infty} n^{r-2-1/p} E\left(\max_{1\leq k\leq n}|S_k| - \varepsilon n^{1/p}\right)^{+} < \infty.
\end{aligned}
\tag{3.24}
$$

Therefore, (2.4) holds true following from (2.3). □

Proof of Theorem 2.3. Similar to the proof of Theorem 2.1, by $\sum_{i=1}^{\infty}|a_i| < \infty$ and $EY < \infty$, $\sum_{i=1}^{\infty} a_i Y_{i+n}$ converges almost surely. It can be seen that

$$
\begin{aligned}
E\left(\sup_{n\geq 1}\left|\frac{S_n}{n^{1/r}}\right|^p\right) &= \int_0^{\infty} P\left(\sup_{n\geq 1}\left|\frac{S_n}{n^{1/r}}\right| > t^{1/p}\right)dt \\
&\leq 2^{p/r} + \int_{2^{p/r}}^{\infty} P\left(\sup_{n\geq 1}\left|\frac{S_n}{n^{1/r}}\right| > t^{1/p}\right)dt \\
&\leq 2^{p/r} + \int_{2^{p/r}}^{\infty} P\left(\sup_{k\geq 1}\max_{2^{k-1}\leq n<2^k}\left|\frac{S_n}{n^{1/r}}\right| > t^{1/p}\right)dt \\
&\leq 2^{p/r} + \int_{2^{p/r}}^{\infty} \sum_{k=1}^{\infty} P\left(\max_{2^{k-1}\leq n<2^k}\left|\frac{S_n}{n^{1/r}}\right| > t^{1/p}\right)dt \\
&\leq 2^{p/r} + \int_{2^{p/r}}^{\infty} \sum_{k=1}^{\infty} P\left(\max_{1\leq n\leq 2^k}|S_n| > 2^{(k-1)/r}t^{1/p}\right)dt \quad \left(\text{let } s = 2^{(k-1)p/r}t\right) \\
&= 2^{p/r} + 2^{p/r}\sum_{k=1}^{\infty} 2^{-kp/r}\int_{2^{kp/r}}^{\infty} P\left(\max_{1\leq n\leq 2^k}|S_n| > s^{1/p}\right)ds.
\end{aligned}
\tag{3.25}
$$

For $s^{1/p} > 0$, let

$$
\begin{aligned}
Y_{sj} &= -s^{1/p}I\left(Y_j < -s^{1/p}\right) + Y_j I\left(\left|Y_j\right| \leq s^{1/p}\right) + s^{1/p}I\left(Y_j > s^{1/p}\right), \quad j \geq 1, \\
\widetilde{Y}_{sj} &= Y_{sj} - EY_{sj}, \quad j \geq 1, \\
Y_{sj}^{*} &= s^{1/p}I\left(Y_j < -s^{1/p}\right) - s^{1/p}I\left(Y_j > s^{1/p}\right) + Y_j I\left(\left|Y_j\right| > s^{1/p}\right), \quad j \geq 1.
\end{aligned}
\tag{3.26}
$$

Since $Y_j = Y_{sj}^{*} + Y_{sj}$, $j \geq 1$, then

$$
\begin{aligned}
&\sum_{k=1}^{\infty} 2^{-kp/r}\int_{2^{kp/r}}^{\infty} P\left(\max_{1\leq n\leq 2^k}|S_n| > s^{1/p}\right)ds \\
&\quad\leq \sum_{k=1}^{\infty} 2^{-kp/r}\int_{2^{kp/r}}^{\infty} P\left(\max_{1\leq n\leq 2^k}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{sj}^{*}\right| > \frac{s^{1/p}}{2}\right)ds \\
&\qquad+ \sum_{k=1}^{\infty} 2^{-kp/r}\int_{2^{kp/r}}^{\infty} P\left(\max_{1\leq n\leq 2^k}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+n} \widetilde{Y}_{sj}\right| > \frac{s^{1/p}}{4}\right)ds \\
&\qquad+ \sum_{k=1}^{\infty} 2^{-kp/r}\int_{2^{kp/r}}^{\infty} P\left(\max_{1\leq n\leq 2^k}\left|\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+n} EY_{sj}\right| > \frac{s^{1/p}}{4}\right)ds \\
&\quad=: H_1 + H_2 + H_3.
\end{aligned}
\tag{3.27}
$$

