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Research Article

On a Discrete Inverse Problem for Two Spectra

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A version of the inverse spectral problem for two spectra of finite-order real Jacobi matrices (tridiagonal symmetric matrices) is investigated. The problem is to reconstruct the matrix using two sets of eigenvalues: one for the original Jacobi matrix and one for the matrix obtained by deleting the last row and last column of the Jacobi matrix.

1. Introduction

The Jacobi matrices (tridiagonal symmetric matrices) appear in variety of applications. A distinguishing feature of the Jacobi matrices from others is that they are related to certain three-term recursion equations (second-order linear difference equations). Therefore, these matrices can be viewed as the discrete analogue of Sturm-Liouville operators, and their investigation have many similarities with Sturm-Liouville theory [1].

An $N \times N$ (real) Jacobi matrix J is a matrix of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{1.1}$$

where for each n, a_n and b_n are arbitrary real numbers such that a_n is different from zero:

$$a_n,\ b_n \in \mathbb{R}, \qquad a_n \neq 0. \tag{1.2}$$

Let J_1 be the truncated matrix obtained by deleting from J the last row and last column:

$$J_1 = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} \end{bmatrix}. \tag{1.3}$$

Denote the eigenvalues of the matrices J and J_1 by $\lambda_1, \dots, \lambda_N$ and $\mu_1, \dots, \mu_{N-1}$, respectively. The (finite) sequences $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ are called the *two spectra* of the matrix J.

The subject of the present paper is the solution of the inverse problem consisting of the following parts.

(i) Is the matrix J determined uniquely by its two spectra?

(ii) To indicate an algorithm for the construction of the matrix J from its two spectra;

(iii) To find necessary and sufficient conditions for two given sequences of real numbers $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ to be the two spectra for some matrix of the form (1.1) with entries from class (1.2).

This problem was solved earlier in [2, 3]. In the present paper we offer another and more effective, as it seems to us, method of solution for this problem.

Other versions of the inverse problem for two spectra are investigated in [1, 4–9].

The paper consists, besides this introductory section, of two sections. Section 2 is auxiliary and presents briefly the solution of the inverse problem for finite Jacobi matrices in terms of the eigenvalues and normalizing numbers. A solution to this problem is presented in [1, Section 4.6] and [10]. In Section 3, we solve our main problem formulated above. At the basis of this solution is the formula

$$\beta_k = \frac{a}{\prod_{j=1,\, j\neq k}^{N} (\lambda_j - \lambda_k) \prod_{j=1}^{N-1} (\mu_j - \lambda_k)}, \tag{1.4}$$

where

$$\frac{1}{a} = \sum_{m=1}^{N} \frac{1}{\prod_{j=1,\, j\neq m}^{N} (\lambda_j - \lambda_m) \prod_{j=1}^{N-1} (\mu_j - \lambda_m)}. \tag{1.5}$$

These formulae express the normalizing numbers β_k of a finite Jacobi matrix in terms of two of its spectra. The formulae (1.4) and (1.5) give a conditional solution (i.e., assuming that there exists a matrix of the form (1.1) which has the sequences $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ as two of its spectra) of the inverse problem in terms of two spectra because once we know the numbers $\{\lambda_k\}_{k=1}^N$ and $\{\beta_k\}_{k=1}^N$, we can form the matrix J by the prescription given in Section 2. Next, we give necessary and sufficient conditions for two sequences of real numbers $\{\lambda_k\}_{k=1}^N$ and $\{\mu_k\}_{k=1}^{N-1}$ to be two spectra of a Jacobi matrix of the form (1.1) with entries in the class (1.2), that is, we solve the main problem of this paper. The conditions consist of the following single and simple condition:

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 \cdots < \lambda_{N-1} < \mu_{N-1} < \lambda_N, \tag{1.6}$$

that is, the numbers λ_k and μ_k interlace.

