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Research Article

Iterative Schemes for Zero Points of Maximal Monotone Operators and Fixed Points of Nonexpansive Mappings and Their Applications

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Two iterative schemes for finding a common element of the set of zero points of maximal monotone operators and the set of fixed points of nonexpansive mappings in the sense of Lyapunov functional in a real uniformly smooth and uniformly convex Banach space are obtained. Two strong convergence theorems are obtained which extend some previous work. Moreover, the applications of the iterative schemes are demonstrated.

1. Introduction and preliminaries

In this paper, we will present two iterative schemes with errors which are proved to be strongly convergent to a common element of the set of zero points of maximal monotone operators and the set of fixed points of nonexpansive mappings with respect to the Lyapunov functional in real uniformly smooth and uniformly convex Banach spaces. Moreover, it is shown that some results proposed by Martinez-Yanes and Xu in [1] and Solodov and Svaiter in [2] are special cases of ours. Finally, we will demonstrate the applications of our iterative schemes on both finding the minimizer of a proper convex and lower semicontinuous function and solving the variational inequalities.

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined as follows:

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad \forall x \in E, \tag{1.1}$$

where $\langle x, x^* \rangle$ denotes the value of $x^* \in E^*$ at $x \in E$. We use symbols "$\xrightarrow{s}$" and "$\xrightarrow{w}$" to represent strong and weak convergence in E or in E^*, respectively.

A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{x \in E : Tx \neq \emptyset\}$ and range $R(T) = \bigcup \{Tx : x \in D(T)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for all $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$. A monotone operator T is said to be a maximal monotone if $R(J + rT) = E^*$ for all $r > 0$. For a monotone operator T, we denote by $T^{-1}0 = \{x \in D(T) : 0 \in Tx\}$ the set of zero points of T. For a single-valued mapping $S : E \to E$, we denote by $\mathrm{Fix}(S) = \{x \in E : Sx = x\}$ the set of fixed points of S.

Lemma 1.1 (see [3, 4]). *The duality mapping J has the following properties.*

(1) *If E is a real reflexive and smooth Banach space, then $J : E \to E^*$ is single-valued.*

(2) *For all $x \in E$ and $\lambda \in R$, $J(\lambda x) = \lambda Jx$.*

(3) *If E is a real uniformly convex and uniformly smooth Banach space, then $J^{-1} : E^* \to E$ is also a duality mapping. Moreover, $J : E \to E^*$ and $J^{-1} : E^* \to E$ are uniformly continuous on each bounded subset of E or E^*, respectively.*

Lemma 1.2 (see [4]). *Let E be a real smooth and uniformly convex Banach space and let $T : E \to 2^{E^*}$ be a maximal monotone operator. Then $T^{-1}0$ is a closed and convex subset of E and the graph of T, $G(T)$, is demiclosed in the following sense: for all $\{x_n\} \subset D(T)$ with $x_n \xrightarrow{w} x$ in E and for all $y_n \in Tx_n$ with $y_n \xrightarrow{s} y$ in E^*, $x \in D(T)$ and $y \in Tx$.*

Definition 1.3. Let E be a real smooth and uniformly convex Banach space and let $T : E \to 2^{E^*}$ be a maximal monotone operator. For all $r > 0$, define the operator $Q_r^T : E \to E$ by $Q_r^T x = (J + rT)^{-1} Jx$ for all $x \in E$.

Definition 1.4 (see [5]). Let E be a real smooth Banach space. Then the Lyapunov functional $\varphi : E \times E \to R^+$ is defined as follows:

$$\varphi(x, y) = \|x\|^2 - 2\langle x, j(y) \rangle + \|y\|^2 \quad \forall x, y \in E,\ j(y) \in Jy. \tag{1.2}$$

Lemma 1.5 (see [5]). *Let E be a real reflexive, strictly convex and smooth Banach space, let C be a nonempty closed and convex subset of E, and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\varphi(x_0, x) = \min\{\varphi(z, x) : z \in C\}$.*

Define the mapping Q_C of E onto C by $Q_C x = x_0$ for all $x \in E$. Q_C is called the generalized projection operator from E onto C. It is easy to see that Q_C is coincident with the metric projection P_C in a Hilbert space.

