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Controlling Chaos Through Growth Rate Adjustment

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An extremely fast growth-rate controlling mechanism has been developed to stabilize a discrete system. Theoretical analysis and computer simulations have been provided to show the effectiveness and efficiency of this new mechanism in practice.

Keywords: Adaptive adjustment; Growth-rate controlling; Chaos; Nonlinear dynamics; Stabilization; Controlling chaos

INTRODUCTION

Stabilizing unstable dynamical systems through feedback adjustment applied either to an available system parameter or to the system variable(s) directly have dominated the recent researches in the field of chaos control (see Socolar *et al.*, 1994; Vieira and Lichtenberg, 1996; McGuire *et al.*, 1997; Parmananda *et al.*, 1999, Huang, 2000, 2001 and references therein). The feedback adjustment methods possess many unique advantages over other approaches such as (i) demanding neither *a priori* information about the system itself nor any external generated control signal(s), (ii) always forcing the original system to converge to its generic periodic points and (iii) easy to implement in practice. In this article, a nonlinear feedback mechanism through controlling the growth rate is developed, which has been shown in theory and by numerical simulations to be effective in stabilizing unstable periodic points of chaotic discrete systems. Comparing to the other feedback methods, this mechanism stabilize a chaotic system at a extremely high speed.

The paper is organized as follows. The second section proposes a growth-rate controlling mechanism for an one-dimensional discrete system and proves theoretically a necessary and sufficient condition for the success of this mechanism. The third section then further discusses the possible application of this new mechanism in stabilizing unstable periodic points. The fourth section generalizes the growth-rate controlling mechanism to general high-order discrete systems and provides a set of necessary conditions. Finally, in the fifth section, we further discuss some sufficient conditions for the application of growth-rate controlling mechanism in stabilization of general second-order discrete systems.

GROWTH-RATE CONTROLLING MECHANISM

Consider an one-dimensional discrete system defined by a first order difference equation:

$$x_t = \theta(x_{t-1}), \tag{1}$$

where $\theta(x_t)$ is a nonlinear function well defined in a domain $I = [x_{\min}, x_{\max}]$, where $x_{\min}$ and $x_{\max}$ are lower bound and upper bound of the domain that satisfy the condition either $0 < x_{\min} < x_{\max}$ or $x_{\min} < x_{\max} < 0$. The function θ, however, can be either "single humped" or "multiple humped", either continuous or discontinuous, either smooth or non-smooth (in the sense of C^1), but must intersect smoothly the diagonal axis $x_{t+1} = x_t$ at least once. That is, there exists at least one $\bar{x}$ such that $\theta(\bar{x}) = \bar{x}$ with $\theta'(\bar{x})$ existing.

By *growth-rate controlling mechanism*, we mean the following modification to the original system (1):

$$x_t = \theta(x_{t-1}) \triangleq x_{t-1}[1 + \gamma(\theta(x_{t-1}) - x_{t-1})], \tag{2}$$

where γ is a *control parameter* that can take any real values.

Remarks

(1) In social science, the concept of growth rate has been used extensively in almost every field that involves

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discrete dynamics. Generally, for an one-dimensional discrete dynamics given by Eq. (1), the *growth rate* is defined as

$$g_t \triangleq \frac{x_t - x_{t-1}}{x_{t-1}} = \frac{\theta(x_{t-1})}{x_{t-1}} - 1. \quad (3)$$

Apparently, if θ is chaotic, or more generally, unstable in the sense that it cannot converge to a fixed point, the growth rates derived from the process will also fluctuate chaotically and permanently. The *growth-rate controlling mechanism* defined by Eq. (2) is mathematically equivalent to

$$\frac{x_t - x_{t-1}}{x_{t-1}} = \gamma(\theta(x_{t-1}) - x_{t-1}), \quad (4)$$

which in fact controls the growth rates of the original system Eq. (1) adaptively.

(2) It is due to the fact that the growth rate defined by Eq. (3) is invalid for $x_t = 0$ that the origin is excluded from the domain I in our definition by letting $0 < x_{\min} < x_{\max}$ or $x_{\min} < x_{\max} < 0$. We can see later (in Example 2) that this restriction can be released in practice.

(3) When the control parameter takes the null value ($\gamma = 0$), the controlled system degenerates into $x_{t+1} = x$, which is "stable" for all x, therefore, this mechanism can be implemented safely in the sense that we can always start with a tiny γ and then increase gradually to speed up the stabilization process.

The following theorem briefly summarizes two desired properties of the growth-rate controlling mechanism.

THEOREM 1 *The controlled system $\tilde{\theta}$ defined by Eq. (2) possesses the following mathematical characteristics:*

i) (Generic property) *The processes θ and $\tilde{\theta}$ share exactly the same set of fixed points, that is, for any $\bar{x} \in I = [x_{\min}, x_{\max}]$, if $\theta(\bar{x}) = \bar{x}$, then $\tilde{\theta}(\bar{x}) = \bar{x}$, and vice versa.*

ii) (Necessary and sufficient condition): *For an unstable fixed point $\bar{x}$, as long as $\theta'(\bar{x}) \neq 1$, regardless of the sign of $\theta'(\bar{x})$, there always exists an effective regime for the control parameter: $\Gamma = (\gamma_{\min}, \gamma_{\max})$ such that $|\tilde{\theta}'(\bar{x})| < 1$ for $\gamma \in \Gamma = (\gamma_{\min}, \gamma_{\max})$.*