2. Preliminaries on the Inverse Spectral Problem

In this section, we follow the author's paper [10]. Given a Jacobi matrix J of the form (1.1) with the entries (1.2), consider the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$, that is equivalent to the second-order linear difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \{0, 1, \ldots, N-1\}, \quad a_{-1} = a_{N-1} = 1, \tag{2.1}$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions:

$$y_{-1} = y_N = 0. \tag{2.2}$$

Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of (2.1) satisfying the initial conditions:

$$P_{-1}(\lambda) = 0, \qquad P_0(\lambda) = 1, \tag{2.3}$$

$$Q_{-1}(\lambda) = -1, \qquad Q_0(\lambda) = 0. \tag{2.4}$$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree n and is called a polynomial of first kind and $Q_n(\lambda)$ is a polynomial of degree $n-1$ and is known as a polynomial of second kind. The equality

$$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) \tag{2.5}$$

holds so that the eigenvalues of the matrix J coincide with the zeros of the polynomial $P_N(\lambda)$. If $P_N(\lambda_0) = 0$, then $\{P_n(\lambda_0)\}_{n=0}^{N-1}$ is an eigenvector of J corresponding to the eigenvalue λ_0. Any eigenvector of J corresponding to the eigenvalue λ_0 is a constant multiple of $\{P_n(\lambda_0)\}_{n=0}^{N-1}$.

As shown in [10, Section 8], the equations

$$P_{N-1}(\lambda)Q_N(\lambda) - P_N(\lambda)Q_{N-1}(\lambda) = 1, \tag{2.6}$$

$$P_{N-1}(\lambda)P_N'(\lambda) - P_N(\lambda)P_{N-1}'(\lambda) = \sum_{n=0}^{N-1} P_n^2(\lambda) \tag{2.7}$$

hold, where the prime denotes the derivative with respect to λ.

Since the real Jacobi matrix J of the form (1.1), (1.2) is self-adjoint, its eigenvalues are real. Let λ_0 be a zero of the polynomial $P_N(\lambda)$. The zero λ_0 is an eigenvalue of the matrix J by (2.5), and hence it is real. Putting $\lambda = \lambda_0$ in (2.7) and using $P_N(\lambda_0) = 0$, we get

$$P_{N-1}(\lambda_0)P'_N(\lambda_0) = \sum_{n=0}^{N-1} P_n^2(\lambda_0). \tag{2.8}$$

The right-hand side of (2.8) is different from zero because the polynomials $P_n(\lambda)$ have real coefficients and hence are real for real values of λ, and besides $P_0(\lambda) = 1$. Therefore, $P'_N(\lambda_0) \neq 0$, that is, the zero λ_0 of the polynomial $P_N(\lambda)$ is simple. Hence the $P_N(\lambda)$, as a polynomial of degree N, has N distinct zeros. Thus, any real Jacobi matrix J of the form (1.1), (1.2) has precisely N real and distinct eigenvalues.

Let $R(\lambda) = (J - \lambda I)^{-1}$ be the resolvent of the matrix J (by I we denote the identity matrix of needed dimension) and e_0 the N-dimensional column vector with the components $1, 0, \ldots, 0$. The rational function

$$w(\lambda) = -\langle R(\lambda)e_0, e_0\rangle = \left\langle (\lambda I - J)^{-1} e_0, e_0 \right\rangle, \tag{2.9}$$

we call the *resolvent function* of the matrix J, where $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{C}^N$. This function is known also as the Weyl-Titchmarsh function of J.

In [10, Section 5] it is shown that the entries $R_{nm}(\lambda)$ of the matrix $R(\lambda) = (J - \lambda I)^{-1}$ (resolvent of J) are of the form

$$R_{nm}(\lambda) = \begin{cases} P_n(\lambda)[Q_m(\lambda) + M(\lambda)P_m(\lambda)], & 0 \le n \le m \le N-1, \\ \\ P_m(\lambda)[Q_n(\lambda) + M(\lambda)P_n(\lambda)], & 0 \le m \le n \le N-1, \end{cases} \tag{2.10}$$

where

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{2.11}$$

Therefore, according to (2.9) and using initial conditions (2.3) and (2.4), we get

$$w(\lambda) = -R_{00}(\lambda) = -M(\lambda) = \frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{2.12}$$

We often will use the following well-known simple useful lemma. We bring it here for easy reference.