For H_1, similar to (3.18), by Markov's inequality and Lemma 1.6, one has

$$\begin{aligned}
H_1 &\leq 2\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\left(\max_{1\leq n\leq 2^k} \left| \sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{sj}^{*} \right| \right) ds \\
&\leq 2\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} \sum_{i=1}^{\infty} |a_i| E\left(\max_{1\leq n\leq 2^k} \sum_{j=i+1}^{i+n} |Y_j| I\left(|Y_j| > s^{1/p} \right) \right) ds \\
&\leq C_1 \sum_{k=1}^{\infty} 2^{k-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\left[Y I\left(Y > s^{1/p} \right) \right] ds \\
&= C_1 \sum_{k=1}^{\infty} 2^{k-kp/r} \sum_{m=k}^{\infty} \int_{2^{mp/r}}^{2^{(m+1)p/r}} s^{-1/p} E\left[Y I\left(Y > s^{1/p} \right) \right] ds \\
&\leq C_2 \sum_{k=1}^{\infty} 2^{k-kp/r} \sum_{m=k}^{\infty} 2^{mp/r-m/r} E\left[Y I\left(Y > 2^{m/r} \right) \right] \\
&= C_2 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E\left[Y I\left(Y > 2^{m/r} \right) \right] \sum_{k=1}^{m} 2^{k-kp/r} \\
&\leq \begin{cases} C_3 \sum_{m=1}^{\infty} 2^{m-m/r} E\left[Y I\left(Y > 2^{m/r} \right) \right], & \text{if } p < r, \\ C_4 \sum_{m=1}^{\infty} m 2^{m-m/r} E\left[Y I\left(Y > 2^{m/r} \right) \right], & \text{if } p = r, \\ C_5 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E\left[Y I\left(Y > 2^{m/r} \right) \right], & \text{if } p > r. \end{cases}
\end{aligned} \tag{3.28}$$

For the case $p < r$, if $0 < r < 1$, then

$$\begin{aligned}
\sum_{m=1}^{\infty} 2^{m-m/r} E\left[Y I\left(Y > 2^{m/r} \right) \right] &= \sum_{m=1}^{\infty} 2^{m-m/r} \sum_{k=m}^{\infty} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r} \right) \right] \\
&= \sum_{k=1}^{\infty} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r} \right) \right] \sum_{m=1}^{k} 2^{m(1-1/r)} \\
&\leq C_1 \sum_{k=1}^{\infty} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r} \right) \right] \\
&\leq C_1 EY.
\end{aligned} \tag{3.29}$$

If $r = 1$, then

$$\begin{aligned}
\sum_{m=1}^{\infty} 2^{m-m/r} E\left[Y I\left(Y > 2^{m/r}\right)\right] &= \sum_{m=1}^{\infty} E[Y I(Y > 2^{m})] \\
&= \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} E\left[Y I\left(2^{k} < Y \leq 2^{k+1}\right)\right] \\
&= \sum_{k=1}^{\infty} E\left[Y I\left(2^{k} < Y \leq 2^{k+1}\right)\right] \sum_{m=1}^{k} 1 \\
&= \sum_{k=1}^{\infty} k E\left[Y I\left(2^{k} < Y \leq 2^{k+1}\right)\right] \\
&\leq C_1 \sum_{k=1}^{\infty} E\left[Y \log(1+Y) I\left(2^{k} < Y \leq 2^{k+1}\right)\right] \\
&\leq C_1 E\left[Y \log(1+Y)\right].
\end{aligned} \tag{3.30}$$