Proof

If $\bar{x} \in I = [x_{\min}, x_{\max}]$ is a fixed point such that $\theta(\bar{x}) = \bar{x}$, then,

$$\tilde{\theta}(\bar{x}) = \bar{x}[1 + \gamma(\theta(\bar{x}) - \bar{x})] = \bar{x},$$

that is, $\bar{x}$ is also a fixed point of the adjusted system. Conversely, if $\tilde{\theta}(\bar{x}) = \bar{x}$, and $\bar{x} \neq 0$, we also have $\theta(\bar{x}) = \bar{x}$.*

The derivative of θ is given by $\tilde{\theta}'(\bar{x}) = 1 + \gamma\bar{x}(\theta'(\bar{x}) - 1)$, and the stabilization means to enforce $|\tilde{\theta}'(\bar{x})| < 1$. The following four possibilities should be distinguished:

Case I: $\bar{x} > 0$ and $\theta'(\bar{x}) > 1$

The inequality $\tilde{\theta}'(\bar{x}) < 1$ requires that $1 + \gamma\bar{x}(\theta'(\bar{x}) - 1) < 1$, or, equivalently,

$$\gamma < \gamma_{\max} = 0.$$

But the inequality relation $\tilde{\theta}'(\bar{x}) > -1$ demands that $\gamma\bar{x}(\theta'(\bar{x}) - 1) > -2$, that is,

$$\gamma > \gamma_{\min} = \frac{2}{\bar{x}(1 - \theta'(\bar{x}))}.$$

Case II: $\bar{x} > 0$ and $\theta'(\bar{x}) < -1$

The inequality $\tilde{\theta}'(\bar{x}) < 1$ implies that $\gamma > \gamma_{\min} = 0$ and the inequality $\tilde{\theta}'(\bar{x}) > -1$ suggests that

$$\gamma < \gamma_{\max} = \frac{2}{\bar{x}(1 - \theta'(\bar{x}))}.$$

Case III: $\bar{x} < 0$ and $\tilde{\theta}'(\bar{x}) > 1$

By similar argument, the stabilization condition $|\tilde{\theta}'(\bar{x})| < 1$ means

$$0 = \gamma_{\min} < \gamma < \gamma_{\max} = \frac{2}{\bar{x}(1 - \theta'(\bar{x}))}.$$

Case IV: $\bar{x} < 0$ and $\theta'(\bar{x}) < -1$

$|\tilde{\theta}'(\bar{x})| < 1$ holds true if and only if

$$\frac{2}{\bar{x}(1 - \theta'(\bar{x}))} = \gamma_{\min} < \gamma < \gamma_{\max} = 0,$$

which conclude the proof. □

The conclusion in Theorem 1 can be verified by numerical simulations.

Example 1 Consider a cubic process defined by

$$x_t = g(x_{t-1}) \triangleq 1 + (x_{t-1} - 1)(4x_{t-1} - 7)^2, \quad (5)$$

which has three fixed points $\bar{x}_1 = 1$, $\bar{x}_2 = 3/2$ and $\bar{x}_3 = 2$, with $g'(\bar{x}_1) = 9$, $g'(\bar{x}_2) = -3$ and $g'(\bar{x}_3) = 9$.

While Fig. 1(a) shows the functional graph of Eq. (5), Fig. 1(b) shows a typical trajectory starting at an initial value of $\bar{x}_0 = 0.3333$.

The implementation of the growth-rate controlling mechanism to Eq. (5) amounts to modify the original

*Since the origin $\bar{x} = 0$ is always a trivial fixed point for the controlled system (2), if we do not exclude it from the domain I, the converse can not be true.

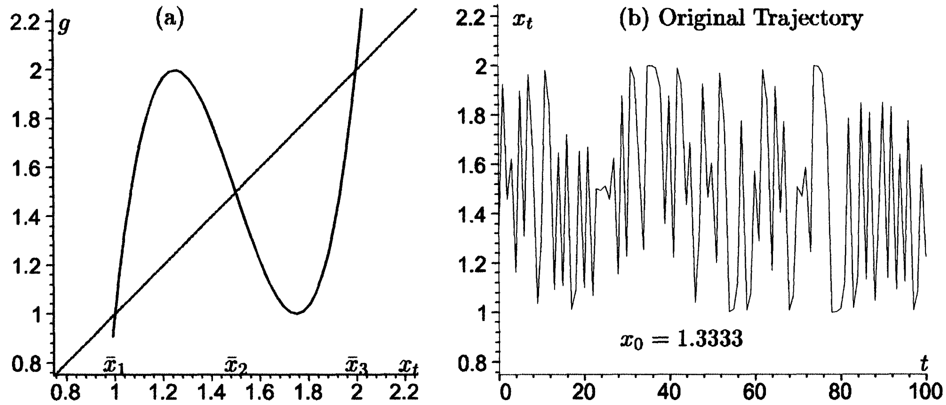

FIGURE 1 Illustration of $x_t = g(x_{t-1})$.

system into:

$$x_t = \tilde{g}(x_{t-1}) \triangleq x_{t-1}[1 + \gamma(g(x_{t-1}) - x_{t-1})]$$
$$= x_{t-1}[1 + 8\gamma(x_{t-1} - 1)(x_{t-1} - 2)(2x_{t-1} - 3)]. \quad (6)$$

For the fixed point $\bar{x}_2 = 3/2$ (with $g'(\bar{x}_2) = -3$), it follows from the proof of Theorem 1 that $\gamma_{\min} = 0$ and $\gamma_{\max} = 2/(\bar{x}(1 - g'(\bar{x}))) = 1/3$. Therefore, a positive range for the control parameter can stabilize the system to the unstable fixed point $\bar{x}_2$. While Fig. 2(a) shows the graph of Eq. (6) for $\gamma = 1/6$, Fig. 2(b) shows a trajectory starting with the same initial point as in Fig. 1(b). We see that, the convergence is achieved in less than five iterations.