Lemma 1.6 (see [5]). *Let E be a real reflexive, strictly convex and smooth Banach space, let C be a nonempty closed and convex subset of E, and let $x \in E$. Then, for all $y \in C$,*

$$\varphi(y, Q_C x) + \varphi(Q_C x, x) \leq \varphi(y, x). \tag{1.3}$$

Lemma 1.7 (see [6]). *Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\varphi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \xrightarrow{s} 0$ as $n \to \infty$.*

Lemma 1.8 (see [7]). *Let E be a real reflexive, strictly convex and smooth Banach space and let $T : E \to 2^{E^*}$ be a maximal monotone operator with $T^{-1}0 \neq \varnothing$. Then for all $x \in E$, $y \in T^{-1}0$ and $r > 0$, one has $\varphi(y, Q_r^T x) + \varphi(Q_r^T x, x) \leq \varphi(y, x)$.*

Lemma 1.9 (see [5]). *Let E be a real smooth Banach space, let C be a convex subset of E, let $x \in E$, and let $x_0 \in C$. Then $\varphi(x_0, x) = \inf\{\varphi(z, x) : z \in C\}$ if and only if $\langle z - x_0, Jx_0 - Jx\rangle \geq 0$ for all $z \in C$.*

Definition 1.10. Let E be a real Banach space. Then $S : E \to E$ is said to be nonexpansive with respect to the Lyapunov functional if $\varphi(Sx, Sy) \leq \varphi(x, y)$ for all $x, y \in E$.

Remark 1.11. If E is a real Hilbert space H, then S is a nonexpansive mapping in the usual sense: $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in H$.

Lemma 1.12. *Let E be a real smooth and uniformly convex Banach space. If $S : E \to E$ is a mapping which is nonexpansive with respect to the Lyapunov functional, then* $\mathrm{Fix}(S)$ *is a convex and closed subset of E.*

Proof. In fact, we only need to prove the case that $\mathrm{Fix}(S) \neq \varnothing$. For all $x, y \in \mathrm{Fix}(S)$ and $t \in [0,1]$, let $z = tx + (1-t)y$. Then we have

$$\begin{aligned}
\varphi(z, Sz) &= t\big(\|x\|^2 - 2\langle x, JSz\rangle + \|Sz\|^2\big) + (1-t)\big(\|y\|^2 - 2\langle y, JSz\rangle + \|Sz\|^2\big) \\
&\quad - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&= t\varphi(x, Sz) + (1-t)\varphi(y, Sz) - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&\leq t\varphi(x, z) + (1-t)\varphi(y, z) - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&= \varphi(z, z) = 0.
\end{aligned} \tag{1.4}$$

By using Lemma 1.7, we know that $z = Sz$, which implies that $\mathrm{Fix}(S)$ is a convex subset of E. For all $x_n \in \mathrm{Fix}(S)$ such that $x_n \xrightarrow{s} x$, it follows that $\varphi(Sx_n, Sx) \leq \varphi(x_n, x) \to 0$. Lemma 1.7 implies that $Sx_n \xrightarrow{s} Sx$ as $n \to \infty$. So $x \in \mathrm{Fix}(S)$. □

2. Strong convergence theorems

Throughout this section, we assume that E is a real uniformly smooth and uniformly convex Banach space, $S : E \to E$ is nonexpansive with respect to the Lyapunov functional and weakly sequentially continuous and $T : E \to 2^{E^*}$ is a maximal monotone operator with $T^{-1}0 \bigcap \mathrm{Fix}(S) \neq \varnothing$.

Theorem 2.1. *The sequence $\{x_n\}$ generated by the following scheme:*

$$\begin{gathered}
x_0 \in E, \quad r_0 > 0, \\
y_n = Q_{r_n}^T(x_n + e_n), \\
Jz_n = \alpha_n Jx_n + (1-\alpha_n)Jy_n, \\
u_n = Sz_n, \\
H_n = \{v \in E : \varphi(v, u_n) \leq \varphi(v, z_n) \leq \alpha_n\varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\
W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\
x_{n+1} = Q_{H_n \cap W_n} x_0 \quad \forall n \geq 0,
\end{gathered} \tag{2.1}$$

converges strongly to $Q_{T^{-1}0\bigcap \mathrm{Fix}(S)}x_0$ *provided*

(i) $\{\alpha_n\} \subset [0,1)$ *is a sequence such that* $\alpha_n \leq 1-\delta$, *for some* $\delta \in (0,1)$;

(ii) $\{r_n\} \subset (0,+\infty)$ *is a sequence such that* $\inf_{n\geq 0} r_n > 0$;

(iii) $\{e_n\} \subset E$ *is a sequence such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

Proof. We split the proof into five steps.
Step 1. Both H_n and W_n are closed and convex subsets of E.

