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Research Article

Some New Weakly Contractive Type Multimaps and Fixed Point Results in Metric Spaces

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Some new weakly contractive type multimaps in the setting of metric spaces are introduced, and we prove some results on the existence of fixed points for such maps under certain conditions. Our results extend and improve several known results including the corresponding recent fixed point results of Pathak and Shahzad (2009), Latif and Abdou (2009), Latif and Albar (2008), Ćirić (2008), Feng and Liu (2006), and Klim and Wardowski (2007).

1. Introduction

Let (X, d) be a metric space. Let 2^X denote a collection of nonempty subsets of X, $\mathrm{Cl}(X)$ a collection of nonempty closed subsets of X, and $\mathrm{CB}(X)$ a collection of nonempty closed bounded subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \tag{1.1}$$

for every $A, B \in \mathrm{CB}(X)$, where $d(x,B) = \inf_{y\in B} d(x,y)$.

An element $x \in X$ is called a *fixed point* of a multivalued map (multimap) $T : X \to 2^X$ if $x \in T(x)$. We denote $\mathrm{Fix}(T) = \{x \in X : x \in T(x)\}$.

A sequence $\{x_n\}$ in X is called an *orbit* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geq 1$. A map $f : X \to \mathbb{R}$ is called *lower semicontinuous* if for any sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Using the concept of Hausdorff metric, Nadler [1] established the following multivalued version of the Banach contraction principle.

Theorem 1.1. *Let* (X, d) *be a complete metric space and let* $T : X \to \mathrm{CB}(X)$ *be a map such that for a fixed constant* $h \in (0, 1)$ *and for each* $x, y \in X$,

$$H(T(x), T(y)) \leq hd(x, y). \tag{1.2}$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

This result has been generalized in many directions. For instance, Mizoguchi and Takahashi [2] have obtained the following general form of the Nadler's theorem.

Theorem 1.2. *Let* (X, d) *be a complete metric space and let* $T : X \to \mathrm{CB}(X)$. *Assume that there exists a function* $k : [0, \infty) \to [0, 1)$ *such that for every* $t \in [0, \infty)$,

$$\limsup_{r \to t^+} k(r) < 1, \tag{1.3}$$

and for all $x, y \in X$,

$$H(T(x), T(y)) \leq k(d(x, y))d(x, y). \tag{1.4}$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

Many authors have been using the Hausdorff metric to obtain fixed point results for multivalued maps. But, in fact, for most cases the existence part of the results can be proved without using the concept of Hausdorff metric. Recently, Feng and Liu [3] extended Nadler's fixed point theorem without using the concept of the Hausdorff metric. They proved the following result.

Theorem 1.3. *Let* (X, d) *be a complete metric space and let* $T : X \to \mathrm{Cl}(X)$ *be a map such that for any fixed constants* $h, b \in (0, 1)$, $h < b$, *and for each* $x \in X$, *there is* $y \in T(x)$ *satisfying the following conditions:*

$$\begin{aligned} bd(x, y) &\leq d(x, T(x)), \\ d(y, T(y)) &\leq hd(x, y). \end{aligned} \tag{1.5}$$

Then $\mathrm{Fix}(T) \neq \emptyset$ *provided that a real-valued function* g *on* X, $g(x) = d(x, T(x))$, *is lower semicontinuous.*

Recently, Klim and Wardowski [4] generalized Theorem 1.3 as follows.

Theorem 1.4. *Let* (X, d) *be a complete metric space and let* $T : X \to \mathrm{Cl}(X)$. *Assume that the following conditions hold:*

(I) *if there exist a number* $b \in (0,1)$ *and a function* $k : [0,\infty) \to [0,b)$ *such that for each* $t \in [0,\infty)$,

$$\limsup_{r \to t^+} k(r) < b, \tag{1.6}$$

(II) *for any* $x \in X$ *there is* $y \in T(x)$ *satisfying*

$$\begin{aligned} bd(x,y) &\le d(x,T(x)), \\ d(y,T(y)) &\le k(d(x,y))d(x,y). \end{aligned} \tag{1.7}$$

Then $\mathrm{Fix}(T) \neq \emptyset$ *provided that a real-valued function* g *on* X, $g(x) = d(x,T(x))$, *is lower semicontinuous.*

The above results have been generalized in many directions; see for instance [5–9] and references therein.