Now consider the fixed point $\bar{x}_3 = 2$ (with $g'(\bar{x}_3) = 9$), we have

$$\gamma_{\min} = \frac{2}{\bar{x}(1 - g'(\bar{x}))} = \frac{-1}{8}, \ \gamma_{\max} = 0.$$

On the other hand, for the fixed point $\bar{x}_1 = 1$ (with $g'(\bar{x}_1) = 9$), we also have

$$\gamma_{\min} = \frac{2}{\bar{x}(1 - g'(\bar{x}))} = -\frac{1}{4}, \ \gamma_{\max} = 0.$$

It can be noticed that, there exists a overlapping section for the effective regimes of the control parameter γ for $\bar{x}_1$ and $\bar{x}_3$. When $\gamma \in \Gamma_1 = (-1/8, 0)$, the controlled system may converge either to $\bar{x}_1$ or $\bar{x}_3$ depending on the initial condition. This phenomenon is shown in Fig. 3(b) for $\gamma = -1/16$. As shown in Fig. 3(a), almost all trajectories starting around $\bar{x}_1$ will finally converge to $\bar{x}_1$, and almost all trajectories staring around $\bar{x}_3$ will and finally converge to $\bar{x}_3$ due to the fact that $0 < \tilde{g}'(\bar{x}_i) < 1$, for $i = 1, 3$.

But when $\gamma \in \Gamma_2 = (-1/4, -1/8]$, the controlled system will converge to the fixed point $\bar{x}_3$ only. This is shown in Fig. 4(b), from which we can see that, there exists a trapping set $J = [\bar{x}_2, x^u)$ in the domain such that the trajectory starting with almost all initial point in J stays inside it and fluctuates asymptotically, where x^u is determined from inverse iterate $x^u = \tilde{g}^{-1}(\bar{x}_2)$. However, almost all the trajectories will converge to $\bar{x}_1$ if $x_0 > x^u$ (not illustrated).

STABILIZATION OF UNSTABLE PERIODIC POINTS

It deserves to mention that the generic property held by controlled system does not hold for periodic orbits (i.e. fixed points of higher order). Even though a one-to-one correspondence exists between the periodic points of θ and those of $\tilde{\theta}$, the exact locations of these periodic points are actually different. This results from the fact that the

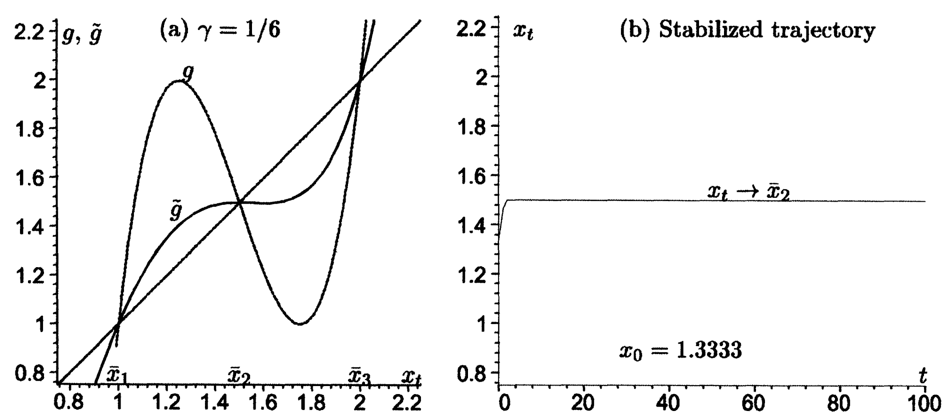

FIGURE 2 Illustration of $x_t = \tilde{g}(x_{t-1})$, $\gamma = 1/6$.

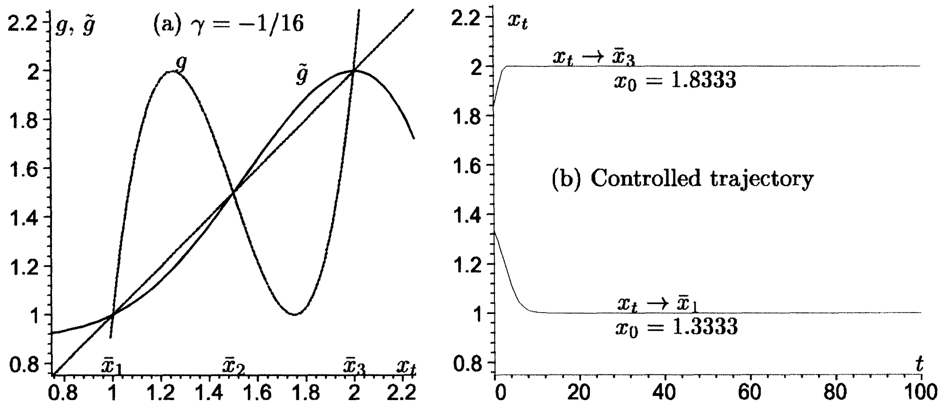

FIGURE 3 Illustration of $x_t = \tilde{g}(x_{t-1})$, $\gamma = -1/16$.

solutions to $\bar{x}^{(k)} = \theta^k(\bar{x}^{(k)})$ and to $\bar{x}^{(k)} = \tilde{\theta}^k(\bar{x}^{(k)})$ are no longer the same if k is greater than one. Although this is acceptable in most applications, there are situations where the original periodic orbits are preferred. In this regard, we need to modify the growth-rate controlling mechanism accordingly so to ensure all trajectories converge to a desired periodic orbit.

