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Research Article

A Note on Wave Equation and Convolutions

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We study the first-order nonhomogenous wave equation. We extend the convolution theorem into a general case with a double convolution as the nonhomogenous term. The uniqueness and continuity of the solution are proved and we provide some examples in order to validate our results.

1. Introduction

The wave equation occurs in many branches of physics, in applied mathematics as well as in engineering, and it is also considered as one of the three fundamental equations in mathematical physics. The homogenous wave equation with constant coefficient can be solved by many ways such as separation of variables [1], the methods of characteristics [2, 3], and Laplace transform and Fourier transform [4]. The nonhomogenous wave equation was also studied in [3] by using the methods of eigenfunction expansion.

In this study, we consider the nonhomogenous wave equation

$$u_t + cu_x = F(x,t), \tag{1.1}$$

replace the nonhomogenous term by a single convolution and double convolutions, and prove that if F_1 and F_2 are solutions for the nonhomogenous equations, then $F_1(x) * F_2(x)$ and $F_1(x,t) *^x *^y F_2(x,t)$ are also solutions.

Definition 1.1. Let $F_1(x)$ and $F_2(x)$ be integrable functions, then the convolution of $F_1(x)$ and $F_2(x)$, as

$$F_1(x)*F_2(x) = \int_0^x F_1(x-\zeta)F_2(\zeta)d\zeta, \tag{1.2}$$

is called a single convolution, provided that the integral exists, see [4].

Definition 1.2. Let $F_1(x,y)$ and $F_2(x,y)$ be integrable functions, then the convolution of $F_1(x,y)$ and $F_2(x,y)$, as

$$F_1(x,y)*^x*^yF_2(x,y) = \int_0^y\int_0^x F_1(x-\zeta,y-\eta)F_2(\zeta,\eta)d\zeta\, d\eta, \tag{1.3}$$

is called a double convolution, and we use the symbol $*^x*^y$ to define the double convolution with respect to x and y, provided that integrals exist, see [5].

2. Main results

We start by studying the first-order nonhomogenous partial differential equation, where the nonhomogenous initial condition with convolution terms is given as

$$\begin{aligned} u_t + cu_x &= F(x,t), \\ u(x,0) &= g(x), \quad \text{where } -\infty < x < \infty,\ t > 0, \end{aligned} \tag{2.1}$$

where the nonhomogenous term of (2.1) is a single convolution defined as follows:

$$F(x,t) = F_1(x,t)*^xF_2(x,t) = \int_0^x F_1(x-\zeta,t)F_2(\zeta,t)d\zeta, \tag{2.2}$$

where t in (2.2) is constant, and we consider the initial condition as a convolution that is given by

$$g(x) = g_1(x)*g_2(x). \tag{2.3}$$

Now, the characteristic equations of (2.1) are in the following form:

$$\frac{dt}{d\beta} = 1, \qquad \frac{dx}{d\beta} = c, \qquad \frac{dv}{d\beta} = F(x,t), \tag{2.4}$$

if the initial conditions are given by

$$t(\alpha,0) = 0, \quad x(\alpha,0) = \alpha, \quad v(\alpha,0) = g(\alpha), \tag{2.5}$$

then we solve (2.4) for t and x, respectively, thus we obtain

$$t(\alpha,\beta) = \beta, \qquad x(\alpha,\beta) = \alpha + c\beta. \tag{2.6}$$

Then the last equation of (2.4), for v, becomes

$$\frac{dv}{d\beta} = F(\alpha+c\beta,\beta), \qquad v(\alpha,0) = g(\alpha), \tag{2.7}$$

by solving (2.7), we have

$$v(\alpha,\beta) = \int_0^{\beta} F(\alpha+c\tau,\tau)d\tau+g(\alpha). \tag{2.8}$$