Lemma 2.1. *Let $A(\lambda)$ and $B(\lambda)$ be polynomials with complex coefficients and* $\deg A < \deg B$. *Next, suppose that $B(\lambda) = b(\lambda - z_1)\cdots(\lambda - z_N)$, where $z_1, \ldots, z_N$ are distinct complex numbers and b is* *a nonzero complex number. Then, there exist uniquely determined complex numbers* $a_1, \ldots, a_N$ *such that*

$$\frac{A(\lambda)}{B(\lambda)} = \sum_{k=1}^{N} \frac{a_k}{\lambda - z_k}, \tag{2.13}$$

for all values of λ *different from* $z_1, \ldots, z_N$*. The numbers* a_k *are given by the equation*

$$a_k = \lim_{\lambda \to z_k} (\lambda - z_k) \frac{A(\lambda)}{B(\lambda)} = \frac{A(z_k)}{B'(z_k)}, \quad k \in \{1, \ldots, N\}. \tag{2.14}$$

Proof. For each $k \in \{1, \ldots, N\}$, define the polynomial

$$L_k(\lambda) = b \prod_{j=1, j \neq k}^{N} (\lambda - z_j) = \frac{B(\lambda)}{\lambda - z_k}, \tag{2.15}$$

of degree $N - 1$ and set

$$F(\lambda) = A(\lambda) - \sum_{k=1}^{N} a_k L_k(\lambda), \tag{2.16}$$

where a_k is defined by (2.14). Obviously $F(\lambda)$ is a polynomial and $\deg F \leq N - 1$ (recall that $\deg A < \deg B = N$). Since

$$L_k(z_j) = 0 \quad \text{for } j \neq k, \qquad L_k(z_k) = B'(z_k) \neq 0, \tag{2.17}$$

we have

$$F(z_j) = A(z_j) - \sum_{k=1}^{N} a_k L_k(z_j) = A(z_j) - a_j L_j(z_j) = A(z_j) - \frac{A(z_j)}{B'(z_j)} B'(z_j) = 0, \tag{2.18}$$

for all $j \in \{1, \ldots, N\}$. Thus, the polynomial $F(\lambda)$ of degree $\leq N - 1$ has N distinct zeros $z_1, \ldots, z_N$. Then $F(\lambda) \equiv 0$ and we get

$$A(\lambda) = \sum_{k=1}^{N} a_k L_k(\lambda) = \sum_{k=1}^{N} a_k \frac{B(\lambda)}{\lambda - z_k} = B(\lambda) \sum_{k=1}^{N} \frac{a_k}{\lambda - z_k}. \tag{2.19}$$

This proves (2.13). Note that the decomposition (2.13) is unique as for the a_k in this decomposition (2.14) necessarily holds. □

Denote by $\lambda_1, \ldots, \lambda_N$ all the zeros of the polynomial $P_N(\lambda)$ (which coincide by (2.5) with the eigenvalues of the matrix J and which are real and distinct):

$$P_N(\lambda) = c(\lambda - \lambda_1) \cdots (\lambda - \lambda_N), \tag{2.20}$$

where c is a nonzero constant. Therefore applying Lemma 2.1 to (2.12), we can get for the resolvent function $w(\lambda)$ the following decomposition:

$$w(\lambda) = \sum_{k=1}^{N} \frac{\beta_k}{\lambda - \lambda_k}, \tag{2.21}$$

where

$$\beta_k = \frac{Q_N(\lambda_k)}{P_N'(\lambda_k)}. \tag{2.22}$$

Further, putting $\lambda = \lambda_k$ in (2.6) and (2.7) and taking into account that $P_N(\lambda_k) = 0$, we get

$$P_{N-1}(\lambda_k)Q_N(\lambda_k) = 1, \tag{2.23}$$

$$P_{N-1}(\lambda_k)P_N'(\lambda_k) = \sum_{n=0}^{N-1} P_n^2(\lambda_k), \tag{2.24}$$

respectively. It follows from (2.23) that $Q_N(\lambda_k) \neq 0$ and therefore $\beta_k \neq 0$. Comparing (2.22), (2.23), and (2.24), we find that

$$\beta_k = \left\{ \sum_{n=0}^{N-1} P_n^2(\lambda_k) \right\}^{-1}, \tag{2.25}$$

whence we obtain, in particular, that $\beta_k > 0$.

Since $\{P_n(\lambda_k)\}_{n=0}^{N-1}$ is an eigenvector of the matrix J corresponding to the eigenvalue λ_k, it is natural, according to the formula (2.25), to call β_k the *normalizing number* of the matrix J corresponding to the eigenvalue λ_k.

The collection of the eigenvalues and normalizing numbers:

$$\left\{\lambda_k, \beta_k (k = 1, \ldots, N)\right\}, \tag{2.26}$$

of the matrix J of the form (1.1), (1.2) is called the *spectral data* of this matrix.