Redefine the growth-rate controlling mechanism as

$$x_t = \tilde{\theta}_{(m)}(x_{t-1}) \triangleq x_{t-1}[1 + \gamma(\theta^m(x_{t-1}) - x_{t-1})], \quad (7)$$

where $\theta^m \triangleq \underbrace{\theta \circ \theta \circ \ldots \circ \theta}_{m\,\text{times}}$ denotes mth recurrent process of θ. By similar arguments as provided in Theorem 1, the set of fixed points of $\tilde{\theta}_{(m)}$ are identical to the ones of θ^m, which implies that, with a suitable choice of γ, the growth-rate controlling mechanism defined by Eq. (7) will lead to a stable periods-m orbit inherited from θ. This can be illustrated with the following example.

Example 2 Consider the Logistic equation defined by

$$x_t = f(x_{t-1}) \triangleq 4x_{t-1}(1 - x_{t-1}), \quad (8)$$

which has two fixed points: $\bar{x}_1 = 0$ and $\bar{x}_2 = 3/4$, with $f'(\bar{x}_1) = 4$ and $f'(\bar{x}_2) = -2$, respectively.

For the controlled system:

$$x_t = \tilde{f}(x_{t-1}) \triangleq x_{t-1}[1 + \gamma(f(x_{t-1}) - x_{t-1})]$$
$$= x_{t-1}[1 + \gamma x_{t-1}(3 - 4x_{t-1})].$$

The effective regime for the control parameter γ can be obtained directly from

$$|\tilde{f}'(\bar{x}_2)| = \left|1 - \frac{9}{4}\gamma\right| < 1,$$

which requires that $0 < \gamma < 8/9$. This can be verified through the bifurcation diagram of γ shown in Fig. 5(a) (while the controlled system corresponding to $\gamma = 0.5$, 1 and 1.5 are shown in Fig. 5(b)). When γ increases above 8/9, there is a range of γ values with which the trajectory of the controlled system starting with almost any initial point will be stabilized to a periods-2 orbits. However, these periods-2 orbits are not generic, i.e. they are not inherited from the original system. Actually, solving equation $f^2(x) = x$ will reveal two trivial periods-2 fixed points: 0 and 3/4, and a pair of non-trivial periods-2 fixed points: $\bar{x}_1^{(2)} = (1/8)(5 - \sqrt{5})$ and $\bar{x}_2^{(2)} = (5/8) + (1/8)\sqrt{5}$. With $f^{2'}(x) = 16(1 - 2x)(8x^2 - 8x + 1)$, we have $f^{2'}(\bar{x}_1^{(2)}) = f^{2'}(\bar{x}_2^{(2)}) = -4$.

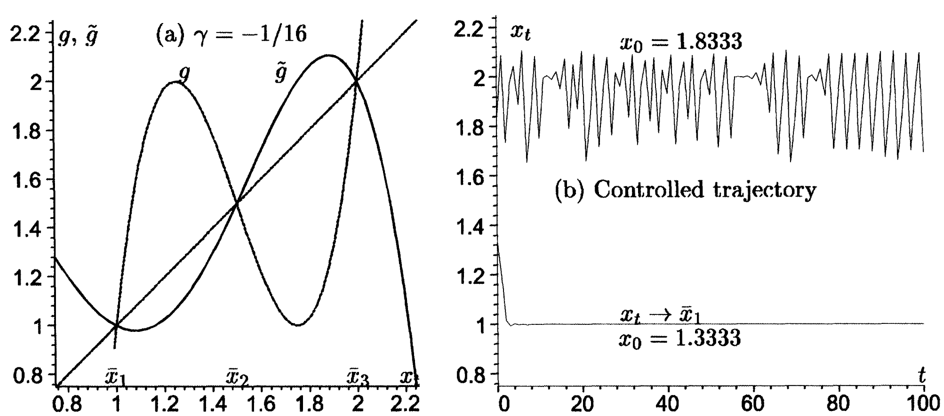

FIGURE 4 Illustration of $x_t = \tilde{g}(x_{t-1})$, $\gamma = -3/16$.

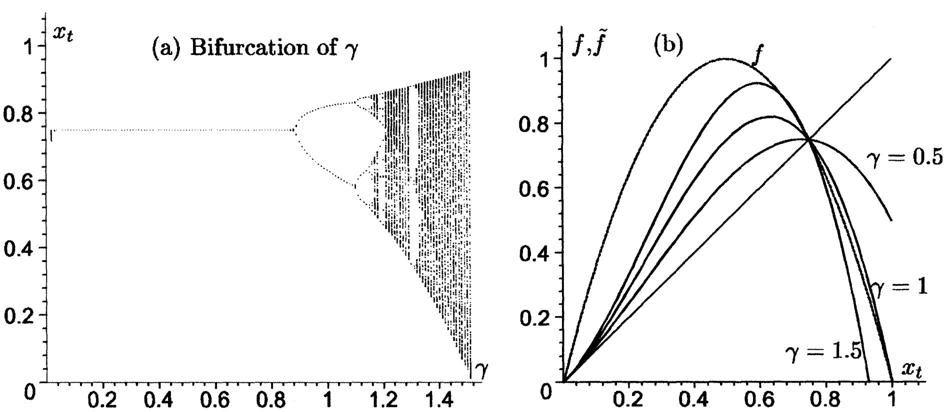

FIGURE 5 Illustration of $x_t = \tilde{f}(x_{t-1})$.