Solving (2.8), for α, β, we get the solution of (2.1) as

$$u(x,t) = \int_0^{t} F(x-ct+c\tau,\tau)d\tau+g(x-ct). \tag{2.9}$$

In particular, if we take the nonhomogenous first-order partial differential equation with the nonhomogenous initial condition

$$\begin{aligned} u_t+u_x &= x^2+2\cos(x)-2, \\ u(x,0) &= \frac{1}{2}e^x-\frac{1}{2}\cos(x)+\frac{1}{2}\sin(x), \quad -\infty<x<\infty,\ t>0, \end{aligned} \tag{2.10}$$

where $c^2 = 1$, then it is easy to prove that

$$\begin{aligned} (x)*^x\sin(x) &= x^2+2\cos(x)-2, \\ e^x*\cos(x) &= \frac{1}{2}e^x-\frac{1}{2}\cos(x)+\frac{1}{2}\sin(x), \end{aligned} \tag{2.11}$$

and then we apply (2.10) as

$$u(x,t) = \int_0^t [(x-t+\tau)^2+2\cos(x-t+\tau)-2]d\tau+\frac{1}{2}e^{x-t}-\frac{1}{2}\cos(x-t)+\frac{1}{2}\sin(x-t). \tag{2.12}$$

If we calculate the integral in (2.12), we obtain

$$\int_0^t [(x-t+\tau)^2+2\cos(x-t+\tau)-2]d\tau = x^2t-xt^2+\frac{1}{3}t^3-2\sin(x-t)+2\sin(x)-2t. \tag{2.13}$$

Thus the solution of (2.10) can be written in the form

$$u(x,t) = \frac{1}{2}e^{x-t}-\frac{1}{2}\cos(x-t)+x^2t-xt^2+\frac{1}{3}t^3-\frac{3}{2}\sin(x-t)+2\sin(x)-2t. \tag{2.14}$$

In the following, we replace the nonhomogenous single convolution term in (2.1) by a double convolution term as

$$\begin{aligned} u_t+cu_x &= F(x,t), \\ u(x,0) &= g(x), \quad -\infty<x<\infty,\ t>0, \end{aligned} \tag{2.15}$$

where $F(x,t)$ is defined as

$$F(x,t) = f_1(x,t)*^x*^t f_2(x,t) = \int_0^t\int_0^x f_1(x-\zeta,t-\tau)f_2(\zeta,\tau)d\zeta\, d\tau, \tag{2.16}$$

then the solution will be in the form of

$$u(x,t) = \int_0^t F(x-ct+c\tau,\tau)d\tau + g(x-ct). \tag{2.17}$$

The proof of (2.15) is similar to that of (2.1).

In particular, take the nonhomogenous first-order partial differential equation with the nonhomogenous initial condition as

$$\begin{aligned} u_t + u_x &= -2\cos(x) + t\sin(x) + x\sin(t) - 2\cos(t) + 2\cos(x+t) + 2, \\ u(x,0) &= -\frac{1}{2}\sinh(x) + \frac{1}{2}x\cosh(x) + \frac{1}{2}x\sinh(x), \quad -\infty < x < \infty,\ t > 0, \end{aligned} \tag{2.18}$$

where $c^2 = 1$, then it is easy to prove that

$$\begin{aligned} (x+t) *^x *^t \sin(x+t) &= -2\cos(x) + t\sin(x) + x\sin(t) - 2\cos(t) + 2\cos(x+t) + 2, \\ e^x * \sinh(x) &= -\frac{1}{2}\sinh(x) + \frac{1}{2}x\cosh(x) + \frac{1}{2}x\sinh(x), \end{aligned} \tag{2.19}$$

where $c^2 = 1$. Now, apply (2.17) as

$$\begin{aligned} u(x,t) =& \int_0^t [-2\cos(x-t+\tau) + \tau\sin(x-t+\tau) - 2\cos(\tau)]d\tau \\ &\times \int_0^t [(x-t+\tau)\sin(\tau) + 2\cos(x-t+2\tau) + 2]d\tau \\ &- \frac{1}{2}\sinh(x-t) + \frac{1}{2}(x-t)\cosh(x-t) + \frac{1}{2}(x-t)\sinh(x-t). \end{aligned} \tag{2.20}$$

Then we get the solution of (2.18) as

$$\begin{aligned} u(x,t) =& -\frac{1}{2}\sinh(x-t) + \frac{1}{2}(x-t)\cosh(x-t) + x + t \\ &+ \frac{1}{2}(x-t)\sinh(x-t) - \sin(x) - \sin(t) - t\cos(x) - x\cos(t) + \sin(x+t). \end{aligned} \tag{2.21}$$