Otherwise for $r > 1$, it has

$$\begin{aligned}
\sum_{m=1}^{\infty} 2^{m-m/r} E\left[Y I\left(Y > 2^{m/r}\right)\right] &= \sum_{m=1}^{\infty} 2^{m-m/r} \sum_{k=m}^{\infty} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r}\right)\right] \\
&= \sum_{k=1}^{\infty} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r}\right)\right] \sum_{m=1}^{k} 2^{m-m/r} \\
&\leq C_1 \sum_{k=1}^{\infty} 2^{k-k/r} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r}\right)\right] \\
&\leq C_1 \sum_{k=1}^{\infty} E\left[Y^{r} I\left(2^{k/r} < Y \leq 2^{(k+1)/r}\right)\right] \leq C_1 E Y^{r}.
\end{aligned} \tag{3.31}$$

Similarly, for the case $p = r$, if $0 < r < 1$, then

$$\begin{aligned}
\sum_{m=1}^{\infty} m 2^{m-m/r} E\left[Y I\left(Y > 2^{m/r}\right)\right] &= \sum_{m=1}^{\infty} m 2^{m-m/r} \sum_{k=m}^{\infty} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r}\right)\right] \\
&= \sum_{k=1}^{\infty} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r}\right)\right] \sum_{m=1}^{k} m 2^{m(1-1/r)} \\
&\leq \sum_{k=1}^{\infty} E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r}\right)\right] k \sum_{m=1}^{k} 2^{m(1-1/r)} \\
&\leq C_1 \sum_{k=1}^{\infty} k E\left[Y I\left(2^{k/r} < Y \leq 2^{(k+1)/r}\right)\right] \\
&\leq C_2 E\left[Y \log(1+Y)\right].
\end{aligned} \tag{3.32}$$

If $r = 1$, then

$$\begin{aligned}
\sum_{m=1}^{\infty} m2^{m-m/r} E\left[YI\left(Y > 2^{m/r}\right)\right] &= \sum_{m=1}^{\infty} m \sum_{k=m}^{\infty} E\left[YI\left(2^k < Y \le 2^{k+1}\right)\right] \\
&= \sum_{k=1}^{\infty} E\left[YI\left(2^k < Y \le 2^{k+1}\right)\right] \sum_{m=1}^{k} m \\
&\le C_2 \sum_{k=1}^{\infty} k^2 E\left[YI\left(2^k < Y \le 2^{k+1}\right)\right] \\
&\le C_2 \sum_{k=1}^{\infty} E\left[Y \log^2(1+Y) I\left(2^k < Y \le 2^{k+1}\right)\right] \\
&\le C_2 E\left[Y \log^2(1+Y)\right].
\end{aligned} \tag{3.33}$$

Otherwise, for $r > 1$, it follows

$$\begin{aligned}
&\sum_{m=1}^{\infty} m2^{m-m/r} E\left[YI\left(Y > 2^{m/r}\right)\right] \\
&\quad = \sum_{m=1}^{\infty} m2^{m-m/r} \sum_{k=m}^{\infty} E\left[YI\left(2^{k/r} < Y \le 2^{(k+1)/r}\right)\right] \\
&\quad = \sum_{k=1}^{\infty} E\left[YI\left(2^{k/r} < Y \le 2^{(k+1)/r}\right)\right] \sum_{m=1}^{k} m2^{m-m/r} \\
&\quad \le \sum_{k=1}^{\infty} E\left[YI\left(2^{k/r} < Y \le 2^{(k+1)/r}\right)\right] k \sum_{m=1}^{k} 2^{m-m/r} \\
&\quad \le C_1 \sum_{k=1}^{\infty} k2^{k-k/r} E\left[YI\left(2^{k/r} < Y \le 2^{(k+1)/r}\right)\right] \le C_2 E\left[Y^r \log(1+Y)\right].
\end{aligned} \tag{3.34}$$