In [10], Kada et al. introduced the concept of w-distance on a metric space as follows.

A function $\omega : X \times X \to [0,\infty)$ is called w*-distance* on X if it satisfies the following for each $x,y,z \in X$:

(w_1) $\omega(x,z) \le \omega(x,y) + \omega(y,z)$;

(w_2) a map $\omega(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous; that is, if there is a sequence $\{y_n\}$ in X with $y_n \to y \in X$, then $\omega(x,y) \le \liminf_{n\to\infty}\omega(x,y_n)$;

(w_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z,x) \le \delta$ and $\omega(z,y) \le \delta$ imply $d(x,y) \le \epsilon$.

Note that, in general for $x,y \in X$, $\omega(x,y) \neq \omega(y,x)$ and neither of the implications $\omega(x,y) = 0 \Leftrightarrow x = y$ necessarily hold. Clearly, the metric d is a w-distance on X. Let $(Y,\|\cdot\|)$ be a normed space. Then the functions $\omega_1,\omega_2 : Y \times Y \to [0,\infty)$ defined by $\omega_1(x,y) = \|y\|$ and $\omega_2(x,y) = \|x\|+\|y\|$ for all $x,y \in Y$ are w-distances [10]. Many other examples and properties of the w-distance can be found in [10, 11].

The following lemmas concerning w-distance are crucial for the proofs of our results.

Lemma 1.5 (see [10]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* X *and let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,\infty)$ *converging to* 0. *Then, for the* w*-distance* ω *on* X*, the following conditions hold for every* $x,y,z \in X$:

(a) *if* $\omega(x_n,y) \le \alpha_n$ *and* $\omega(x_n,z) \le \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$; *in particular, if* $\omega(x,y) = 0$ *and* $\omega(x,z) = 0$, *then* $y = z$;

(b) *if* $\omega(x_n,y_n) \le \alpha_n$ *and* $\omega(x_n,z) \le \beta_n$ *for any* $n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to* z;

(c) *if* $\omega(x_n,x_m) \le \alpha_n$ *for any* $n,m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence;*

(d) *if* $\omega(y,x_n) \le \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Lemma 1.6 (see [12]). *Let* K *be a closed subset of* X *and* ω *be a* w*-distance on* X. *Suppose that there exists* $u \in X$ *such that* $\omega(u,u) = 0$. *Then* $\omega(u,K) = 0 \Leftrightarrow u \in K$, *where* $\omega(u,K) = \inf_{y\in K}\omega(u,y)$.

Using the concept of w-distance, the authors of this paper most recently extended and generalized Theorem 1.4 and [8, Theorem 3.3] as follows.

Theorem 1.7 (see [13]). *Let (X,d) be a complete metric space with a w-distance ω. Let $T : X \to \mathrm{Cl}(X)$ be a multivalued map satisfying that for any constant $b \in (0,1)$ and for each $x \in X$ there is $y \in J_b^x$ such that*

$$\omega(y,T(y)) \le k(\omega(x,y))\omega(x,y), \tag{1.8}$$

where $J_b^x = \{y \in T(x) : b\omega(x,y) \le \omega(x,T(x))\}$ and k is a function from $[0,\infty)$ to $[0,b)$ with $\limsup_{r\to t^+} k(r) < b$, for every $t \in [0,\infty)$. Suppose that a real-valued function g on X defined by $g(x) = \omega(x,T(x))$ is lower semicontinuous. Then there exists $v_o \in X$ such that $g(v_o) = 0$. Further, if $\omega(v_o,v_o) = 0$, then $v_0 \in \mathrm{Fix}(T)$.

Let $A \in (0,+\infty]$. Let $\eta : [0,A) \to R$ satisfy that

(i) $\eta(0) = 0$ and $\eta(t) > 0$ for each $t \in (0,A)$;

(ii) η is nondecreasing on $[0,A)$;

(iii) η is subadditive; that is,

$$\eta(t_1 + t_2) \le \eta(t_1) + \eta(t_2) \quad \forall t_1, t_2 \in (0,A). \tag{1.9}$$

We define $\Omega[0,A) = \{\eta : \eta$ satisfies (i)–(iii) above$\}$.