Theorem 2.1. *Consider the Cauchy problem as*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= F(x,t), \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= p(x), \qquad u_t(x,0) = q(x), \end{aligned} \tag{2.22}$$

where

$$F(x,t) = F_1(x,t) *^x F_2(x,t), \tag{2.23}$$

and the initial condition is defined as

$$p(x) = h_1(x) * h_2(x), \qquad q(x) = g_1(x) * g_2(x), \tag{2.24}$$

then the solution is given by

$$u(x,t) = \frac{1}{2}p(x+ct) + \frac{1}{2}p(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} q(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\beta)}^{x+c(t-\beta)} F(y,\beta)dy\,d\beta. \tag{2.25}$$

Proof. The nonhomogenous term of (2.1) is a convolution defined by

$$f_1(x,t) *^x f_2(x,t) = \int_0^x f_1(x-\zeta,t) f_2(\zeta,t) d\zeta, \tag{2.26}$$

where t is considered constant, see [6], and the nonhomogenous initial condition of (2.1) is also a single convolution defined by

$$h_1(x) *^x h_2(x) = \int_0^x h_1(x-\zeta) h_2(\zeta) d\zeta, \tag{2.27}$$

also see [6], and

$$g_1(x) *^x g_2(x) = \int_0^x g_1(x-\zeta) g_2(\zeta) d\zeta, \tag{2.28}$$

then by applying the method of reduction to first-order equation, we can write (2.1) in the following form:

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u(x,t) = F(x,t). \tag{2.29}$$

Let $w = (\partial_t - c\partial_x)u(x,t)$, then we introduce (2.1) with the initial condition to the first order as

$$\begin{aligned} w_t + cw_x &= F(x,t), \\ w(x,0) = u_t(x,0) - cu_x(x,0) &= q(x) - cp'(x), \end{aligned} \tag{2.30}$$

thus we can write the characteristic equations as

$$\frac{dt}{d\beta} = 1, \qquad \frac{dx}{d\beta} = c, \qquad \frac{dz}{d\beta} = F(x,t), \tag{2.31}$$

if the initial condition is provided as

$$t(\alpha,0) = 0, \quad x(\alpha,0) = \alpha, \quad z(\alpha,0) = q(\alpha) - cp'(\alpha). \tag{2.32}$$

Now, first of all, we solve (2.31) for t and x, then we get

$$t(\alpha,\beta) = \beta, \qquad x(\alpha,\beta) = c\beta + \alpha, \tag{2.33}$$

the last equation of (2.31), for z, becomes

$$\frac{dz}{d\beta} = F(\alpha + c\beta, \beta), \qquad z(\alpha,0) = q(\alpha) - cp'(\alpha), \tag{2.34}$$

by solving (2.34), we obtain

$$z(\alpha,\beta) = \int_0^\beta F(\alpha+c\tau,\tau)d\tau + q(\alpha) - cp'(\alpha). \tag{2.35}$$

The solution of (2.35), for α, β, will be given by

$$w(x,t) = \int_0^t F(x-ct+c\tau,\tau)d\tau + q(x-ct) - cp'(x-ct). \tag{2.36}$$

Similarly, we consider to solve the equation below by the same method:

$$\begin{aligned} u_t - cu_x &= w(x,t), \\ u(x,0) &= p(x), \end{aligned} \tag{2.37}$$

where the characteristic equations are

$$\frac{dt}{d\beta} = 1, \qquad \frac{dx}{d\beta} = -c, \qquad \frac{dz}{d\beta} = w(x,t), \tag{2.38}$$

and the initial condition is

$$t(\alpha,0) = 0, \quad x(\alpha,0) = \alpha, \quad z(\alpha,0) = p(\alpha). \tag{2.39}$$

Now, we solve (2.38), for t, x, as

$$t(\alpha,\beta) = \beta, \qquad x(\alpha,\beta) = \alpha - c\beta, \tag{2.40}$$

the last term of (2.38), for z, becomes

$$\frac{dz}{d\beta} = w(\alpha-c\beta,\beta), \qquad z(\alpha,0) = p(\alpha). \tag{2.41}$$