On the other hand, for the case $p > r$, if $0 < p < 1$, then

$$\begin{aligned}
\sum_{m=1}^{\infty} 2^{mp/r-m/r} E\left[YI\left(Y > 2^{m/r}\right)\right] &= \sum_{m=1}^{\infty} 2^{m(p-1)/r} \sum_{k=m}^{\infty} E\left[YI\left(2^{k/r} < Y \le 2^{(k+1)/r}\right)\right] \\
&= \sum_{k=1}^{\infty} E\left[YI\left(2^{k/r} < Y \le 2^{(k+1)/r}\right)\right] \sum_{m=1}^{k} 2^{m(p-1)/r} \\
&\le C_1 \sum_{k=1}^{\infty} E\left[YI\left(2^{k/r} < Y \le 2^{(k+1)/r}\right)\right] \\
&\le C_1 EY.
\end{aligned} \tag{3.35}$$

If $p = 1$, then

$$\begin{aligned}
\sum_{m=1}^{\infty} 2^{mp/r-m/r} E\left[\Upsilon I\left(\Upsilon > 2^{m/r}\right)\right] &= \sum_{m=1}^{\infty}\sum_{k=m}^{\infty} E\left[\Upsilon I\left(2^{k/r} < \Upsilon \le 2^{(k+1)/r}\right)\right] \\
&= \sum_{k=1}^{\infty} E\left[\Upsilon I\left(2^{k/r} < \Upsilon \le 2^{(k+1)/r}\right)\right]\sum_{m=1}^{k} 1 \\
&= \sum_{k=1}^{\infty} kE\left[\Upsilon I\left(2^{k/r} < \Upsilon \le 2^{(k+1)/r}\right)\right] \\
&\le C_1 E\Upsilon \log(1+\Upsilon).
\end{aligned} \tag{3.36}$$

For $p > 1$, it has

$$\begin{aligned}
\sum_{m=1}^{\infty} 2^{mp/r-m/r} E\left[\Upsilon I\left(\Upsilon > 2^{m/r}\right)\right] &= \sum_{m=1}^{\infty} 2^{m(p-1)/r} \sum_{k=m}^{\infty} E\left[\Upsilon I\left(2^{k/r} < \Upsilon \le 2^{(k+1)/r}\right)\right] \\
&= \sum_{k=1}^{\infty} E\left[\Upsilon I\left(2^{k/r} < \Upsilon \le 2^{(k+1)/r}\right)\right]\sum_{m=1}^{k} 2^{m(p-1)/r} \\
&\le C_1 \sum_{k=1}^{\infty} 2^{k(p-1)/r} E\left[\Upsilon I\left(2^{k/r} < \Upsilon \le 2^{(k+1)/r}\right)\right] \\
&\le C_1 \sum_{k=1}^{\infty} E\left[\Upsilon^p I\left(2^{k/r} < \Upsilon \le 2^{(k+1)/r}\right)\right] \le C_1 E\Upsilon^p.
\end{aligned} \tag{3.37}$$

Consequently, by (3.28), the conditions of Theorem 2.3 and inequalities above, we obtain that

$$H_1 \le \begin{cases} C_1 \sum_{m=1}^{\infty} 2^{m-m/r} E\left[\Upsilon I\left(\Upsilon > 2^{m/r}\right)\right], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m 2^{m-m/r} E\left[\Upsilon I\left(\Upsilon > 2^{m/r}\right)\right], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E\left[\Upsilon I\left(\Upsilon > 2^{m/r}\right)\right], & \text{if } p > r \end{cases}$$