Remark 1.8. (a) It follows from (ii) property of η that for each $t_1, t_2 \in (0,A)$;

$$\eta(t_1) < \eta(t_2) \Longrightarrow t_1 < t_2. \tag{1.10}$$

(b) If $\eta \in \Omega[0,A)$ and η is continuous at 0, then due to the following two facts η must be continuous at each point of $[0,A)$. First, every sub-additive and continuous function η at 0 such that $\eta(0) = 0$ is right upper and left lower semicontinuous [14]. Second, each nondecreasing function is left upper and right lower semicontinuous.

(c) For any $\eta \in \Omega[0,A)$ and for each sequence α_n in $[0,A)$ satisfying $\lim_{n\to\infty}\eta(\alpha_n) = 0$, we have $\lim_{n\to\infty}\alpha_n = 0$.

For a metric space (X,d), we denote $\delta(X) = \sup\{d(x,y) : x,y \in X\}$. In the sequel, we consider $A = \delta(X)$ if $\delta(X) = \infty$ and $A > \delta(X)$ if $\delta(X) < \infty$.

Assuming that the function η is continuous and satisfies the conditions (i) and (ii) above, Zhang [15] proved some fixed point results for single-valued maps which satisfy some contractive type condition involving such function η. Recently, using $\eta \in \Omega[0,A)$, Pathak and Shahzad [9] generalized Theorem 1.4.

In this paper, we prove some results on the existence of fixed points for contractive type multimaps involving the function $\eta \circ \omega$, where $\eta \in \Omega[0,A)$ and the function ω is a w-distance on a metric space X. Our results either generalize or improve several known fixed point results in the setting of metric spaces, (see Remarks 2.3 and 2.6).

2. The Results

Theorem 2.1. *Let* (X, d) *be a complete metric space with a* w*-distance* ω*. Let* $T : X \to \mathrm{Cl}(X)$ *be a multimap. Assume that the following conditions hold:*

(I) *there exist a number* $b \in (0, 1)$ *and a function* $k : [0, \infty) \to [0, b)$ *such that for each* $t \in [0, \infty)$

$$\limsup_{r \to t^+} k(r) < b; \tag{2.1}$$

(II) *there exists a function* $\eta \in \Omega[0, A)$ *such that for any* $x \in X$*, there exists* $y \in T(x)$ *satisfying*

$$\begin{gathered} b\eta(\omega(x, y)) \le \eta(\omega(x, T(x))), \\ \eta(\omega(y, T(y))) \le k(\omega(x, y))\eta(\omega(x, y)); \end{gathered} \tag{2.2}$$

(III) *the map* $g : X \to \mathbb{R}$ *defined by* $g(x) = \omega(x, T(x))$ *is lower semicontinuous.*

Then there exists $v_0 \in X$ *such that* $g(v_0) = 0$*. Further if* $\omega(v_0, v_0) = 0$*, then* $v_0 \in T(v_0)$*.*

Proof. Let $x_0 \in X$ be any initial point. Then from (II) we can choose $x_1 \in T(x_0)$ such that

$$b\eta(\omega(x_0, x_1)) \le \eta(\omega(x_0, T(x_0))), \tag{2.3}$$

$$\eta(\omega(x_1, T(x_1))) \le k(\omega(x_0, x_1))\eta(\omega(x_0, x_1)), \quad k(\omega(x_0, x_1)) < b. \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned} \eta(\omega(x_0, T(x_0))) - \eta(\omega(x_1, T(x_1))) &\ge b\eta(\omega(x_0, x_1)) - k(\omega(x_0, x_1))\eta(\omega(x_0, x_1)) \\ &= [b - k(\omega(x_0, x_1))]\eta(\omega(x_0, x_1)) > 0. \end{aligned} \tag{2.5}$$

Similarly, there exists $x_2 \in T(x_1)$ such that

$$b\eta(\omega(x_1, x_2)) \le \eta(\omega(x_1, T(x_1))), \tag{2.6}$$

$$\eta(\omega(x_2, T(x_2))) \le k(\omega(x_1, x_2))\eta(\omega(x_1, x_2)), \quad k(\omega(x_1, x_2)) < b. \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\begin{aligned} \eta(\omega(x_1, T(x_1))) - \eta(\omega(x_2, T(x_2))) &\ge b\eta(\omega(x_1, x_2)) - k(\omega(x_1, x_2))\eta(\omega(x_1, x_2)) \\ &= [b - k(\omega(x_1, x_2))]\eta(\omega(x_1, x_2)) > 0. \end{aligned} \tag{2.8}$$