Noting the facts that

$$\begin{aligned}
&\varphi(v,z_n) \leq \alpha_n \varphi(v,x_n) + (1-\alpha_n)\varphi(v,x_n+e_n) \\
&\Longleftrightarrow \|z_n\|^2 - \alpha_n\|x_n\|^2 - (1-\alpha_n)\|x_n+e_n\|^2 \leq 2\langle v, Jz_n - \alpha_n Jx_n - (1-\alpha_n)J(x_n+e_n)\rangle, \\
&\varphi(v,u_n) \leq \varphi(v,z_n) \Longleftrightarrow \|z_n\|^2 - \|u_n\|^2 \geq 2\langle v, Jz_n - Ju_n\rangle,
\end{aligned} \tag{2.2}$$

we can easily know that H_n is a closed and convex subset of E. It is obvious that W_n is also a closed and convex subset of E.
Step 2. $T^{-1}0\bigcap \mathrm{Fix}(S) \subset H_n \cap W_n$ for each nonnegative integer n.

To observe this, take $p \in T^{-1}0\bigcap \mathrm{Fix}(S)$. From the definition of the maximal monotone operator, we know that there exists $y_0 \in E$ such that $y_0 = Q_{r_0}^T(x_0+e_0)$. It follows from Lemma 1.8 that $\varphi(p,y_0) \leq \varphi(p,x_0+e_0)$. Then

$$\varphi(p,u_0) \leq \varphi(p,z_0) \leq \alpha_0\varphi(p,x_0) + (1-\alpha_0)\varphi(p,y_0) \leq \alpha_0\varphi(p,x_0) + (1-\alpha_0)\varphi(p,x_0+e_0), \tag{2.3}$$

which implies that $p \in H_0$.

On the other hand, it is clear that $p \in W_0 = E$. Then $p \in H_0 \cap W_0$ and therefore $x_1 = Q_{H_0\cap W_0}x_0$ are well defined.

Suppose that $p \in H_{n-1} \cap W_{n-1}$ and x_n is well defined for some $n \geq 1$. Then there exists $y_n \in E$ such that $y_n = Q_{r_n}^T(x_n+e_n)$. Lemma 1.8 implies that $\varphi(p,y_n) \leq \varphi(p,x_n+e_n)$. Thus

$$\begin{aligned}
\varphi(p,u_n) &\leq \varphi(p,z_n) \leq \alpha_n\varphi(p,x_n) + (1-\alpha_n)\varphi(p,y_n) \\
&\leq \varphi(p,z_n) \leq \alpha_n\varphi(p,x_n) + (1-\alpha_n)\varphi(p,y_n)
\end{aligned} \tag{2.4}$$

which implies that $p \in H_n$. It follows from Lemma 1.9 that

$$\langle p - x_n, Jx_0 - Jx_n\rangle = \langle p - Q_{H_{n-1}\cap W_{n-1}}x_0, Jx_0 - JQ_{H_{n-1}\cap W_{n-1}}x_0\rangle \leq 0, \tag{2.5}$$

which implies that $p \in W_n$. Hence $x_{n+1} = Q_{H_n\cap W_n}x_0$ is well defined. Then, by induction, the sequence generated by (2.1) is well defined and $T^{-1}0\bigcap \mathrm{Fix}(S) \subset H_n \cap W_n$ for each $n \geq 0$.

Step 3. $\{x_n\}$ is a bounded sequence of E.

In fact, for all $p \in T^{-1}0 \bigcap \mathrm{Fix}(S) \subset H_n \cap W_n$, it follows from Lemma 1.6 that

$$\varphi(p, Q_{W_n}x_0) + \varphi(Q_{W_n}x_0, x_0) \le \varphi(p, x_0). \tag{2.6}$$

From the definition of W_n and Lemmas 1.5 and 1.9, we know that $x_n = Q_{W_n}x_0$, which implies that $\varphi(p, x_n) + \varphi(x_n, x_0) \le \varphi(p, x_0)$. Therefore, $\{x_n\}$ is bounded.

Step 4. $\omega(x_n) \subset T^{-1}0 \bigcap \mathrm{Fix}(S)$, where $\omega(x_n)$ denotes the set consisting all of the weak limit points of $\{x_n\}$.