$$\leq \begin{cases} \text{for } p<r, \begin{cases} C_4 EY<\infty, & \text{if } 0<r<1, \\ C_5 E\left[Y\log(1+Y)\right]<\infty, & \text{if } r=1, \\ C_6 EY^r<\infty, & \text{if } r>1, \end{cases} \\ \text{for } p=r, \begin{cases} C_7 E\left[Y\log(1+Y)\right]<\infty, & \text{if } 0<r<1, \\ C_8 E\left[Y\log^2(1+Y)\right]<\infty, & \text{if } r=1, \\ C_9 E\left[Y^r\log(1+Y)\right]<\infty, & \text{if } r>1, \end{cases} \\ \text{for } p>r, \begin{cases} C_{10} EY<\infty, & \text{if } 0<p<1, \\ C_{11} E\left[Y\log(1+Y)\right]<\infty, & \text{if } p=1, \\ C_{12} EY^p<\infty, & \text{if } p>1. \end{cases} \end{cases} \tag{3.38}$$

Since $\{\widetilde{Y}_{sj}, 1 \leq j < \infty\}$ is a mean zero AANA sequence and $E\widetilde{Y}_{sj}^2 \leq EY_{sj}^2$, $Y_{sj}^2 = Y_j^2 I(|Y_j| \leq s^{1/p}) + s^{2/p} I(|Y_j| > s^{1/p})$, $j \geq 1$, similar to the proof of (3.11), we obtain that

$$\begin{aligned} H_2 &\leq C_1 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-2/p} E\left\{ \max_{1\leq n\leq 2^k} \left(\sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+n} \widetilde{Y}_{sj} \right)^2 \right\} ds \\ &\leq C_1 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-2/p} \left(\sum_{i=1}^{\infty} |a_i| \right)^2 \sup_{i\geq 1} E\left\{ \max_{1\leq n\leq 2^k} \left(\sum_{j=i+1}^{i+n} \widetilde{Y}_{sj} \right)^2 \right\} ds \\ &\leq C_2 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-2/p} \sup_{i\geq 1} \sum_{j=i+1}^{i+2^k} E\widetilde{Y}_{sj}^2 ds \\ &\leq C_3 \sum_{k=1}^{\infty} 2^{-kp/r+k} \int_{2^{kp/r}}^{\infty} s^{-2/p} E\left[Y^2 I\left(Y \leq s^{1/p} \right) \right] ds \\ &\quad + C_4 \sum_{k=1}^{\infty} 2^{-kp/r+k} \int_{2^{kp/r}}^{\infty} P\left(Y > s^{1/p} \right) ds \\ &=: C_3 H_{21} + C_4 H_{22}. \end{aligned} \tag{3.39}$$

Similar to the proof of Theorem 1.1 of Chen and Gan [7], by $p < 2$ and the conditions of Theorem 2.3, we have that

$$\begin{aligned} H_{21} &= \sum_{k=1}^{\infty} 2^{-kp/r+k} \sum_{m=k}^{\infty} \int_{2^{mp/r}}^{2^{(m+1)p/r}} s^{-2/p} E\left[Y^2 I\left(Y \leq s^{1/p} \right) \right] ds \\ &\leq \sum_{k=1}^{\infty} 2^{-kp/r+k} \sum_{m=k}^{\infty} 2^{mp/r-2m/r} E\left[Y^2 I\left(Y \leq 2^{(m+1)/r} \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} 2^{m(p-2)/r} E\left[Y^2 I\left(Y \le 2^{(m+1)/r}\right)\right] \sum_{k=1}^{m} 2^{k(1-p/r)} \\
&\le \begin{cases} C_1 \sum_{m=1}^{\infty} 2^{m(r-2)/r} E\left[Y^2 I\left(Y \le 2^{(m+1)/r}\right)\right], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m 2^{m(r-2)/r} E\left[Y^2 I\left(Y \le 2^{(m+1)/r}\right)\right], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{m(p-2)/r} E\left[Y^2 I\left(Y \le 2^{(m+1)/r}\right)\right], & \text{if } p > r \end{cases} \\
&\le \begin{cases} C_4 E Y^r < \infty, & \text{if } p < r, \\ C_5 E\left[Y^r \log(1+Y)\right] < \infty, & \text{if } p = r, \\ C_6 E Y^p < \infty, & \text{if } p > r. \end{cases}
\end{aligned}
\tag{3.40}
$$