From (2.4) and (2.6), it follows that

$$\eta(\omega(x_1, x_2)) \leq \frac{1}{b}\eta(\omega(x_1, T(x_1))) \leq \frac{1}{b}k(\omega(x_0, x_1))\eta(\omega(x_0, x_1)) \leq \eta(\omega(x_0, x_1)). \tag{2.9}$$

Continuing this process, we get an orbit $\{x_n\}$ of T at x_0 satisfying the following:

$$\begin{gathered} b\eta(\omega(x_n, x_{n+1})) \leq \eta(\omega(x_n, T(x_n))), \\ \eta(\omega(x_{n+1}, T(x_{n+1}))) \leq k(\omega(x_n, x_{n+1}))\eta(\omega(x_n, x_{n+1})), \quad k(\omega(x_n, x_{n+1})) < b. \end{gathered} \tag{2.10}$$

From (2.10), we get

$$\eta(\omega(x_n, T(x_n))) - \eta(\omega(x_{n+1}, T(x_{n+1}))) \geq [b - k(\omega(x_n, x_{n+1}))]\eta(\omega(x_n, x_{n+1})). \tag{2.11}$$

Note that for each n,

$$\begin{gathered} \eta(\omega(x_{n+1}, T(x_{n+1}))) < \eta(\omega(x_n, T(x_n))), \\ \eta(\omega(x_n, x_{n+1})) \leq \eta(\omega(x_{n-1}, x_n)). \end{gathered} \tag{2.12}$$

Thus the sequences of non-negative real numbers $\{\eta(\omega(x_n, T(x_n)))\}$ and $\{\eta(\omega(x_n, x_{n+1}))\}$ are decreasing. Now, since η is nondecreasing, it follows that $\{\omega(x_n, T(x_n))\}$ and $\{\omega(x_n, x_{n+1})\}$ are decreasing sequences and are bounded from below, thus convergent. Now, by the definition of the function k, there exists $\alpha \in [0, b)$ such that

$$\limsup_{n\to\infty} k(\omega(x_n, x_{n+1})) = \alpha. \tag{2.13}$$

Thus, for any $b_0 \in (\alpha, b)$, there exists $n_0 \in \mathbb{N}$ such that

$$k(\omega(x_n, x_{n+1})) < b_0, \quad \forall n > n_0, \tag{2.14}$$

and thus for all $n > n_0$, we have

$$k(\omega(x_n, x_{n+1})) \times \cdots \times k(\omega(x_{n_0+1}, x_{n_0+2})) < b_0^{n-n_0}. \tag{2.15}$$

Also, it follows from (2.11) that for all $n > n_0$,

$$\eta(\omega(x_n, T(x_n))) - \eta(\omega(x_{n+1}, T(x_{n+1}))) \geq \beta\eta(\omega(x_n, x_{n+1})), \tag{2.16}$$

where $\beta = b - b_0$. Note that for all $n > n_0$, we have

$$\begin{aligned}
\eta(\omega(x_{n+1}, T(x_{n+1}))) &\le k(\omega(x_n, x_{n+1}))\eta(\omega(x_n, x_{n+1})) \\
&\le \frac{1}{b}k(\omega(x_n, x_{n+1}))\eta(\omega(x_n, T(x_n))) \\
&\le \frac{1}{b}\frac{1}{b}k(\omega(x_n, x_{n+1}))k(\omega(x_{n-1}, x_n))\eta(\omega(x_{n-1}, T(x_{n-1}))) \\
&\;\;\vdots \\
&\le \frac{1}{b^n}[k(\omega(x_n, x_{n+1})) \times \cdots \times k(\omega(x_1, x_2))]\eta(\omega(x_1, T(x_1))) \\
&= \frac{k(\omega(x_n, x_{n+1})) \times \cdots \times k(\omega(x_{n_0+1}, x_{n_0+2}))}{b^{n-n_0}} \\
&\quad \times \frac{k(\omega(x_{n_0}, x_{n_0+1})) \times \cdots \times k(\omega(x_1, x_2))\eta(\omega(x_1, T(x_1)))}{b^{n_0}}.
\end{aligned} \tag{2.17}$$