Determination of the spectral data of a given Jacobi matrix is called the *direct spectral problem* for this matrix.

Thus, the spectral data consist of the eigenvalues and associated normalizing numbers derived by decomposing the resolvent function (Weyl-Titchmarsh function) into partial fractions using the eigenvalues. The resolvent function $w(\lambda)$ of the matrix J can be constructed by using (2.12). Another convenient formula for computing the resolvent function is (see [10, Section 5])

$$w(\lambda) = -\frac{\det\left(J^{(1)} - \lambda I\right)}{\det(J - \lambda I)}, \tag{2.27}$$

where $J^{(1)}$ is the matrix obtained from J by deleting the first row and first column of J.

It follows from (2.27) that $\lambda w(\lambda)$ tends to 1 as $\lambda \to \infty$. Therefore multiplying (2.21) by λ and passing then to the limit as $\lambda \to \infty$, we find

$$\sum_{k=1}^{N} \beta_k = 1. \tag{2.28}$$

The *inverse spectral problem* is stated as follows.

(i) To see if it is possible to reconstruct the matrix J, given its spectral data (2.26). If it is possible, to describe the reconstruction procedure;

(ii) To find the necessary and sufficient conditions for a given collection (2.26) to be spectral data for some matrix J of the form (1.1) with entries belonging to the class (1.2).

The solution of this problem is well known (see [1, Section 4.6] and [10]) and let us bring here the final result.

Given a collection (2.26), where $\lambda_1, \ldots, \lambda_N$ are real and distinct and $\beta_1, \ldots, \beta_N$ are positive, define the numbers:

$$s_l = \sum_{k=1}^{N} \beta_k \lambda_k^l, \quad l = 0, 1, 2, \ldots, \tag{2.29}$$

and using these numbers introduce the determinants:

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n = 0, 1, 2, \ldots. \tag{2.30}$$

Lemma 2.2. *For the determinants D_n defined by* (2.30) *and* (2.29), *we have $D_n > 0$ for $n \in \{0, 1, \ldots, N-1\}$ and $D_n = 0$ for $n \geq N$.*

Proof. Denote by A the $(n+1) \times (n+1)$ matrix corresponding to the determinant D_n given by (2.30). Then for arbitrary real column vector $x = (x_0, x_1, \ldots, x_n)^T$, we have

$$\langle Ax, x \rangle = \sum_{j,m=0}^{n} s_{j+m} x_m x_j = \sum_{j,m=0}^{n} \left(\sum_{k=1}^{N} \beta_k \lambda_k^{j+m} \right) x_m x_j = \sum_{k=1}^{N} \beta_k \left(\sum_{j=0}^{n} x_j \lambda_k^j \right)^2 = \sum_{k=1}^{N} \beta_k [G(\lambda_k)]^2 \geq 0, \tag{2.31}$$

where

$$G(\lambda) = \sum_{j=0}^{n} x_j \lambda^j. \tag{2.32}$$

Further, it follows that if $\langle Ax, x\rangle = 0$, then

$$G(\lambda_k) = 0, \quad k = 1, \ldots, N. \tag{2.33}$$

If $n \leq N-1$, then $\deg G \leq N-1$ and (2.33) is possible only if $G(\lambda) \equiv 0$ (recall that $\lambda_1, \ldots, \lambda_N$ are distinct). But then $x_0 = x_1 = \cdots = x_n = 0$. Therefore,

$$\langle Ax, x\rangle > 0, \tag{2.34}$$

for all nonzero real vectors $x = (x_0, x_1, \ldots, x_n)^T$ if $n \leq N-1$. Then as is well known from Linear Algebra, we have $\det A > 0$. Thus we have proved that $D_n > 0$ for $n \leq N-1$.

To prove that $D_n = 0$ for $n \geq N$, let us define the linear functional Ω on the linear space of all polynomials in λ with complex coefficients as follows: if $G(\lambda)$ is a polynomial, then the value $\langle \Omega, G(\lambda)\rangle$ of the functional Ω on the element (polynomial) G is

$$\langle \Omega, G(\lambda)\rangle = \sum_{k=1}^{N} \beta_k G(\lambda_k). \tag{2.35}$$

Let $m \geq 0$ be a fixed integer and set

$$T(\lambda) = \lambda^m(\lambda-\lambda_1)\cdots(\lambda-\lambda_N) = t_m\lambda^m + t_{m+1}\lambda^{m+1} + \cdots + t_{m+N-1}\lambda^{m+N-1} + \lambda^{m+N}. \tag{2.36}$$