Now consider a growth-rate controlling mechanism defined by Eq. (7):

$$
\begin{aligned}
x_t =& \tilde{f}_{(2)}(x_{t-1}) \triangleq x_{t-1}[1 + \gamma(f^2(x_{t-1}) - x_{t-1})] \\
=& x_{t-1}(1 - \gamma x_{t-1}(4x_{t-1} - 3) \\
& \times (16x_{t-1}^2 - 20x_{t-1} + 5)).
\end{aligned} \tag{9}
$$

For the fixed point $\bar{x}_1^{(2)} = (5/8) - (1/8)\sqrt{5}$, we have

$$\gamma_{\min}^{(1)} = 0,\ \gamma_{\max}^{(1)} = \frac{2}{\bar{x}_2^{(2)}(1 - f^{2'}(\bar{x}_1^{(2)}))} = \frac{16}{5(5 - \sqrt{5})}.$$

But for the fixed point $\bar{x}_2^{(2)} = (5/8) + (1/8)\sqrt{5}$, we have

$$\gamma_{\min}^{(2)} = 0,\ \gamma_{\max}^{(2)} = \frac{2}{\bar{x}_2^{(2)}(1 - f^{2'}(\bar{x}_2^{(2)}))} = \frac{16}{5(5 + \sqrt{5})}.$$

Therefore, when γ is in the range of $\Gamma_1 = (0, \gamma_{\max}^{(2)})$, the controlled system (9) will converge to either $\bar{x}_1^{(2)}$ or $\bar{x}_2^{(2)}$ depending on where the initial point starts at. This is shown in Fig. 6 for $\gamma = 2/5$. On the other hand, when γ is in the range of $\Gamma_2 = [\gamma_{\max}^{(2)}, \gamma_{\max}^{(1)})$, the controlled system (9) will converge to $\bar{x}_1^{(2)}$ only, as shown in Fig. 7. However, if we notice that $f(\bar{x}_2^{(2)}) = \bar{x}_2^{(1)}$ and $f(\bar{x}_2^{(1)}) = \bar{x}_2^{(2)}$, the controlled system is actually stabilized to a stable periods-2 orbit $(\bar{x}_2^{(1)}, \bar{x}_2^{(2)})$ for $\gamma \in (0, \gamma_{\max}^{(1)})$. We notice that, all convergencies are achieved in a few iterations.

GENERALIZATION TO HIGH-ORDER DISCRETE SYSTEMS

From numerical simulations given in Example 1 and 2, we see that, the growth-rate controlling mechanism is able to stabilize a chaotic system to converge to its generic fixed point or periodic orbits in extremely high speed.

A possible generalization of growth-rate controlling mechanism is to stabilize a high-order discrete system given by

$$x_t = \theta(x_{t-1}, x_{t-2}, \ldots x_{t-n}), \tag{10}$$

where $n > 1$ and $x_t \in I = [x_{\min}, x_{\max}]$, where $x_{\min}$ and $x_{\max}$ are lower bound and upper bound of the domain that satisfy the condition either $0 < x_{\min} < x_{\max}$ or $x_{\min} < x_{\max} < 0$.

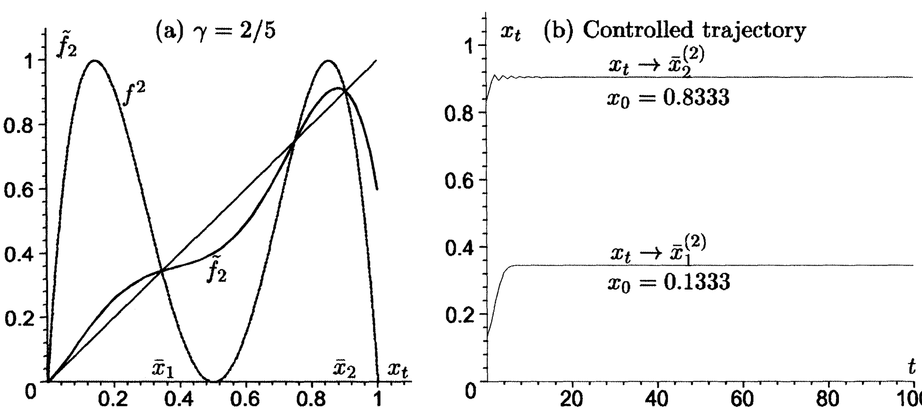

FIGURE 6 Illustration of $x_t = \tilde{f}_2(x_{t-1})$, $\gamma = 2/5$.

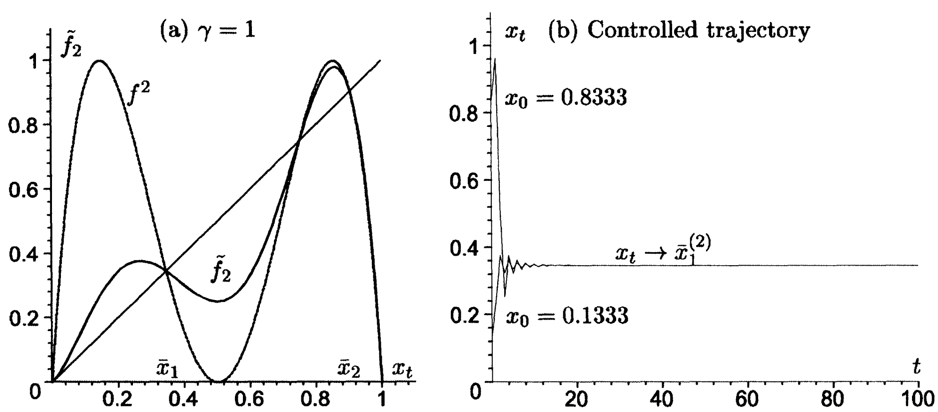

FIGURE 7 Illustration of $x_t = \tilde{f}_2(x_{t-1})$, $\gamma = 1$.