We solve (2.41) as

$$\begin{aligned} z(\alpha,\beta) &= \int_0^\beta w(\alpha-c\theta,\theta)d\theta + p(\alpha) \\ &= \int_0^\beta \left[\int_0^\theta F((-c\theta+\alpha)-c\theta+c\tau,\tau)d\tau\right]d\theta + \int_0^\beta [q(\alpha-2c\theta)-cp'(\alpha-2c\theta)]d\theta + p(\alpha), \end{aligned} \tag{2.42}$$

and we also solve (2.42), for α, β, where $\alpha = x+ct$ and $\beta = t$, then the solution is given by

$$\begin{aligned} u(x,t) = &\int_0^t \left[\int_0^\theta F(x+ct-2c\theta+c\tau,\tau)d\tau\right]d\theta \\ &+ \int_0^t q(x+ct-2c\theta)d\theta - \int_0^t cp'(x+ct-2c\theta)d\theta + p(x+ct). \end{aligned} \tag{2.43}$$

If we let $y = x + ct - 2c\theta$, then we get

$$\int_0^t q(x+ct-2c\theta)d\theta = \frac{1}{2c}\int_{x-ct}^{x+ct} q(y)dy, \tag{2.44}$$

similarly, we have

$$-\int_0^t cp'(x+ct-2c\theta)d\theta = -\frac{1}{2}p(x+ct) + \frac{1}{2}p(x-ct). \tag{2.45}$$

Then (2.44) and the last term of (2.43) can be written as

$$\frac{1}{2}p(x+ct) + \frac{1}{2}p(x-ct), \tag{2.46}$$

for the first term, we change the variable. Let $y = x + ct - 2c\theta + c\tau$ and $\beta = \tau$, then we can write the first term of (2.43) in the following form:

$$\int_0^t \left[\int_0^\theta F(x+ct-2c\theta+c\tau,\tau)d\tau\right]d\theta = -\int_0^t \int_{x-c(t-\beta)}^{x+c(t-\beta)} F(y,\beta)J\,dy\,d\beta, \tag{2.47}$$

where J is the Jacobian

$$J = -\frac{1}{2C}dy\,d\beta = d\tau d\theta, \tag{2.48}$$

this completes the proof of the theorem. □

Now, if we take the nonhomogenous wave equation with nonhomogenous initial condition as

$$\begin{aligned} u_{tt} - u_{xx} &= 6e^x - x^3 - 3x^2 - 6x - 6, \qquad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= \frac{1}{2}x\sin(x), \qquad u_t(x,0) = \frac{1}{2}x\cosh(x) - \frac{1}{2}\sinh(x), \end{aligned} \tag{2.49}$$

where $c^2 = 1$, and since

$$\begin{aligned} e^x *^x x^3 &= 6e^x - x^3 - 3x^2 - 6x - 6, \\ \sin(x) * \cos(x) &= \frac{1}{2}x\sin(x), \\ \sinh(x) * \sinh(x) &= \frac{1}{2}x\cosh(x) - \frac{1}{2}\sinh(x), \end{aligned} \tag{2.50}$$

we apply Theorem 2.1, then we have

$$\begin{aligned} u(x,t) = {} & \frac{1}{4}(x-t)\sin(x-t) + \frac{1}{4}(x+t)\sin(x+t) + \frac{1}{4}\int_{x-t}^{x+t}[y\cosh(y) - \sinh(y)]dy \\ & + \frac{1}{2}\int_0^t \int_{x-c(t-\beta)}^{x+c(t-\beta)} [6e^y - y^3 - 3y^2 - 6y - 6]dy\,d\beta, \end{aligned} \tag{2.51}$$

if we calculate the last two integrals, we obtain the solution of (2.49) as

$$\begin{aligned} u(x,t) = {} & 3e^{(x-t)} + 3e^{(x+t)} - \frac{1}{4}t^4 - 3xt^2 - \frac{3}{2}x^2t^2 - 3t^2 - 6e^x \\ & - \frac{1}{4}xt^4 - \frac{1}{2}x^3t^2 - \frac{1}{8}\sinh(x-t)x + \frac{1}{8}\sinh(x-t)t \\ & + \frac{1}{4}\cosh(x-t) + \frac{1}{8}\sinh(x+t)x + \frac{1}{8}\sinh(x+t)t \\ & - \frac{1}{4}\cosh(x+t) + \frac{1}{4}(x-t)\sin(x-t) + \frac{1}{4}(x+t)\sin(x+t). \end{aligned} \tag{2.52}$$

In the following theorem, we prove the uniqueness and continuity of the above-mentioned solution.