Then, according to (2.35),

$$\left\langle \Omega, \lambda^l T(\lambda)\right\rangle = 0, \quad l = 0, 1, 2, \ldots. \tag{2.37}$$

Consider (2.37) for $l = 0, 1, 2, \ldots, N+m$, and substitute (2.36) in it for $T(\lambda)$. Taking into account that

$$\left\langle \Omega, \lambda^l\right\rangle = \sum_{k=1}^{N} \beta_k \lambda_k^l = s_l, \quad l = 0, 1, 2, \ldots, \tag{2.38}$$

we get

$$\begin{gathered} t_m s_{l+m} + t_{m+1}s_{l+m+1} + \cdots + t_{m+N-1}s_{l+m+N-1} + s_{l+m+N} = 0, \\ l = 0, 1, 2, \ldots, N+m. \end{gathered} \tag{2.39}$$

Therefore, $(0, \ldots, 0, t_m, t_{m+1}, \ldots, t_{m+N-1}, 1)$ is a nontrivial solution of the homogeneous system of linear algebraic equations:

$$\begin{gathered} x_0 s_l + x_1 s_{l+1} + \cdots + x_m s_{l+m} + x_{m+1}s_{l+m+1} + \cdots + x_{m+N-1}s_{l+m+N-1} + x_{m+N}s_{l+m+N} = 0, \\ l = 0, 1, 2, \ldots, N+m, \end{gathered} \tag{2.40}$$

with the unknowns $x_0, x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+N-1}, x_{m+N}$. Therefore, the determinant of this system, which coincides with D_{N+m}, must be equal to zero. □

Theorem 2.3. *Let an arbitrary collection* (2.26) *of numbers be given. In order for this collection to be the spectral data for a Jacobi matrix J of the form* (1.1) *with entries belonging to the class* (1.2), *it is necessary and sufficient that the following two conditions be satisfied:*

(i) *The numbers $\lambda_1, \ldots, \lambda_N$ are real and distinct.*

(ii) *The numbers $\beta_1, \ldots, \beta_N$ are positive and such that $\beta_1 + \cdots + \beta_N = 1$.*

Under the conditions (i) and (ii) we have $D_n > 0$ for $n \in \{0, 1, \ldots, N-1\}$ and the entries a_n and b_n of the matrix J for which the collection (2.26) *is spectral data, are recovered by the formulae*

$$a_n = \frac{\pm\sqrt{D_{n-1}D_{n+1}}}{D_n}, \quad n \in \{0, 1, \ldots, N-2\}, \ D_{-1} = 1, \tag{2.41}$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \quad n \in \{0, 1, \ldots, N-1\}, \quad \Delta_{-1} = 0, \ \Delta_0 = s_1, \tag{2.42}$$

where D_n is defined by (2.30) *and* (2.29), *and Δ_n is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$.*

It follows from the above solution of the inverse problem that the matrix (1.1) is not uniquely restored from the spectral data. This is linked with the fact that the a_n are determined from (2.41) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs + and −. Namely, let $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ be a given finite sequence, where for each $n \in \{0, 1, \ldots, N-2\}$ the σ_n is + or −. We have 2^{N-1} such different sequences. Now to determine a_n uniquely from (2.41) for $n \in \{0, 1, \ldots, N-2\}$, we can choose the sign σ_n when extracting the square root. In this way, we get precisely 2^{N-1} distinct Jacobi matrices possessing the same spectral data. The inverse problem is solved uniquely from the data consisting of the spectral data and a sequence $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ of signs + and −. Thus, we can say that the inverse problem with respect to the spectral data is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix. In particular, the inverse problem is solvable uniquely in the class of entries $a_n > 0$, $b_n \in \mathbb{R}$.

3. Inverse Problem for Two Spectra

Let J be an $N \times N$ Jacobi matrix of the form (1.1) with entries satisfying (1.2). Define J_1 to be the truncated Jacobi matrix given by (1.3). We denote the eigenvalues of the matrices J and J_1 by $\lambda_1 < \cdots < \lambda_N$ and $\mu_1 < \cdots < \mu_{N-1}$, respectively. We call the collections $\{\lambda_k (k = 1, \ldots, N)\}$ and $\{\mu_k (k = 1, \ldots, N-1)\}$ the *two spectra* of the matrix J.