Thus

$$\eta(\omega(x_{n+1}, T(x_{n+1}))) \le \left(\frac{b_0}{b}\right)^{n-n_0} q, \tag{2.18}$$

where $q = k(\omega(x_{n_0}, x_{n_0+1})) \times \cdots \times k(\omega(x_1, x_2))\eta(\omega(x_1, T(x_1)))/b^{n_0}$. Now, since $b_0 < b$, we have $\lim_{n\to\infty}(b_0/b)^{n-n_0} = 0$, and we get the decreasing sequence $\{\eta(\omega(x_n, T(x_n)))\}$ converging to 0. Thus we have

$$\omega(x_n, T(x_n)) \longrightarrow 0. \tag{2.19}$$

Note that for all $n > n_0$,

$$\eta(\omega(x_n, x_{n+1})) < \gamma^n \eta(\omega(x_0, x_1)), \tag{2.20}$$

where $\gamma = b_0/b < 1$. Now, for any $n, m \in \mathbb{N}$, $m > n > n_0$,

$$\eta(\omega(x_n, x_m)) \le \sum_{j=n}^{m-1} \eta\left(\omega\left(x_j, x_{j+1}\right)\right) < \frac{\gamma^n}{1-\gamma}\eta(\omega(x_0, x_1)). \tag{2.21}$$

Clearly, $\lim_{n,m\to\infty}\eta(\omega(x_n, x_m)) = 0$, and thus we get that

$$\lim_{n,m\to\infty} \omega(x_n, x_m) = 0, \tag{2.22}$$

that is, $\{x_n\}$ is Cauchy sequence in X. Due to the completeness of X, there exists some $v_0 \in X$ such that $\lim_{n\to\infty} x_n = v_0$. Due to the fact that the function g is lower semicontinuous and (2.19), we have

$$0 \le g(v_0) \le \liminf_{n\to\infty} g(x_n) = \liminf_{n\to\infty} \omega(x_n, T(x_n)) = 0, \tag{2.23}$$

thus, $g(v_0) = \omega(v_0, T(v_0)) = 0$. Since $\omega(v_0, v_0) = 0$, and $T(v_0)$ is closed, it follows from Lemma 1.6 that $v_0 \in T(v_0)$. □

If we consider a constant map $k(t) = h,\ t \in (0,\infty),\ h \in (0,b)$ in Theorem 2.1, then we obtain the following result.

Corollary 2.2. *Let (X,d) be a complete metric space with a w-distance ω. Let $T : X \to \mathrm{Cl}(X)$ be a multimap satisfying that for any constants $b, h \in (0,1),\ h < b$ and for each $x \in X$ there is $y \in J_b^x$ such that*

$$\eta(\omega(y, T(y))) \le h\eta(\omega(x, y)), \tag{2.24}$$

where $J_b^x = \{y \in T(x) : b\eta(\omega(x,y)) \le \eta(\omega(x, T(x)))\}$. Suppose that a real-valued function g on X defined by $g(x) = \omega(x, T(x))$ is lower semicontinuous. Then there exists $v_o \in X$ such that $g(v_o) = 0$. Further, if $\omega(v_o, v_o) = 0$, then $v_0 \in \mathrm{Fix}(T)$.

Remark 2.3. (a) Theorem 2.1 extends and generalizes Theorem 1.7. Indeed, if we consider $\eta(t) = t$ for each $t \in [0, A)$ in Theorem 2.1, then we can get Theorem 1.7 due to Latif and Abdou [13, Theorem 2.2].

(b) Theorem 2.1 contains Theorem 2.2 of Pathak and Shahzad [9] as a special case.

(c) Corollary 2.2 extends and generalizes Theorem 3.3 of Latif and Albar [8].

We have also the following fixed point result which generalizes [13, Theorem 2.4].

Theorem 2.4. *Suppose that all the hypotheses of Theorem 2.1 except (III) hold. Assume that*

$$\inf\{\eta(\omega(x,v)) + \eta(\omega(x, T(x))) : x \in X\} > 0, \tag{2.25}$$

for every $v \in X$ with $v \notin T(v)$ and the function η is continuous at 0. Then $\mathrm{Fix}(T) \neq \emptyset$.