From the facts $x_n = Q_{W_n}x_0$, $x_{n+1} \in W_n$ and Lemma 1.6, we have

$$\varphi(x_{n+1}, x_n) + \varphi(x_n, x_0) \le \varphi(x_{n+1}, x_0)\cdot \tag{2.7}$$

Therefore, $\lim_{n\to\infty}\varphi(x_n, x_0)$ exists. Then $\varphi(x_{n+1}, x_n) \to 0$, which implies from Lemma 1.7 that $x_{n+1} - x_n \xrightarrow{s} 0$ as $n \to \infty$. Since $x_{n+1} \in H_n$, we have

$$\varphi(x_{n+1}, u_n) \le \varphi(x_{n+1}, z_n), \tag{2.8}$$

$$\varphi(x_{n+1}, z_n) \le \alpha_n\varphi(x_{n+1}, x_n) + (1 - \alpha_n)\varphi(x_{n+1}, x_n + e_n)\cdot \tag{2.9}$$

Notice that

$$\varphi(x_{n+1}, x_n + e_n) - \varphi(x_{n+1}, x_n) = \|x_n + e_n\|^2 - \|x_n\|^2 + 2\langle x_{n+1}, Jx_n - J(x_n + e_n)\rangle. \tag{2.10}$$

Since $J : E \to E^*$ is uniformly continuous on each bounded subset of E and $\|e_n\| \to 0$, we know from (2.10) that $\varphi(x_{n+1}, x_n + e_n) \to 0$, which implies that $\varphi(x_{n+1}, z_n) \to 0$ by (2.9). Moreover, (2.8) implies that $\varphi(x_{n+1}, u_n) \to 0$ as $n \to \infty$. Using Lemma 1.7 , we know that $x_{n+1} - z_n \xrightarrow{s} 0$, $x_{n+1} - u_n \xrightarrow{s} 0$ as $n \to \infty$. Since both $J : E \to E^*$ and $J^{-1} : E^* \to E$ are uniformly continuous on bounded subsets, we have $x_n - y_n \xrightarrow{s} 0$ as $n \to \infty$. From Step 3, we know that $\omega(x_n) \ne \emptyset$. Then, for all $q \in \omega(x_n)$, there exists a subsequence of $\{x_n\}$, for simplicity, we still denote it by $\{x_n\}$ such that $x_n \xrightarrow{w} q$ as $n \to \infty$. Therefore, $u_n \xrightarrow{w} q$, $z_n \xrightarrow{w} q$ and $y_n \xrightarrow{w} q$ as $n \to \infty$. Since $S : E \to E$ is weakly continuous and $u_n = Sz_n$, then $q \in \mathrm{Fix}(S)$. From the iterative scheme (2.1), we know that there exists $v_n \in Ty_n$ such that $r_n v_n = J(x_n + e_n) - Jy_n$. Then $v_n \xrightarrow{s} 0$ as $n \to \infty$. Lemma 1.2 implies that $q \in T^{-1}0$.

Step 5. $x_n \xrightarrow{s} q^* = Q_{T^{-1}0 \bigcap \mathrm{Fix}(S)}x_0$ as $n \to \infty$.

Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$ which is weakly convergent to $q \in T^{-1}0 \bigcap \mathrm{Fix}(S)$. Since $x_{n+1} = Q_{H_n \cap W_n}x_0$ and $q^* \in T^{-1}0 \bigcap \mathrm{Fix}(S) \subset H_n \cap W_n$, we have $\varphi(x_{n+1}, x_0) \le \varphi(q^*, x_0)$. Then it follows that

$$\begin{aligned} \varphi(x_n, q^*) &= \varphi(x_n, x_0) + \varphi(x_0, q^*) - 2\langle x_n - x_0, Jq^* - Jx_0\rangle \\ &\le \varphi(q^*, x_0) + \varphi(x_0, q^*) - 2\langle x_n - x_0, Jq^* - Jx_0\rangle, \end{aligned} \tag{2.11}$$

which yields

$$\begin{aligned}\limsup_{n\to\infty}\varphi(x_{n_i},q^*) &\le \varphi(q^*,x_0)+\varphi(x_0,q^*)-2\langle q-x_0,Jq^*-Jx_0\rangle\\ &= 2\langle q^*-q,Jq^*-Jx_0\rangle \le 0.\end{aligned} \tag{2.12}$$