On the other hand, by the proof of (3.28) and (3.38), it follows

$$
\begin{aligned}
H_{22} &\le \sum_{k=1}^{\infty} 2^{k-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\left[Y I\left(Y > s^{1/p}\right)\right] ds \\
&\le \begin{cases} \text{for } p < r, \begin{cases} C_1 E Y < \infty, & \text{if } 0 < r < 1, \\ C_2 E\left[Y \log(1+Y)\right] < \infty, & \text{if } r = 1, \\ C_3 E Y^r < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p = r, \begin{cases} C_4 E\left[Y \log(1+Y)\right] < \infty, & \text{if } 0 < r < 1, \\ C_5 E\left[Y \log^2(1+Y)\right] < \infty, & \text{if } r = 1, \\ C_6 E\left[Y^r \log(1+Y)\right] < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p > r, \begin{cases} C_7 E Y < \infty, & \text{if } 0 < p < 1, \\ C_8 E\left[Y \log(1+Y)\right] < \infty, & \text{if } p = 1, \\ C_9 E Y^p < \infty, & \text{if } p > 1. \end{cases} \end{cases}
\end{aligned}
\tag{3.41}
$$

Similar to the proof of (3.23), by the property $EY_j = 0$ and the proofs of (3.28) and (3.38), one has

$$
\begin{aligned}
H_3 &\le 4 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} \left(\max_{1 \le n \le 2^k} \left| \sum_{i=1}^{\infty} a_i \sum_{j=i+1}^{i+n} E Y_{sj} \right| \right) ds \\
&\le 8 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} \left(\max_{1 \le n \le 2^k} \sum_{i=1}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E\left[|Y_j| I\left(|Y_j| > s^{1/p} \right) \right] \right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\le C_1 \sum_{k=1}^{\infty} 2^{k-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\Big[Y I\Big(Y > s^{1/p}\Big)\Big] ds \\
&\le \begin{cases}
\text{for } p < r, \begin{cases} C_1 EY < \infty, & \text{if } 0 < r < 1, \\ C_2 E\big[Y \log(1+Y)\big] < \infty, & \text{if } r = 1, \\ C_3 EY^r < \infty, & \text{if } r > 1, \end{cases} \\
\text{for } p = r, \begin{cases} C_4 E\big[Y \log(1+Y)\big] < \infty, & \text{if } 0 < r < 1, \\ C_5 E\Big[Y \log^2(1+Y)\Big] < \infty, & \text{if } r = 1, \\ C_6 E\big[Y^r \log(1+Y)\big] < \infty, & \text{if } r > 1, \end{cases} \\
\text{for } p > r, \begin{cases} C_7 EY < \infty, & \text{if } 0 < p < 1, \\ C_8 E\big[Y \log(1+Y)\big] < \infty, & \text{if } p = 1, \\ C_9 EY^p < \infty, & \text{if } p > 1. \end{cases}
\end{cases}
\end{aligned}
\tag{3.42}
$$

Consequently, by (3.25), (3.27), (3.28), (3.38), (3.39), (3.40), (3.41), and (3.42), we finally obtain (2.6). □