Theorem 2.2. *If $h \in C^2(\mathbb{R})$, and $k \in C^1(\mathbb{R})$, then the initial value problem*

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0, \quad |x| \le \infty, \quad |t| \le \infty, \\ u(x,0) = h(x), \qquad u_t(x,0) = k(x), \end{aligned} \tag{2.53}$$

where the nonhomogenous initial conditions are convolution terms as

$$h(x) = f_1(x) * f_2(x), \qquad k(x) = g_1(x) * g_2(x), \tag{2.54}$$

a solution of the form

$$u(x,t) = \frac{1}{2}h(x+ct) + \frac{1}{2}h(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} k(y)\,dy, \tag{2.55}$$

which is well posed and its unique solution is given by D'Alembert formula.

Proof. It is easy to see that D'Alembert formula satisfies the wave equation and the initial condition. We will focus on proving the uniqueness. The general solution is given by

$$u(x,t) = F(x-ct) + G(x+ct), \tag{2.56}$$

and the initial condition determines F and G. If we had two solutions, $p_1 = F_1 + G_1$ and $p_2 = F_2 + G_2$ solve (2.44) with the initial condition $h(x)$ and $k(x)$, then the function

$$p(x,t) = p_1(x,t) - p_2(x,t) = (F_1 - F_2) + (G_1 - G_2) \tag{2.57}$$

satisfies the following equation:

$$\begin{aligned} p_{tt} - c^2 p_{xx} = 0, \quad |x| \le \infty, \quad |t| \le \infty, \\ p(x,0) = 0, \qquad p_t(x,0) = 0, \end{aligned} \tag{2.58}$$

where F and G are uniquely determined from the initial conditions, and we have that $p(x,t) = 0$ satisfies the above system, then $F_1 \equiv F_2$ and $G_1 \equiv G_2$. For continuous dependence on data, let u, v correspond to the solution with initial data h, k and v_0, v_1, respectively, suppose

$$|h(x) - v_0(x)| \leq \delta, \qquad |k(x) - v_1(x)| \leq \delta, \qquad \forall x, \tag{2.59}$$

then for $0 \leq t \leq T$, we have

$$\begin{aligned} |u(x,t) - v(x,t)| &\leq \frac{1}{2}|h(x+ct) - v_0(x+ct)| \\ &\quad + \frac{1}{2}|h(x-ct) - v_0(x-ct)| + \frac{1}{2c}\int_{x-ct}^{x+ct} |k(y) - v_1(y)|\, dy \\ &\leq \frac{1}{2}\delta + \delta\frac{1}{2} + \frac{1}{2c}\delta(2ct) < \delta(1+T) \end{aligned} \tag{2.60}$$

for $\epsilon > 0$, if $\delta(1+T) < \epsilon$, then $|u(x,t) - v(x,t)| < \epsilon$. □

Theorem 2.3. *Consider the Cauchy problem*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= F(x,t), \qquad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= h(x), \qquad u_t(x,0) = k(x), \end{aligned} \tag{2.61}$$

where the nonhomogenseity of the term of (2.61), defined as the double convolution

$$F(x,t) = f_1(x,t) *^x *^t f_2(x,t), \tag{2.62}$$

is well posed.

Proof. It is easy to verify the uniqueness since the difference of two solutions satisfies the homogenous wave equation with zero initial condition and this was already discussed in the above theorem, we need only to consider the equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= F(x,t), \qquad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= 0, \qquad u_t(x,0) = 0. \end{aligned} \tag{2.63}$$

The above equation has a solution of the form

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\beta)}^{x+c(t-\beta)} F(y,\beta)\, dy\, d\beta. \tag{2.64}$$

Now, suppose that $F(x,t)$ and $G(x,t)$ satisfy

$$|F(x,t) - G(x,t)| < \delta, \qquad -\infty < x < \infty,\ t \geq 0. \tag{2.65}$$