Hence $\varphi(x_{n_i},q^*)\to 0$ as $i\to\infty$. It follows from Lemma 1.7 that $x_{n_i}\xrightarrow{s} q^*$ as $i\to\infty$. This means that the whole sequence $\{x_n\}$ converges weakly to q^* and each weakly convergent subsequence of $\{x_n\}$ converges strongly to q^*. Therefore, $x_n\xrightarrow{s} q^*=Q_{T^{-1}0\bigcap \mathrm{Fix}(S)}x_0$ as $n\to\infty$. □

Remark 2.2. If E is reduced to a real Hilbert space H and $S\equiv I$, then $Q^T_{r_n}$ equals to $J^T_{r_n}=(I+r_nT)^{-1}$. So the iterative scheme (2.1) is reduced to the following one introduced by Yanes and Xu in [1]:

$$\begin{gathered}x_0\in H \text{ chosen arbitrarily},\\ y_n=\alpha_nx_n+(1-\alpha_n)J^T_{r_n}(x_n+e_n),\\ H_n=\left\{v\in H:\left\|y_n-v\right\|^2\le\left\|x_n-v\right\|^2+2(1-\alpha_n)\langle x_n-v,e_n\rangle+\left\|e_n\right\|^2\right\},\\ W_n=\left\{z\in H:\langle z-x_n,x_0-x_n\rangle\le 0\right\},\\ x_{n+1}=P_{H_n\cap W_n}x_0,\quad \forall n\ge 0.\end{gathered} \tag{2.13}$$

They proved that, if $T^{-1}0\neq\varnothing$, then the sequence $\{x_n\}$ generated by (2.13) converges strongly to $P_{T^{-1}0}x_0$ provided

(i) $\{\alpha_n\}\subset[0,1)$ is a sequence such that $\alpha_n\le 1-\delta$ for some $\delta\in(0,1)$;

(ii) $\{r_n\}\subset(0,+\infty)$ is a sequence such that $\inf_n r_n>0$;

(iii) $\{e_n\}\subset E$ is a sequence such that $\|e_n\|\to 0$.

Remark 2.3. If E is reduced to a real Hilbert space H, $\alpha_n\equiv 0$, $e_n\equiv 0$ and $S\equiv I$, then (2.1) includes the following iterative scheme introduced by Solodov and Svaiter in [2]:

$$\begin{gathered}x_0\in H,\\ 0=v_n+\frac{1}{r_n}(y_n-x_n),\quad v_n\in Ty_n,\\ H_n=\left\{z\in H:\langle z-y_n,v_n\rangle\le 0\right\},\\ W_n=\left\{z\in H:\langle z-x_n,x_0-x_n\rangle\le 0\right\},\\ x_{n+1}=P_{H_n\cap W_n}x_0,\quad \forall n\ge 0.\end{gathered} \tag{2.14}$$

They proved that, if $T^{-1}0\neq\varnothing$ and $\liminf_{n\to\infty}r_n>0$, then the sequence generated by (2.14) converges strongly to $P_{T^{-1}0}x_0$.

Corollary 2.4. *Suppose that E and S are the same as those in Theorem 2.1. For $i = 1, 2, \ldots, m$, let $T_i : E \to 2^{E^*}$ be maximal monotone operators. Denote $D: = \bigcap_{i=1}^{m} T_i^{-1}0 \bigcap \mathrm{Fix}(S)$ and suppose that $D \neq \emptyset$. Then the sequence $\{x_n\}$ generated by*

$$
\begin{gathered}
x_0 \in E, \quad r_{0,i} > 0, \quad i = 1, 2, \ldots, m, \\
y_{n,i} = Q_{r_{n,i}}^{T_i}(x_n + e_n), \quad i = 1, 2, \ldots, m, \\
Jz_{n,i} = \alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) Jy_{n,i}, \quad i = 1, 2, \ldots, m, \\
u_{n,i} = Sz_{n,i}, \quad i = 1, 2, \ldots, m, \\
H_{n,i} = \{v \in E : \varphi(v, u_{n,i}) \leq \varphi(v, z_{n,i}) \leq \alpha_{n,i}\varphi(v, x_n) + (1 - \alpha_{n,i})\varphi(v, x_n + e_n)\}, \quad i = 1, 2, \ldots, m, \\
H_n := \bigcap_{i=1}^{m} H_{n,i}, \\
W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\
x_{n+1} = Q_{H_n \cap W_n} x_0 \quad \forall n \geq 0,
\end{gathered}
\tag{2.15}
$$

converges strongly to $Q_D x_0$ provided

(i) *$\{\alpha_{n,i}\} \subset [0, 1)$ is a sequence such that $\alpha_{n,i} \leq 1 - \delta$, for some $\delta \in (0, 1)$, $i = 1, 2, \ldots, m$ and $n \geq 0; 1, 2, \ldots,$*