The implementation of the growth-rate controlling mechanism to Eq. (10) amounts to modify it into

$$\begin{aligned} x_t &= \tilde{\theta}_h(x_{t-1}, x_{t-2}, \ldots x_{t-n}) \\ &= x_{t-h}(1 + \gamma_h(f(x_{t-1}, x_{t-2}, \ldots, x_{t-n}) - x_{t-h})), \end{aligned} \quad (11)$$

where γ_h is the controlling parameter and $1 \le h \le n$.

The controlled system (11) is equivalent to the following growth-rate control:

$$\frac{x_t - x_{t-h}}{x_{t-h}} \triangleq \gamma_h(\theta(x_{t-1}, x_{t-2}, \ldots x_{t-n}) - x_{t-h}).$$

Let θ_i' denote the partial derivative of θ with respect to x_{t-i} evaluated at an unstable fixed point $\bar{x}$, that is,

$$\theta_i' = \left.\frac{\partial \theta}{\partial x_{t-i}}\right|_{x_{t-1}=x_{t-2}\ldots=x_{t-n}=\bar{x}}, \text{ for } i = 1, 2, \ldots, n. \quad (12)$$

We have following conclusions.

THEOREM 2 *The controlled system $\tilde{\theta}_h$ possesses the following mathematical characteristics:*

i) (Generic property): *The processes θ defined by Eq. (10) and $\tilde{\theta}_h$ defined by Eq. (11) share exactly the same set of fixed points, that is, for any $\bar{x} \in I = [x_{\min}, x_{\max}]$, if $\theta(\bar{x}) = \bar{x}$, then $\tilde{\theta}_h(\bar{x}) = \bar{x}$, and vice versa.*
ii) (Necessary conditions): *For the process $\tilde{\theta}_h$ to converge to an unstable fixed point $\bar{x}$, the original process must satisfy the condition:*

$$\sum_{i=1}^{n} \theta_i' \neq 1. \quad (13)$$

Moreover, if h is even, it must also satisfy the condition:

$$\theta_1' - \theta_2' + \ldots + (-1)^{n-1}\theta_n' \neq (-1)^{n-1}. \quad (14)$$

Proof

i) It follows the same discussions in Theorem 1.
ii) The studies of high-order discrete system such as systems (10) and (11) are usually conducted in a multi-dimensional space through variable transformations:

$$\left.\begin{aligned} y_{1,t} &= x_{t-n+1} \\ y_{2,t} &= x_{t-n+2} \\ &\cdots \\ y_{n-1,t} &= x_{t-1} \\ y_{n,t} &= x_t \end{aligned}\right\}. \quad (15)$$

with which the controlled system (11) is converted into a multi-dimensional discrete dynamical system as follows:

$$\left.\begin{aligned} y_{1,t} &= y_{2,t-1} \\ y_{2,t} &= y_{3,t-1} \\ &\cdots \\ y_{n-1,t} &= y_{n,t-1} \\ y_{n,t} &= y_{n-h+1,t-1}(1 + \gamma_h(\theta(y_{n,t-1}, y_{n-1,t-1}, \ldots, y_{1,t-1}) \\ &\quad - y_{n-h+1,t-1})) \end{aligned}\right\}. \quad (16)$$

The Jacobian of Eq. (16) at an unstable fixed point $y_1 = y_2 = \ldots = y_n = \bar{x}$ is then

$$\tilde{\mathcal{J}}(\bar{x}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \ldots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ a_n & a_{n-1} & \ldots & a_2 & a_1 \end{pmatrix}, \quad (17)$$

where

$$a_i = \begin{cases} \gamma_h \bar{x} \theta_i', & \text{if } i \neq h, \\ \gamma_h \bar{x} \theta_h' + (1 - \gamma_h \bar{x}), & \text{if } i = h. \end{cases}$$

The characteristic equation of $\tilde{\mathcal{J}}(\bar{x})$ can thus be expressed as

$$\begin{aligned} g(\lambda) = {}& \lambda^n - a_1\lambda^{n-1} - a_2\lambda^{n-2} \\ & - \ldots - a_{n-1}\lambda - a_n = 0. \end{aligned} \quad (18)$$

The convergency of adjusted system (16) to the unstable fixed point $\bar{x}$ requires that the maximum modulus of characteristic roots for Eq. (18) must be strictly less than unity. However, by noticing that

$$\sum_{i=1}^{n} a_i = 1 + \gamma_h \bar{x}\left(\sum_{i=1}^{n} \theta_i' - 1\right),$$

we may conclude that, regardless the value of γ_h and h, as long as

$$\sum_{i=1}^{n} \theta_i' = 1,$$

the unity is always one of the characteristic roots of the Eq. (18), which implies the impossibility of convergence.

Similarly, substituting $\lambda = -1$ into the characteristic polynomial $g(\lambda)$ gives us

$$(-1)^n - \gamma_h \bar{x} \sum_i (-1)^{n-i} \theta_i' - (-1)^{n-h}(1 - \gamma_h \bar{x})$$

$$= (-1)^n - (-1)^{n-h} - \gamma_h \bar{x}\left(\sum_i (-1)^{n-i} \theta_i' - 1\right).$$

Therefore, when h is even and

$$\sum_i (-1)^{n-i} \theta_i' = 1,$$

the identity $g(-1) = 0$ holds true regardless of the value of γ_h, which implies the failure of convergence as well.

In conclusion, Eq. (13) and (14) are two necessary conditions. □

STABILIZING SECOND-ORDER DISCRETE SYSTEMS

Due to the extra complexity arising from multi-dimensional dynamics, Theorem 2 reveals only some necessary conditions instead of sufficient condition for the success of growth-rate controlling mechanism under high-order discrete systems. It is difficult to provide certain sufficient conditions in general due to the presence of complex roots in characteristic equation defined by Eq. (18). However, for a relative simple second-order discrete system, we are able to give a set of sufficient conditions.