If u, v are solutions corresponding to $F(x,t)$ and $G(x,t)$, respectively, and t satisfies $0 \leq t \leq T$, then

$$\begin{aligned}
|u(x,t) - v(x,t)| &= \frac{1}{2c}\left|\int_0^t \int_{x-c(t-\beta)}^{x+c(t-\beta)} (F-G)(y,\beta)\,dy\,d\beta\right| \\
&\leq \frac{\delta}{2c}\int_0^t 2c(t-\beta)\,d\beta = -\delta\left(\frac{(t-\beta)^2}{2}\bigg|_0^t\right) \\
&= \delta\left(\frac{t^2}{2}\right) \leq \delta\left(\frac{T^2}{2}\right) < \delta(T^2).
\end{aligned} \tag{2.66}$$

Thus for fixed finite time interval $[0,T]$ and $\epsilon > 0$ if $\epsilon > \delta T^2$, then $|u(x,t) - v(x,t)| < \epsilon$. □

Theorem 2.4. *Consider the Cauchy problem as*

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= F(x,t), \quad -\infty < x < \infty,\ t > 0, \\
u(x,0) &= p(x), \qquad u_t(x,0) = q(x),
\end{aligned} \tag{2.67}$$

where

$$F(x,t) = f_1(x,t) *^x *^x f_2(x,t), \tag{2.68}$$

and the initial condition, defined as

$$p(x) = h_1(x) * h_2(x), \qquad q(x) = g_1(x) * g_2(x) \tag{2.69}$$

has a solution in the form of

$$u(x,t) = \frac{1}{2}p(x+ct) + \frac{1}{2}p(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} q(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(x-\beta)}^{x+c(t-\beta)} [F(y,\beta)]\,dy\,d\beta. \tag{2.70}$$

The proof of this theorem is similar to that of Theorem 2.1.

In particular, if we take the nonhomogenous wave equation with the nonhomogenous initial condition such as

$$\begin{aligned}
u_{tt} - u_{xx} &= \frac{1}{2}xt\sin(x+t), \quad -\infty < x < \infty,\ t > 0, \\
u(x,0) &= \frac{1}{2}x\cosh(x) + \frac{1}{2}x\sinh(x) - \frac{1}{2}\sinh(x), \\
u_t(x,0) &= 2\sinh(x) - 2x,
\end{aligned} \tag{2.71}$$

where $c^2 = 1$,then it is easy to prove that the right-hand side of (2.71) can be written as

$$\sin(x+t) *^x *^t \cos(x+t) = \frac{1}{2}xt\sin(x+t), \tag{2.72}$$

and similarly,

$$
\begin{aligned}
e^x *^x \sinh(x) &= \frac{1}{2}x\cosh(x) + \frac{1}{2}x\sinh(x) - \frac{1}{2}\sinh(x), \\
x^2 *^x \cosh(x) &= 2\sinh(x) - 2x.
\end{aligned}
\tag{2.73}
$$

Now, we apply Theorem 2.4 as

$$
\begin{aligned}
u(x,t) = {} & \frac{1}{4}(x-t)\cosh(x-t) + \frac{1}{4}(x-t)\sinh(x-t) - \frac{1}{4}\sinh(x-t) \\
& + \frac{1}{4}(x+t)\cosh(x+t) + \frac{1}{4}(x+t)\sinh(x+t) \\
& - \frac{1}{4}\sinh(x+t)\frac{1}{2}\int_{x-t}^{x+t}[2\sinh(y) - 2y]\,dy + \frac{1}{4}\int_0^t\int_{x-(t-\beta)}^{x+(t-\beta)} y\beta\sin(y+\beta)\,dy\,d\beta,
\end{aligned}
\tag{2.74}
$$

and we integrate the last two terms of (2.74), then we obtain the solution of (2.71) as

$$
\begin{aligned}
u(x,t) = {} & -\cosh(x-t) - 2xt + \cosh(x+t) + \frac{1}{8}\sin(x-t) \\
& - \frac{1}{16}\cos(x-t)x + \frac{1}{16}\cos(x-t)t + \frac{1}{8}xt\sin(x+t) \\
& - \frac{1}{8}\cos(x+t)xt^2 + \frac{1}{16}\cos(x+t)x + \frac{3}{16}\cos(x+t)t \\
& - \frac{1}{8}\sin(x+t) + \frac{1}{8}\sin(x+t)t^2 - \frac{1}{24}\cos(x+t)t^3 - \frac{1}{2}\sinh(x-t) \\
& + \frac{1}{2}\cosh(x-t)(x-t) + \frac{1}{2}\sinh(x-t)(x-t) - \frac{1}{2}\sinh(x+t) \\
& + \frac{1}{2}\cosh(x+t)(x+t) + \frac{1}{2}\sinh(x+t)(x+t).
\end{aligned}
\tag{2.75}
$$

In the following theorem, we extend Theorem 2.4.