(ii) *$\{r_{n,i}\} \subset (0, +\infty)$ is a sequence such that $\inf_{n \geq 0} r_{n,i} > 0$ for $i = 1, 2, \ldots, m$;*

(iii) *$\{e\} \subset E$ is a sequence such that $\|e_n\| \to 0$ as $n \to \infty$.*

Similar to the proof of Theorem 2.1, we have the following result.

Theorem 2.5. *The sequence $\{x_n\}$ generated by*

$$
\begin{gathered}
x_0 \in E, \quad r_0 > 0, \\
y_n = Q_{r_n}^{T}(x_n + e_n), \\
Jz_n = \alpha_n Jx_0 + (1 - \alpha_n) Jy_n, \\
u_n = Sz_n, \\
H_n = \{v \in E : \varphi(v, u_n) \leq \varphi(v, z_n) \leq \alpha_n \varphi(v, x_0) + (1 - \alpha_n)\varphi(v, x_n + e_n)\}, \\
W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\
x_{n+1} = Q_{H_n \cap W_n} x_0 \quad \forall n \geq 0,
\end{gathered}
\tag{2.16}
$$

converges strongly to $Q_{T^{-1}0 \bigcap \mathrm{Fix}(S)} x_0$ provided

(i) *$\{\alpha_n\} \subset [0, 1)$ is a sequence such that $\alpha_n \to 0$ as $n \to \infty$;*

(ii) *$\{r_n\} \subset (0, +\infty)$ is a sequence such that $\inf_{n \geq 0} r_n > 0$;*

(iii) *$\{e_n\} \subset E$ is a sequence such that $\|e_n\| \to 0$ as $n \to \infty$.*

Remark 2.6. If E is reduced to a real Hilbert space H and $S \equiv I$, then the iterative scheme (2.16) is reduced to the following one, which is similar to that in [1]:

$$\begin{gathered}
x_0 \in H \text{ chosen arbitrarily}, \\
y_n = \alpha_n x_0 + (1-\alpha_n) J_{r_n}^{T}(x_n + e_n), \\
H_n = \{ v \in H : \|y_n - v\|^2 \le \|x_n - v\|^2 + \alpha_n (\|x_0\|^2 + 2\langle x_n - x_0, v\rangle) \\
+2(1-\alpha_n)\langle x_n - v, e_n\rangle + (1-\alpha_n)\|e_n\|^2 - \alpha_n \|x_n\|^2 \}, \\
W_n = \{ z \in H : \langle z - x_n, x_0 - x_n\rangle \le 0 \}, \\
x_{n+1} = P_{H_n \cap W_n} x_0 \quad \forall n \ge 0.
\end{gathered} \tag{2.17}$$

Corollary 2.7. *Suppose that E, S, T_i, and D are the same as those in Corollary 2.4. If $D \neq \emptyset$, then the sequence $\{x_n\}$ generated by*

$$\begin{gathered}
x_0 \in E, \quad r_{0,i} > 0, \\
y_{n,i} = Q_{r_{n,i}}^{T_i}(x_n + e_n), \\
J z_{n,i} = \alpha_{n,i} J x_0 + (1-\alpha_{n,i}) J y_{n,i}, \\
u_{n,i} = S z_{n,i}, \\
H_{n,i} = \{ v \in E : \varphi(v, u_{n,i}) \le \varphi(v, z_{n,i}) \le \alpha_{n,i}\varphi(v, x_0) + (1-\alpha_{n,i})\varphi(v, x_n + e_n) \}, \\
H_n := \bigcap_{i=1}^{m} H_{n,i}, \quad i = 1, 2, \dots, m, \\
W_n = \{ z \in E : \langle z - x_n, J x_0 - J x_n\rangle \le 0 \}, \\
x_{n+1} = Q_{H_n \cap W_n} x_0 \quad \forall n \ge 0,
\end{gathered} \tag{2.18}$$

converges strongly to $Q_D x_0$ provided

(i) $\{\alpha_{n,i}\} \subset [0,1)$ *is a sequence such that* $\alpha_{n,i} \to 0$ *as* $n \to \infty$ *for* $i = 1, 2, \dots, m$;