The inverse problem for two spectra consists in the reconstruction of the matrix J by two of its spectra.

We will reduce the inverse problem for two spectra to the inverse problem for eigenvalues and normalizing numbers solved above in Section 2.

First, let us study some necessary properties of the two spectra of the Jacobi matrix J.

Let $P_n(\lambda)$ and $Q_n(\lambda)$ be the polynomials of the first and second kind for the matrix J. By (2.5) we have

$$\det(J-\lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda), \tag{3.1}$$

$$\det(J_1-\lambda I) = (-1)^{N-1} a_0 a_1 \cdots a_{N-2} P_{N-1}(\lambda). \tag{3.2}$$

Note that we have used the fact that $a_{N-1} = 1$. Therefore, the eigenvalues $\lambda_1, \ldots, \lambda_N$ and $\mu_1, \ldots, \mu_{N-1}$ of the matrices J and J_1 coincide with the zeros of the polynymials $P_N(\lambda)$ and $P_{N-1}(\lambda)$, respectively.

Dividing both sides of (2.6) by $P_{N-1}(\lambda)P_N(\lambda)$ gives

$$\frac{Q_N(\lambda)}{P_N(\lambda)} - \frac{Q_{N-1}(\lambda)}{P_{N-1}(\lambda)} = \frac{1}{P_{N-1}(\lambda)P_N(\lambda)}. \tag{3.3}$$

Therefore, by formula (2.12) for the resolvent function $w(\lambda)$, we obtain

$$w(\lambda) = \frac{Q_{N-1}(\lambda)}{P_{N-1}(\lambda)} + \frac{1}{P_{N-1}(\lambda)P_N(\lambda)}. \tag{3.4}$$

Lemma 3.1. *The matrices J and J_1 have no common eigenvalues, that is, $\lambda_k \neq \mu_j$ for all values of k and j.*

Proof. Suppose that λ is an eigenvalue of the matrices J and J_1. Then by (3.1) and (3.2) we have $P_N(\lambda) = P_{N-1}(\lambda) = 0$. But this is impossible by (2.6). □

Lemma 3.2. *The equality (trace formula)*

$$\lambda_N + \sum_{k=1}^{N-1} (\lambda_k - \mu_k) = b_{N-1} \tag{3.5}$$

holds.

Proof. For any matrix $A = [a_{jk}]_{j,k=1}^N$ the spectral trace of A coincides with the matrix trace of A : If $\nu_1, \ldots, \nu_N$ are the eigenvalues of A, then

$$\sum_{k=1}^N \nu_k = \sum_{k=1}^N a_{kk}. \tag{3.6}$$

Therefore, we can write

$$\sum_{k=1}^N \lambda_k = b_0 + b_1 + \cdots + b_{N-2} + b_{N-1}, \qquad \sum_{k=1}^{N-1} \mu_k = b_0 + b_1 + \cdots + b_{N-2}. \tag{3.7}$$

Subtracting the last two equalities side by side, we arrive at (3.5). □

Lemma 3.3. *The eigenvalues of J and J_1 interlace:*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 \cdots < \lambda_{N-1} < \mu_{N-1} < \lambda_N. \tag{3.8}$$

Proof. Let us set

$$\psi(\lambda) = \frac{P_{N-1}(\lambda)}{P_N(\lambda)}, \tag{3.9}$$

so that $\psi(\lambda)$ is a rational function whose poles coincide with the eigenvalues of J and whose zeros coincide with the eigenvalues of J_1. Applying Lemma 2.1 to the rational function $\psi(\lambda)$ we can write

$$\psi(\lambda) = \sum_{k=1}^{N} \frac{\gamma_k}{\lambda - \lambda_k}, \tag{3.10}$$

where

$$\gamma_k = \frac{P_{N-1}(\lambda_k)}{P'_N(\lambda_k)}. \tag{3.11}$$

Next, (2.24) shows that $P_{N-1}(\lambda_k)P'_N(\lambda_k) > 0$, that is, $P_{N-1}(\lambda_k)$ and $P'_N(\lambda_k)$ have the same sign. Then (3.11) implies that $\gamma_k > 0 (k = 1, \ldots, N)$.