Theorem 2.5. *Consider the Cauchy problem as*

$$
\begin{aligned}
u_{tt} - c^2u_{xx} = F(x,t) + G(x,t), \qquad -\infty < x < \infty,\ t > 0, \\
u(x,0) = p(x), \qquad u_t(x,0) = q(x),
\end{aligned}
\tag{2.76}
$$

where

$$
F(x,t) = f_1(x,t) *^x *^x f_2(x,t), \qquad G(x,t) = S_1(x,t) *^x *^t S_2(x,t), \tag{2.77}
$$

and the initial condition given by

$$
p(x) = h_1(x) * h_2(x), \qquad q(x) = g_1(x) * g_2(x) \tag{2.78}
$$

has a solution of the form

$$\begin{aligned}u(x,t) = {} & \frac{1}{2}p(x+ct)+\frac{1}{2}p(x-ct)\\ & +\frac{1}{2c}\int_{x-ct}^{x+ct}q(y)dy+\frac{1}{2c}\int_0^t\int_{x-c(x-\beta)}^{x+c(t-\beta)}[F(y,\beta)+G(y,\beta)]dy\,d\beta.\end{aligned} \tag{2.79}$$

The proof of this theorem is similar to that of Theorem 2.4. In the following theorem, we generalized Theorem 2.5 as follows.

Theorem 2.6. *Consider the Cauchy problem as*

$$\begin{aligned}u_{tt}-c^2u_{xx} &= \sum_{i=1}^{n}F_i(x,0)*^xG_i(x,0), \qquad -\infty<x<\infty,\ t>0,\\ u(x,0) &= p(x), \qquad u_t(x,0)=q(x),\end{aligned} \tag{2.80}$$

where the initial condition defined as

$$p(x) = \sum_{i=1}^{n}h_i(x)*s_i(x), \qquad q(x) = \sum_{i=1}^{n}g_i(x)*k_i(x) \tag{2.81}$$

has a solution in the form of

$$\begin{aligned}u(x,t) = {} & \frac{1}{2}\sum_{i=1}^{n}h_i(x+ct)*s_i(x+ct)+\frac{1}{2}\sum_{i=1}^{n}h_i(x-ct)*s_i(x-ct)\\ & +\frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{i=1}^{n}g_i(y)*k_i(y)dy+\frac{1}{2c}\int_0^t\int_{x-c(x-\beta)}^{x+c(t-\beta)}\sum_{i=1}^{n}F_i(y,0)*^yG_i(y,0)dy\,d\beta.\end{aligned} \tag{2.82}$$

The proof of this theorem is similar to that of Theorem 2.4.

The result in Theorem 2.5 can be extended to a double convolution as in the following theorem.

Theorem 2.7. *Consider the Cauchy problem as*

$$\begin{aligned}u_{tt}-c^2u_{xx} &= \sum_{i=1}^{n}F_i(x,t)*^x*^tG_i(x,t), \qquad -\infty<x<\infty,\ t>0,\\ u(x,0) &= p(x), \qquad u_t(x,0)=q(x),\end{aligned} \tag{2.83}$$

where the initial condition defined as

$$p(x) = \sum_{i=1}^{n}h_i(x)*s_i(x), \qquad q(x) = \sum_{i=1}^{n}g_i(x)*k_i(x) \tag{2.84}$$

has a solution in the following form:

$$u(x,t) = \frac{1}{2}\sum_{i=1}^{n} h_i(x+ct) * s_i(x+ct) + \frac{1}{2}\sum_{i=1}^{n} h_i(x-ct) * s_i(x-ct)$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} \sum_{i=1}^{n} g_i(y) * k_i(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(x-\beta)}^{x+c(t-\beta)} \sum_{i=1}^{n} F_i(y,\beta) *^{y} *^{\beta} G_i(y,\beta)\,dy\,d\beta. \tag{2.85}$$

The proof is similar to that of Theorem 2.5.

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