(ii) $\{r_{n,i}\} \subset (0, +\infty)$ *is a sequence such that* $\inf_{n\ge 0} r_{n,i} > 0$ *for* $i = 1, 2, \dots, m$;

(iii) $\{e_n\} \subset E$ *is a sequence such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

3. Applications to minimization problem

Definition 3.1. Let $f : E \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then the subdifferential ∂f of f is defined by

$$\partial f(z) = \{ v \in E^* : f(y) \ge f(z) + \langle y - z, v\rangle, \ \forall y \in E \} \quad \forall z \in E\cdot \tag{3.1}$$

Theorem 3.2. *Let E, S, $\{\alpha_n\}$, $\{r_n\}$, and $\{e_n\}$ be the same as those in Theorem 2.1. Let $f : E \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered} x_0 \in E, \quad r_0 > 0, \\ y_n = \arg\min_{z\in E} \left\{ f(z) + \frac{1}{2r_n}\|z_n\|^2 - \frac{1}{r_n}\langle z, J(x_n + e_n)\rangle \right\}, \\ Jz_n = \alpha_n Jx_n + (1-\alpha_n) Jy_n, \\ u_n = Sz_n, \\ H_n = \{v \in E : \varphi(v, u_n) \le \varphi(v, z_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0 \quad \forall n \ge 0. \end{gathered} \tag{3.2}$$

If $(\partial f)^{-1}0 \bigcap \operatorname{Fix}(S) \neq \emptyset$, then $\{x_n\}$ converges strongly to $Q_{(\partial f)^{-1}0 \bigcap \operatorname{Fix}(S)} x_0$.

Proof. Since $f : E \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function, the subdifferential ∂f of f is a maximal monotone operator from E into E^*. We also know that

$$y_n = \arg\min_{z\in E} \left\{ f(z) + \frac{1}{2r_n}\|z_n\|^2 - \frac{1}{r_n}\langle z, J(x_n + e_n)\rangle \right\} \tag{3.3}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n} Jy_n - \frac{1}{r_n} J(x_n + e_n). \tag{3.4}$$

Thus we have $y_n = Q_{r_n}^{\partial f}(x_n + e_n)$ and so Theorem 2.1 implies that $\{x_n\}$ converges strongly to $Q_{(\partial f)^{-1}0 \bigcap \operatorname{Fix}(S)} x_0$ as $n \to \infty$. □

Similarly, we have the following theorem.

Theorem 3.3. *Let E, S, $\{\alpha_n\}$, $\{r_n\}$, and $\{e_n\}$ be the same as those in Theorem 2.5. Let $f : E \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered} x_0 \in E, \quad r_0 > 0, \\ y_n = \arg\min_{z\in E} \left\{ f(z) + \frac{1}{2r_n}\|z_n\|^2 - \frac{1}{r_n}\langle z, J(x_n + e_n)\rangle \right\}, \\ Jz_n = \alpha_n Jx_0 + (1-\alpha_n) Jy_n, \\ u_n = Sz_n, \\ H_n = \{v \in E : \varphi(v, u_n) \le \varphi(v, z_n) \le \alpha_n \varphi(v, x_0) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0 \quad \forall n \ge 0. \end{gathered} \tag{3.5}$$

If $(\partial f)^{-1}0 \bigcap \operatorname{Fix}(S) \neq \emptyset$, then $\{x_n\}$ converges strongly to $Q_{(\partial f)^{-1}0 \bigcap \operatorname{Fix}(S)} x_0$.

4. Applications on solving the variational inequalities

Definition 4.1 (see [4]). Let E be a real Banach space. A single-valued operator $A : E \to E^*$ is said to be hemicontinuous if it is continuous along each line segment in E with respect to the $weak^*$ topology of E^*.

Definition 4.2. Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $A : C \to E^*$ be a single-valued monotone operator which is hemicontinuous. Then a point $u \in C$ is said to be a solution of the variational inequality for A if

$$\langle y - u, Au \rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

We denote by $\mathrm{VI}(C, A)$ the set of all solutions of the variational inequality for A.