Remark 3.1. Zhou and Lin [17] obtained the result (2.6) for partial sums of moving average process under φ-mixing sequence. But there is one problem in their proof. On page 694 of Zhou and Lin [17], they presented that

$$
\begin{aligned}
I_1 \le \cdots \le &\begin{cases}
C \sum_{m=1}^{\infty} 2^{m-m/r} E\big[|Y_1| I\big(|Y_1| > 2^{m/r}\big)\big], & \text{if } p < r, \\
C \sum_{m=1}^{\infty} m 2^{m-m/r} E\big[|Y_1| I\big(|Y_1| > 2^{m/r}\big)\big], & \text{if } p = r, \\
C \sum_{m=1}^{\infty} 2^{mp/r-m/r} E\big[|Y_1| I\big(|Y_1| > 2^{m/r}\big)\big], & \text{if } p > r,
\end{cases} \\
\le &\begin{cases}
CE|Y_1|^r < \infty, & \text{if } p < r, \\
CE\big[|Y_1|^r \log(1+|Y_1|)\big] < \infty, & \text{if } p = r, \\
CE|Y_1|^p < \infty, & \text{if } p > r,
\end{cases}
\end{aligned}
\tag{3.43}
$$

where $1 \le r < 2$ and $p > 0$. For the case $p < r$, by taking $r = 1$, we cannot get that

$$
\sum_{m=1}^{\infty} 2^{m-m/r} E\Big[|Y_1| I\Big(|Y_1| > 2^{m/r}\Big)\Big] = \sum_{m=1}^{\infty} E\big[|Y_1| I\big(|Y_1| > 2^m\big)\big] \le CE|Y_1| < \infty. \tag{3.44}
$$

Here, we give a counter example to illustrate this problem. Assume that the density function of nonnegative random variable Y is

$$f(y) = \frac{C}{y^2 \ln^2 y}, \quad y > 2, \ C = \left[\int_2^{\infty} \frac{1}{y^2 \ln^2 y} dy\right]^{-1}. \tag{3.45}$$

Obviously, it can be found that

$$\mathrm{E}Y = C\int_2^{\infty} \frac{1}{y \ln^2 y} dy < \infty. \tag{3.46}$$

But for $r = 1$,

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{m-m/r} E\left[YI\left(Y > 2^{m/r}\right)\right] &= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} E\left[YI\left(2^n < Y \leq 2^{n+1}\right)\right] = \sum_{n=1}^{\infty} E\left[YI\left(2^n < Y \leq 2^{n+1}\right)\right]\sum_{m=1}^{n} 1 \\ &= C\sum_{n=1}^{\infty} n\int_{2^n}^{2^{n+1}} \frac{1}{y\ln^2 y} dy = \frac{C}{\ln 2}\sum_{n=1}^{\infty}\frac{1}{n+1} = \infty.\end{aligned} \tag{3.47}$$

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References

[1] I. A. Ibragimov, "Some limit theorems for stationary processes," *Theory of Probability and Its Applications*, vol. 7, no. 4, pp. 349–382, 1962.
[2] R. M. Burton and H. Dehling, "Large deviations for some weakly dependent random processes," *Statistics & Probability Letters*, vol. 9, no. 5, pp. 397–401, 1990.
[3] D. L. Li, M. B. Rao, and X. C. Wang, "Complete convergence of moving average processes," *Statistics & Probability Letters*, vol. 14, no. 2, pp. 111–114, 1992.
[4] L. X. Zhang, "Complete convergence of moving average processes under dependence assumptions," *Statistics and Probability Letters*, vol. 30, no. 2, pp. 165–170, 1996.
[5] Y. X. Li and L. X. Zhang, "Complete moment convergence of moving-average processes under dependence assumptions," *Statistics & Probability Letters*, vol. 70, no. 3, pp. 191–197, 2004.
[6] T.-S. Kim, M.-H. Ko, and Y.-K. Choi, "Complete moment convergence of moving average processes with dependent innovations," *Journal of the Korean Mathematical Society*, vol. 45, no. 2, pp. 355–365, 2008.
[7] P. Y. Chen and S. X. Gan, "On moments of the maximum of normed partial sums of ρ-mixing random variables," *Statistics & Probability Letters*, vol. 78, no. 10, pp. 1215–1221, 2008.