For a general second-order discrete system $x_t = \theta(x_{t-1}, x_{t-2})$, the growth-rate controlling mechanism is achieved through

$$x_t = \tilde{\theta}(x_{t-1}, x_{t-2})$$
$$= x_{t-h}(1 + \gamma_h(\theta(x_{t-1}, x_{t-2}) - x_{t-h})), \; h = 1, 2. \quad (19)$$

Case I: $h = 1$

The Jacobian of Eq. (19) at any fixed point $\bar{x}$ is thus given by

$$\mathcal{J}(x) = \begin{pmatrix} 0 & 1 \\ \gamma_1 \bar{x} \theta_2' & 1 + \gamma_1 \bar{x}(\theta_1' - 1) \end{pmatrix}. \quad (20)$$

Denote

$$\mathcal{T} = 1 + \gamma_1 \bar{x}(\theta_1' - 1) = \text{trace of } \mathcal{J},$$

$$\mathcal{D} = -\gamma_1 \bar{x} \theta_2' = \text{determinant of } \mathcal{J},$$

then the associated eigenvalues for $\mathcal{J}(x)$ can be simply expressed as (see Huang, 2001 for the detail discussion):

$$\lambda_{1,2} = \frac{1}{2}(\mathcal{T} \pm \sqrt{\mathcal{T}^2 - 4\mathcal{D}}). \quad (21)$$

The local convergency of the system to a particular fixed point $\bar{x}$ (that is, the local stability of $\bar{x}$) is guaranteed if and only if the following three inequalities hold simultaneously

$$\left.\begin{array}{r} \mathcal{D} < 1 \\ \mathcal{T} - \mathcal{D} < 1 \\ \mathcal{T} + \mathcal{D} > -1 \end{array}\right\}, \quad (22)$$

that is,

$$\left.\begin{array}{r} -\gamma_1 \bar{x} \theta_2' < 1 \\ \gamma_1 \bar{x}(\theta_1' + \theta_2' - 1) < 0 \\ \gamma_1 \bar{x}(\theta_1' - \theta_2' - 1) > -2 \end{array}\right\}, \quad (23)$$

Therefore, three inequalities given in Eq. (23) form an effective region in the adaptive parameters (γ_1, γ_2) space, inside which, all $\{\gamma_1, \gamma_2\}$ combinations ensure the success of stabilization of $\bar{x}$.

Case II: $h = 2$

By similar argument, the Jacobian of Eq. (19) at any fixed point $\bar{x}$ is now

$$\mathcal{J}(x) = \begin{pmatrix} 0 & 1 \\ 1 + \gamma_2 \bar{x}(\theta_2' - 1) & \gamma_2 \bar{x} \theta_1' \end{pmatrix}.$$

with $\mathcal{T} = \gamma_2 \bar{x} \theta_1'$ and $\mathcal{D} = -1 - \gamma_2 \bar{x}(\theta_2' - 1)$. Therefore, the conditions (22) become

$$\left.\begin{array}{r} \gamma_2 \bar{x}(1 - \theta_2') < 2 \\ \gamma_2 \bar{x}(\theta_1' + \theta_2' - 1) < 0 \\ \gamma_2 \bar{x}(\theta_1' - \theta_2' + 1) > 0 \end{array}\right\}, \quad (24)$$

The second inequality in Eq. (24) implies that $\theta_1' + \theta_2' \neq 1$ is necessary while the third inequality that $\theta_1' - \theta_2' \neq -1$ is indispensable.

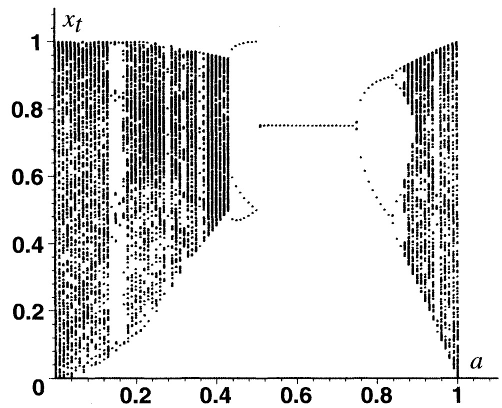

FIGURE 8 Bifurcation of $\theta_t = \theta(x_{t-1}, x_{t-2})$.

Remark A fixed point $\bar{x}$ for the original system $x_t = \theta(x_{t-1}, x_{t-2})$ is stable if the following inequalities hold:

$$\left.\begin{array}{r}\theta_2' > -1\\ \theta_1' + \theta_2' < 1\\ \theta_1' - \theta_2' > -1\end{array}\right\}. \quad (25)$$

The importance of the sufficient conditions (23) and (24) can be illustrated in the following numerical example.

Example 3 Delayed logistic system

Consider a second-order discrete system given by:

$$x_t = \theta(x_{t-1}, x_{t-2})$$
$$= 4ax_{t-1}(1 - x_{t-1}) + 4(1 - a)x_{t-2}(1 - x_{t-2}), \quad (26)$$

where $1 > a > 0$. This system is a higher-order version of famous logistic equation and hence gives a unique non-trivial fixed point $\bar{x} = 3/4$.