Differentiating (3.10) we get

$$\psi'(\lambda) = -\sum_{k=1}^{N} \frac{\gamma_k}{(\lambda - \lambda_k)^2}. \tag{3.12}$$

It follows from (3.12) that $\psi'(\lambda) < 0$ for real values of λ, different from $\lambda_1, \ldots, \lambda_N$. Therefore, $\psi(\lambda)$ is strictly decreasing continuous function on the intervals $(-\infty, \lambda_1)$, $(\lambda_1, \lambda_2), \ldots,$ $(\lambda_{N-1}, \lambda_N)$, (λ_N, ∞). Besides, it follows from (3.10) that

$$\lim_{|\lambda| \to \infty} \psi(\lambda) = 0, \qquad \lim_{\lambda \to \lambda_k^-} \psi(\lambda) = -\infty, \qquad \lim_{\lambda \to \lambda_k^+} \psi(\lambda) = \infty. \tag{3.13}$$

Consequently, the function $\psi(\lambda)$ has no zero in the intervals $(-\infty, \lambda_1)$ and (λ_N, ∞), and exactly one zero in each of the intervals $(\lambda_1, \lambda_2), \ldots, (\lambda_{N-1}, \lambda_N)$. Since the zeros of the function $\psi(\lambda)$ coincide with the eigenvalues of J_1 by (3.9), the proof is complete. □

The following lemma gives a formula for calculating the normalizing numbers $\beta_1, \ldots, \beta_N$ in terms of the two spectra.

Lemma 3.4. *For each $k \in \{1, \ldots, N\}$ the formula*

$$\beta_k = \frac{a}{\prod_{j=1, j \neq k}^{N} (\lambda_j - \lambda_k) \prod_{j=1}^{N-1} (\mu_j - \lambda_k)} \tag{3.14}$$

holds, where

$$\frac{1}{a} = \sum_{m=1}^{N} \frac{1}{\prod_{j=1,\, j \neq m}^{N} (\lambda_j - \lambda_m) \prod_{j=1}^{N-1} (\mu_j - \lambda_m)}. \tag{3.15}$$

Proof. Substituting (2.21) in the left-hand side of (3.4), we can write

$$\sum_{m=1}^{N} \frac{\beta_m}{\lambda - \lambda_m} = \frac{Q_{N-1}(\lambda)}{P_{N-1}(\lambda)} + \frac{1}{P_{N-1}(\lambda) P_N(\lambda)}. \tag{3.16}$$

Multiply both sides of the last equality by $\lambda - \lambda_k$ and pass then to the limit as $\lambda \to \lambda_k$. Taking into account that $P_N(\lambda_k) = 0$, $P'_N(\lambda_k) \neq 0$, $P_{N-1}(\lambda_k) \neq 0$ (see (2.23) and (2.24)), we get

$$\beta_k = \frac{1}{P'_N(\lambda_k) P_{N-1}(\lambda_k)}. \tag{3.17}$$

Next, by (3.1) and (3.2) we have

$$\begin{gathered} (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) = \prod_{j=1}^{N} (\lambda_j - \lambda), \\ (-1)^{N-1} a_0 a_1 \cdots a_{N-2} P_{N-1}(\lambda) = \prod_{j=1}^{N-1} (\mu_j - \lambda). \end{gathered} \tag{3.18}$$

Substituting these in the right-hand side of (3.17), we obtain

$$\beta_k = \frac{a}{\prod_{j=1,\, j \neq k}^{N} (\lambda_j - \lambda_k) \prod_{j=1}^{N-1} (\mu_j - \lambda_k)}, \tag{3.19}$$

where

$$a = (a_0 a_1 \cdots a_{N-2})^2. \tag{3.20}$$

Replacing k by m in (3.19) and then summing this equation over $m = 1, \ldots, N$ and taking into account (2.28), we get (3.15). The lemma is proved. □

Theorem 3.5. *(uniqueness result). The two spectra $\{\lambda_k\}_{k=1}^{N}$ and $\{\mu_k\}_{k=1}^{N-1}$ of the Jacobi matrix J of the form* (1.1) *in the class*

$$a_n > 0, \quad b_n \in \mathbb{R}, \tag{3.21}$$

uniquely determine the matrix J.

Proof. Given the two spectra $\{\lambda_k\}_{k=1}^{N}$ and $\{\mu_k\}_{k=1}^{N-1}$ of the matrix J, we determine uniquely the normalizing numbers $\beta_k(k = 1,\ldots,N)$ of the matrix J by (3.14) and (3.15). Since the collection of the eigenvalues and normalizing numbers $\{\lambda_k,\beta_k(k = 1,\ldots,N)\}$ of the matrix J determines J uniquely in the class (3.21), the proof is complete. □

The following theorem solves the inverse problem in terms of the two spectra. Its proof given below contains an effective procedure for the construction of the Jacobi matrix from its two spectra.