[8] P. Y. Chen, T.-C. Hu, and A. Volodin, “Limiting behaviour of moving average processes under φ-mixing assumption,” *Statistics & Probability Letters*, vol. 79, no. 1, pp. 105–111, 2009.
[9] X. C. Zhou, “Complete moment convergence of moving average processes under φ-mixing assumptions,” *Statistics and Probability Letters*, vol. 80, no. 5-6, pp. 285–292, 2010.
[10] P. L. Hsu and H. Robbins, “Complete convergence and the law of large numbers,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 33, no. 2, pp. 25–31, 1947.
[11] Y. S. Chow, “On the rate of moment convergence of sample sums and extremes,” *Bulletin of the Institute of Mathematics*, vol. 16, no. 3, pp. 177–201, 1988.
[12] Q. M. Shao, “A moment inequality and its applications,” *Acta Mathematica Sinica*, vol. 31, no. 6, pp. 736–747, 1988 (Chinese).
[13] S. H. Sung, “A note on the complete convergence of moving average processes,” *Statistics & Probability Letters*, vol. 79, no. 11, pp. 1387–1390, 2009.
[14] S. H. Sung, “Complete convergence for weighted sums of ρ^*-mixing random variables,” *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 630608, 13 pages, 2010.
[15] S. H. Sung, “Convergence of moving average processes for dependent random variables,” *Communications in Statistics-Theory and Methods*, vol. 40, no. 13, pp. 2366–2376, 2011.
[16] P. Y. Chen and Y. M. Li, “Limiting behavior of moving average processes of martingale difference sequences,” *Acta Mathematica Scientia Series A*, vol. 31, no. 1, pp. 179–187, 2011 (Chinese).
[17] X. C. Zhou and J. G. Lin, “On moments of the maximum of partial sums of moving average processes under dependence assumptions,” *Acta Mathematicae Applicatae Sinica, English Series*, vol. 27, no. 4, pp. 691–696, 2011.
[18] K. Joag-Dev and F. Proschan, “Negative association of random variables, with applications,” *The Annals of Statistics*, vol. 11, no. 1, pp. 286–295, 1983.
[19] P. Matuła, “A note on the almost sure convergence of sums of negatively dependent random variables,” *Statistics & Probability Letters*, vol. 15, no. 3, pp. 209–213, 1992.
[20] T. K. Chandra and S. Ghosal, “Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables,” *Acta Mathematica Hungarica*, vol. 71, no. 4, pp. 327–336, 1996.
[21] T. K. Chandra and S. Ghosal, “The strong law of large numbers for weighted averages under dependence assumptions,” *Journal of Theoretical Probability*, vol. 9, no. 3, pp. 797–809, 1996.
[22] Y. Wang, J. Yan, F. Cheng, and C. Su, “The strong law of large numbers and the law of the iterated logarithm for product sums of NA and AANA random variables,” *Southeast Asian Bulletin of Mathematics*, vol. 27, no. 2, pp. 369–384, 2003.
[23] M.-H. Ko, T.-S. Kim, and Z. Lin, “The Hájeck-Rènyi inequality for the AANA random variables and its applications,” *Taiwanese Journal of Mathematics*, vol. 9, no. 1, pp. 111–122, 2005.
[24] D. M. Yuan and J. An, “Rosenthal type inequalities for asymptotically almost negatively associated random variables and applications,” *Science in China, Series A*, vol. 52, no. 9, pp. 1887–1904, 2009.
[25] X. J. Wang, S. H. Hu, and W. Z. Yang, “Convergence properties for asymptotically almost negatively associated sequence,” *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 218380, 15 pages, 2010.
[26] X. J. Wang, S. H. Hu, and W. Z. Yang, “Complete convergence for arrays of rowwise asymptotically almost negatively associated random variables,” *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 717126, 11 pages, 2011.
[27] Q. Y. Wu, *Probability Limit Theory for Mixed Sequence*, Science Press, Beijing, China, 2006.
[28] A. Gut, *Probability: A Graduate Course*, Springer Texts in Statistics, Springer, New York, NY, USA, 2005.