Since $\theta_1'(\bar{x}) = -2a$ and $\theta_2'(\bar{x}) = -2(1 - a)$, we have

$$\left.\begin{array}{r}\theta_1' - \theta_2' = 2 - 4a\\ \theta_1' + \theta_2' = -2\end{array}\right\}. \quad (27)$$

The fixed point $\bar{x} = 3/4$ is itself stable when the conditions in Eq. (25) are satisfied, which leads to $1/2 < a < 3/4$. Therefore, when $0 \le a \le 1/2$ or $3/4 \le a \le 1$, the non-trivial fixed point $\bar{x} = 3/4$ becomes unstable, which is verified by the bifurcation diagram on a shown in Fig. 8.

We now study the effective regime for γ_h under a growth-rate controlling mechanism given by

$$x_t = \tilde{\theta}_h(x_{t-1}, x_{t-2})$$
$$= x_{t-h}(1 + \gamma_h(\theta(x_{t-1}, x_{t-2}) - x_{t-h})),\ h = 1, 2. \quad (28)$$

Since $\theta_1' + \theta_2' \neq 1$, the controlled system $\tilde{\theta}_1$ is able to converge to the unstable fixed point $\bar{x}$ under suitable choice of control parameter γ_1. The conditions in Eq. (23) reduce to

$$\left.\begin{array}{r}\gamma_1 \frac{3}{2}(1 - a) < 1\\ \gamma_1 \frac{9}{4} > 0\\ \gamma_1 \frac{3}{4}(1 - 4a) > -2\end{array}\right\}. \quad (29)$$

Therefore, the effective regime for γ_1 is given by

$$\begin{cases} 0 < \gamma_1 < \frac{2}{3(1-a)} & \text{if } 0 < a < 1/4,\\ 0 < \gamma_1 < \min\left\{\frac{2}{3(1-a)}, \frac{8}{3(4a-1)}\right\} & \text{if } 1/4 < a < 1,\end{cases}$$

which is illustrated in Fig. 9(a).

Since $\theta_1' - \theta_2' = 2 - 4a = -1$ only occurs at $a = 3/4$, it follows from Theorem 2 that $\tilde{\theta}_2$ might be utilized to stabilize the unstable fixed point $\bar{x}$ for the case of $a \neq 3/4$. However, the conditions in Eq. (24) simplify to:

$$\left.\begin{array}{r}\frac{8}{3(3-2a)} > \gamma_2\\ \gamma_2 \frac{3}{4}(-2 - 1) < 0\\ \gamma_2 \frac{3}{4}(3 - 4a) > 0\end{array}\right\}, \quad (30)$$

which, as shown in Fig. 9(b), can only be held

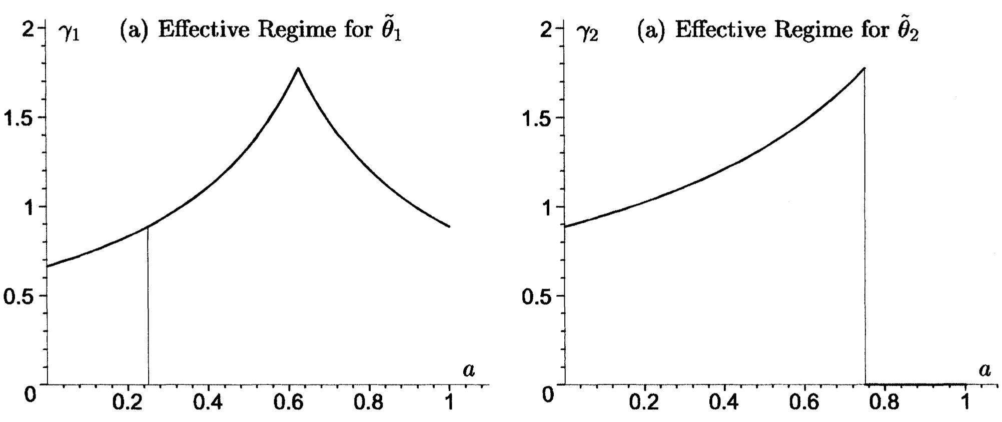

FIGURE 9 Illustration of effective regimes.

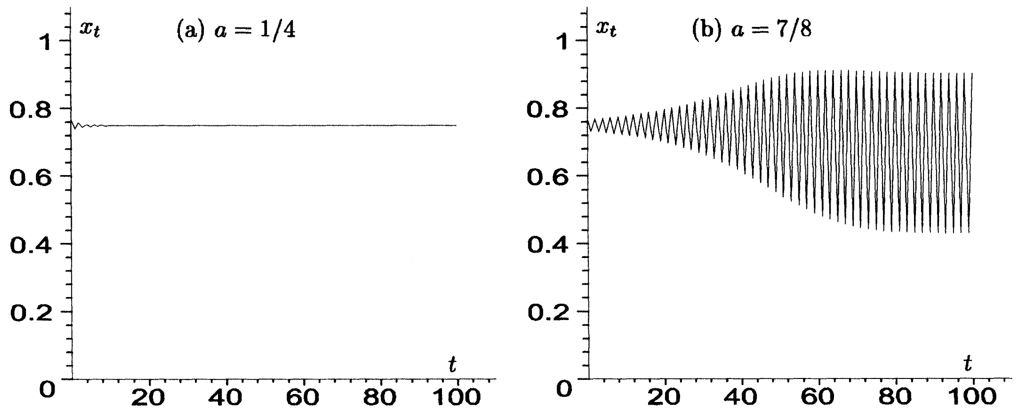

FIGURE 10 Trajectories of $x_t = \tilde{\theta}_2(x_{t-1}, x_{t-2})$, $\gamma = 1/4$.

simultaneously for the case of $a < 3/4$. Therefore, when $a \geq 3/4$, it is impossible to utilize $\tilde{\theta}_2$ to stabilize the original system. This point is demonstrated by the related numerical simulations shown in Fig. 10.

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