Proof. Following the proof of Theorem 2.1, there exists a Cauchy sequence $\{x_n\}$ and $v_0 \in X$ such that $x_{n+1} \in T(x_n)$ and $\lim_{n\to\infty} x_n = v_0$. Since $\omega(x_n, \cdot)$ is lower semicontinuous, it follows, from the proof of Theorem 2.1 that for all $n > n_0$, we have

$$\eta(\omega(x_n, v_0)) \le \liminf_{m\to\infty} \eta(\omega(x_n, x_m)) < \frac{\gamma^n}{1-\gamma}\eta(\omega(x_0, x_1)), \tag{2.26}$$

where $\gamma = b_0/b < 1$. Since $\omega(x_n, T(x_n)) \le \omega(x_n, x_{n+1})$, for all n, and the function η is nondecreasing, we have

$$\eta(\omega(x_n, T(x_n))) \le \eta(\omega(x_n, x_{n+1})), \tag{2.27}$$

and thus by using (2.20), we get

$$\eta(\omega(x_n, T(x_n))) \le \gamma^n \eta(\omega(x_0, x_1)). \tag{2.28}$$

Assume that $v_0 \notin T(v_0)$. Then, we have

$$\begin{aligned}
0 &< \inf\{\eta(\omega(x, v_0)) + \eta(\omega(x, T(x))) : x \in X\} \\
&\le \inf\{\eta(\omega(x_n, v_0)) + \eta(\omega(x_n, T(x_n))) : n > n_0\} \\
&\le \inf\left\{\frac{\gamma^n}{1-\gamma}\eta(\omega(x_0, x_1)) + \gamma^n \eta(\omega(x_0, x_1)) : n > n_0\right\} \\
&= \left\{\frac{2-\gamma}{1-\gamma}\eta(\omega(x_0, x_1))\right\} \inf\{\gamma^n : n > n_0\} = 0,
\end{aligned} \tag{2.29}$$

which is impossible and hence $v_0 \in \mathrm{Fix}(T)$. □

Theorem 2.5. *Let (X, d) be a complete metric space with a w-distance ω. Let $T : X \to \mathrm{Cl}(X)$ be a multimap. Assume that the following conditions hold.*

(I) *there exists a function $k : [0, \infty) \to [0, 1)$ such that for each $t \in [0, \infty)$,*

$$\limsup_{r \to t^+} k(r) < 1; \tag{2.30}$$

(II) *there exists a function $\eta \in \Omega[0, A)$ such that for any $x \in X$, there exists $y \in T(x)$ satisfying*

$$\begin{aligned}
\eta(\omega(x, y)) &= \eta(\omega(x, T(x))), \\
\eta(\omega(y, T(y))) &\le k(\omega(x, y))\eta(\omega(x, y));
\end{aligned} \tag{2.31}$$

(III) *the map $g : X \to \mathbb{R}$ defined by $g(x) = \omega(x, T(x))$ is lower semicontinuous.*

Then there exists $v_0 \in X$ such that $g(v_0) = 0$. Further if $\omega(v_0, v_0) = 0$, then $v_0 \in T(v_0)$.

Proof. Let $x_0 \in X$ be any initial point. Following the same method as in the proof of Theorem 2.1, we obtain the existence of a Cauchy sequence $\{x_n\}$ such that $x_{n+1} \in T(x_n)$, satisfying

$$\begin{aligned}
\eta(\omega(x_n, x_{n+1})) &= \eta(\omega(x_n, T(x_n))), \\
\eta(\omega(x_{n+1}, T(x_{n+1}))) &\le k(\omega(x_n, x_{n+1}))\eta(\omega(x_n, x_{n+1})), \qquad k(\omega(x_n, x_{n+1})) < 1.
\end{aligned} \tag{2.32}$$

Consequently, there exists $v_0 \in X$ such that $\lim_{n\to\infty} x_n = v_0$. Since g is lower semicontinuous, we have

$$0 \le g(v_0) \le \lim_{n\to\infty} \inf g(x_n) = 0, \tag{2.33}$$

thus, $g(v_0) = \omega(v_0, T(v_0)) = 0$. Further by the closedness of $T(v_0)$ and since $\omega(v_0, v_0) = 0$, it follows from Lemma 1.6 that $v_0 \in T(v_0)$. □

Remark 2.6. Theorem 2.5 extends and generalizes fixed point results of Klim and Wardowski [4, Theorem 2.2], Cirić [5, Theorem 7], and improves fixed point result of Pathak and Shahzad [9, Theorem 2.4].