Definition 4.3. Let E be a real Banach space and let C be a nonempty closed and convex subset of E. We denote by $N_C(x)$ the normal cone for C at a point $x \in C$, that is,

$$N_C(x) = \{x^* \in E^* : \langle y - x, x^* \rangle \leq 0, \ y \in C\}. \tag{4.2}$$

In [8], it is proven that the operator $T : E \to 2^{E^*}$ defined by

$$Tx = \begin{cases} Ax + N_C(x), & x \in C, \\ \varnothing, & x \notin C, \end{cases} \tag{4.3}$$

is a maximal monotone operator. It is easy to verify that $T^{-1}(0) = \mathrm{VI}(C, A)$.

Theorem 4.4. *Let E, S be the same as those in Theorem 2.1 and let C be a nonempty closed and convex subset of E. Let $A : C \to E^*$ be a single-valued monotone operator which is hemicontinuous. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} x_0 \in E, \quad r_0 > 0, \\ y_n = \mathrm{VI}\left(C, A + \frac{1}{r_n}(J - J(x_n + e_n))\right), \\ Jz_n = \alpha_n Jx_n + (1 - \alpha_n) Jy_n, \\ u_n = Sz_n, \\ H_n = \{v \in E : \varphi(v, u_n) \leq \varphi(v, z_n) \leq \alpha_n \varphi(v, x_n) + (1 - \alpha_n)\varphi(v, x_n + e_n)\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0 \quad \forall n \geq 0, \end{gathered} \tag{4.4}$$

which converges strongly to $Q_{\mathrm{VI}(C,A) \bigcap \mathrm{Fix}(S)} x_0$ provided

(i) $\{\alpha_n\} \subset [0, 1)$ *is a sequence such that* $\alpha_n \leq 1 - \delta$, *for some* $\delta \in (0, 1)$;

(ii) $\{r_n\} \subset (0, +\infty)$ *is a sequence such that* $\inf_{n \geq 0} r_n > 0$;

(iii) $\{e_n\} \subset E$ *is a sequence such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

Proof. Note that

$$\begin{aligned} y_n &= \mathrm{VI}\left(C, A + \frac{1}{r_n}(J - J(x_n + e_n))\right) \\ &\Longleftrightarrow \left\langle y - y_n, Ay_n + \frac{1}{r_n}(Jy_n - J(x_n + e_n))\right\rangle \geq 0 \quad \forall y \in C \\ &\Longleftrightarrow J(x_n + e_n) \in r_n T y_n + J y_n \\ &\Longleftrightarrow y_n = (J + r_n T)^{-1} J(x_n + e_n) = Q^T_{r_n}(x_n + e_n), \end{aligned} \tag{4.5}$$

where T is the same as that in Definition 4.3. Then the result follows from Theorem 2.1. □

Similarly, we have the following result.

Theorem 4.5. *Let E, C, S and A be the same as those in Theorem 4.4. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} x_0 \in E, \quad r_0 > 0, \\ y_n = \mathrm{VI}\left(C, A + \frac{1}{r_n}(J - J(x_n + e_n))\right), \\ J z_n = \alpha_n J x_0 + (1 - \alpha_n) J y_n, \\ u_n = S z_n, \\ H_n = \{v \in E : \varphi(v, u_n) \leq \varphi(v, z_n)\} \leq \alpha_n \varphi(v, x_0) + (1 - \alpha_n)\varphi(v, x_n + e_n)\}, \\ W_n = \{z \in E : \langle z - x_n, J x_0 - J x_n\rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_0 \quad \forall n \geq 0, \end{gathered} \tag{4.6}$$

which converges strongly to $Q_{\mathrm{VI}(C,A) \cap \mathrm{Fix}(S)} x_0$ provided

(i) $\{\alpha_n\} \subset [0, 1)$ *is a sequence such that* $\alpha_n \to 0$ *as* $n \to \infty$;

(ii) $\{r_n\} \subset (0, +\infty)$ *is a sequence such that* $\inf_{n \geq 0} r_n > 0$;

(iii) $\{e_n\} \subset E$ *is a sequence such that* $\|e_n\| \to 0$ *as* $n \to \infty$.

Remark 4.6. It will be interesting to consider similar problems when a single mapping "S" is replaced by an amenable semigroup S of mappings that are nonexpansive with respect to the Lyapunov functional and to combine the iterative scheme for the fixed point set determined by a left regular sequence of means as demonstrated in the recent work [9] with that of Theorem 2.1.

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