Theorem 3.6. *In order for giving two collections of real numbers* $\{\lambda_k\}_{k=1}^{N}$ *and* $\{\mu_k\}_{k=1}^{N-1}$ *to be the spectra of two matrices* J *and* J_1*, respectively, of the forms* (1.1) *and* (1.3) *with the entries in the class* (1.2)*, it is necessary and sufficient that the following inequalities be satisfied:*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 \cdots < \lambda_{N-1} < \mu_{N-1} < \lambda_N. \tag{3.22}$$

Proof. The necessity of the condition (3.22) has been proved above in Lemma 3.3. To prove the sufficiency, suppose that two collections of real numbers $\{\lambda_k\}_{k=1}^{N}$ and $\{\mu_k\}_{k=1}^{N-1}$ are given which satisfy the inequalities in (3.22). We construct $\beta_k(k = 1,\ldots,N)$ according to these data by (3.14) and (3.15). It follows from (3.22) that

$$\prod_{j=1,\ sj\neq k}^{N}(\lambda_j - \lambda_k)\prod_{j=1}^{N-1}(\mu_j - \lambda_k) > 0, \quad k = 1,\ldots,N. \tag{3.23}$$

Therefore, the expression on the right-hand side of (3.14) is positive and hence $\beta_k > 0(k = 1,\ldots,N)$. Next, it follows directly from (3.14) and (3.15) that $\beta_1 + \cdots + \beta_N = 1$.

Consequently, the collection $\{\lambda_k,\beta_k(k = 1,\ldots,N)\}$ satisfies the conditions of Theorem 2.3, and hence there exists a Jacobi matrix J of the form (1.1) with entries from the class (1.2) such that the $\lambda_k(k = 1,\ldots,N)$ are the eigenvalues and the $\beta_k(k = 1,\ldots,N)$ are the corresponding normalizing numbers for J. Having the matrix J, we construct the matrix J_1 by (1.3). It remains to show that $\{\mu_k\}_{k=1}^{N-1}$ is the spectrum of the constructed matrix J_1. Denote the eigenvalues of J_1 by $\tilde{\mu}_1 < \cdots < \tilde{\mu}_{N-1}$. By Lemma 3.3,

$$\lambda_1 < \tilde{\mu}_1 < \lambda_2 < \tilde{\mu}_2 < \cdots < \lambda_{N-1} < \tilde{\mu}_{N-1} < \lambda_N. \tag{3.24}$$

We have to show that $\tilde{\mu}_k = \mu_k(k = 1,\ldots,N-1)$.

By the direct spectral problem, we have (Lemma 3.4):

$$\beta_k = \frac{\tilde{a}}{\prod_{j=1,j\neq k}^{N}(\lambda_j - \lambda_k)\prod_{j=1}^{N-1}(\tilde{\mu}_j - \lambda_k)}, \tag{3.25}$$

where

$$\frac{1}{\tilde{a}} = \sum_{m=1}^{N}\frac{1}{\prod_{j=1,j\neq m}^{N}(\lambda_j - \lambda_m)\prod_{j=1}^{N-1}(\tilde{\mu}_j - \lambda_m)}. \tag{3.26}$$

On the other hand, by our construction of β_k, we have (3.14) and (3.15). Equating the right-hand sides of (3.25) and (3.14), we obtain

$$\tilde{a}\prod_{j=1}^{N-1}(\mu_j-\lambda_k)=a\prod_{j=1}^{N-1}(\tilde{\mu}_j-\lambda_k),\qquad k=1,\dots,N. \tag{3.27}$$

This means that the polynomial

$$\tilde{a}\prod_{j=1}^{N-1}(\lambda-\mu_j)-a\prod_{j=1}^{N-1}(\lambda-\tilde{\mu}_j), \tag{3.28}$$

of degree $\leq N-1$ has N distinct zeros $\lambda_1,\dots,\lambda_N$. Then, this polynomial identically equals zero. Hence, $\tilde{a}=a$ and $\mu_j=\tilde{\mu}_j(j=1,\dots,N)$. The proof is complete. □

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