Following the same method as in the proof of Theorem 2.4, we can obtain the following fixed point result.

Theorem 2.7. *Suppose that all the hypotheses of Theorem 2.5 except (III) hold. Assume that*

$$\inf\{\eta(\omega(x,v)) + \eta(\omega(x,T(x))) : x \in X\} > 0, \tag{2.34}$$

for every $v \in X$ with $v \notin T(v)$ and the function η is continuous at 0, *then* $\mathrm{Fix}(T) \neq \emptyset$.

Now we present an example which satisfies all the conditions of the main results, namely, Theorems 2.1 and 2.5 and thus the set of fixed points of T is nonempty.

Example 2.8. Let $X = [0,1]$ with the usual metric d. Define a w-distance function $\omega : X \times X \to [0,\infty)$, by

$$\omega(x,y) = y \quad \forall x, y \in X. \tag{2.35}$$

Let $T : X \to \mathrm{Cl}(X)$ be defined as

$$T(x) = \begin{cases} \left\{\dfrac{1}{2}x^2\right\}; & x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right], \\ \left\{0, \dfrac{1}{6}, \dfrac{7}{32}\right\}; & x = \dfrac{15}{32}. \end{cases} \tag{2.36}$$

Note that $\delta(X) = 1$. Let $A \in [1,\infty)$, $b = 9/10$. Define a function $\eta : [0, A) \to \mathbb{R}$ by $\eta(t) = t^{1/2}$. Clearly, $\eta \in \Omega[0, A)$. Define $k : [0,\infty) \to [0, b)$ by

$$k(t) = \begin{cases} \left(\dfrac{9}{8}\right)^{1/4} t^{1/2}; & t \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{22}{25}; & t \in \left[\dfrac{1}{2}, \infty\right). \end{cases} \tag{2.37}$$

Note that

$$g(x) = \omega(x, T(x)) = \begin{cases} \left\{\dfrac{1}{2}x^2\right\}; & x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right]. \\ 0; & x = \dfrac{15}{32}. \end{cases} \tag{2.38}$$

Clearly, g is lower semicontinuous. Note that for each $x \in [0, 15/32) \cup (15/32, 1]$, we have

$$b\eta(\omega(x,y)) = \frac{9}{10}\eta\left(\omega\left(x, \frac{1}{2}x^2\right)\right) = \frac{9}{10}\left(\frac{1}{2}x^2\right)^{1/2} \leq \left(\frac{1}{2}x^2\right)^{1/2} = \eta(\omega(x, T(x))),$$

$$\eta(\omega(y, T(y))) = \eta\left(\omega\left(\frac{1}{2}x^2, \frac{1}{2}\left(\frac{1}{2}x^2\right)^2\right)\right) \leq \left(\frac{9}{8}\right)^{1/4}\left(\frac{1}{2}x^2\right)^{1/2}\left(\frac{1}{2}x^2\right)^{1/2} \tag{2.39}$$

$$= k(\omega(x,y))\eta(\omega(x,y)).$$

Thus, for $x \in [0,1]$, $x \neq 15/32$, T satisfies all the conditions of Theorem 2.1. Now, let $x = 15/32$, then we have $T(x) = \{0, 1/6, 7/32\}$. Clearly, there exists $y = 0 \in T(x)$, such that $\eta(\omega(x, T(x))) = 0$. Now

$$b\eta(\omega(x,y)) = \frac{9}{10}\eta\left(\omega\left(\frac{15}{32}, 0\right)\right) = 0 = \eta(\omega(x, T(x))),$$

$$\eta(\omega(y, T(y))) = \eta(\omega(0,0)) = k(\omega(x,y))\eta(\omega(x,y)). \tag{2.40}$$

Thus, all the hypotheses of Theorem 2.1 are satisfied and clearly we have $\mathrm{Fix}(T) = \{0\}$. Now, if we consider $b = 1$, then all the hypotheses of Theorem 2.5 are also satisfied. Note that in the above example the w-distance ω is not a metric d.

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