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Research Article

Fixed Points of Multivalued Maps in Modular Function Spaces

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The purpose of this paper is to study the existence of fixed points for contractive-type and nonexpansive-type multivalued maps in the setting of modular function spaces. We also discuss the concept of w-modular function and prove fixed point results for *weakly*-modular contractive maps in modular function spaces. These results extend several similar results proved in metric and Banach spaces settings.

1. Introduction and Preliminaries

The well-known Banach fixed point theorem on complete metric spaces (specifically, each contraction self-map of a complete metric space has a unique fixed point) has been extended and generalized in different directions. For example, see Edelstein [1, 2], Kasahara [3], Rhoades [4], Siddiq and Ansari [5], and others. One of its generalizations is for nonexpansive single-valued maps on certain subsets of a Banach space. Indeed, these fixed points are not necessarily unique. See, for example, Browder [6–8] and Kirk [9]. Fixed point theorems for contractive and nonexpansive multivalued maps have also been established by several authors. Let H denote the Hausdorff metric on the space of all bounded nonempty subsets of a metric space (X, d). A multivalued map $J : X \to 2^X$ (where 2^X denotes the collection of all nonempty subsets of X) with bounded subsets as values is called contractive [10] if

$$H(J(x), J(y)) \le hd(x, y) \tag{1.1}$$

for all $x, y \in X$ and for a fixed number $h \in [0, 1)$. If the Lipschitz constant $h = 1$, then J is called a multivalued nonexpansive mapping [11]. Nadler [10], Markin [11], Lami-Dozo [12], and others proved fixed point theorems for these maps under certain conditions in the setting of metric and Banach spaces. Note that an element $x \in X$ is called a fixed point of a multivalued map $J : X \to 2^X$ if $x \in J(x)$. Among others, without using the concept of the Hausdorff metric, Husain and Tarafdar [13] introduced the notion of a nonexpansive-type multivalued map and proved a fixed point theorem on compact intervals of the real line. Using such type of notions Husain and Latif [14] extended their result to general Banach space setting.

The fixed point results in modular function spaces were given by Khamsi et al. [15]. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated in modular spaces. For instance, fixed point theorems are proved in [15, 16] for nonexpansive maps.

In this paper, we define nonexpansive-type and contractive-type multivalued maps in modular function spaces, investigate the existence of fixed points of such mappings, and prove similar results found in [17].

Now, we recall some basic notions and facts about modular spaces as formulated by Kozlowski [18]. For more details the reader may consult [15, 16].

Let Ω be a nonempty set and let Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Σ, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$.

Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, that is, all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \le |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 1.1. A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if

(P_1) $\rho(0, E) = 0$ for any $E \in \Sigma$,

(P_2) $\rho(f, E) \le \rho(g, E)$ whenever $|f(\omega)| \le |g(\omega)|$ for any $\omega \in \Omega$, $f, g \in \mathcal{E}$ and $E \in \Sigma$,

(P_3) $\rho(f, \cdot) : \Sigma \to [0, \infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$,

(P_4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,

(P_5) if there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$, and

(P_6) for any $\alpha > 0$, $\rho(\alpha, .)$ is order continuous on $\mathcal{P}$, that is, $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \in \mathcal{P}$ and decreases to $\emptyset$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f, E) = \sup\{\rho(g, E);\ g \in \varepsilon,\ |g(\omega)| \le |f(\omega)|, \text{ for every } \omega \in \Omega\}. \tag{1.2}$$

For the sake of simplicity we write $\rho(f)$ instead of $\rho(f, \Omega)$.

Definition 1.2. A set E is said to be ρ-null if $\rho(\alpha, E) = 0$ for every $\alpha > 0$. A property $p(w)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{w \in \Omega : p(w) \text{does not hold}\}$ is ρ-null.

Definition 1.3. A modular function ρ is called σ-finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $0 < \rho(K_n) < \infty$ and $\Omega = \bigcup K_n$. It is easy to see that the functional

$\rho : \mathcal{M} \to [0,\infty]$ is a modular and satisfies the following properties:

(i) $\rho(f) = 0$ if and only if $f = 0$ ρ-a.e.,

(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$, and

(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ and $f, g \in \mathcal{M}$.

In addition, if the following property is satisfied,

(iii)′ $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ and, $f, g \in \mathcal{M}$,

we say that ρ is a convex modular.

The modular ρ defines a corresponding modular space, that is, the vector space L_ρ given by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}. \tag{1.3}$$

When ρ is convex, the formula

$$\|f\|_p = \inf\left\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq 1\right\} \tag{1.4}$$

defines a norm in the modular space L_ρ which is frequently called the Luxemburg norm. We can also consider the space

$$E_\rho = \{f \in \mathcal{M}; \rho(\alpha f, A_n) \to 0 \text{ as } n \to \infty \text{ for every } A_n \in \Sigma \text{ that decreases to } \emptyset \text{ and } \alpha > 0\}. \tag{1.5}$$

Definition 1.4. A function modular is said to satisfy the Δ_2-condition if $\sup_{n\geq 1}\rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{f_n\}_{n\geq 1} \subset \mathcal{M}$, $D_k \in \Sigma$ decreases to $\emptyset$ and $\sup_{n\geq 1}\rho(f_n, D_k) \to 0$ as $k \to \infty$.

We know from [18] that $E_\rho = L_\rho$ when ρ satisfies the Δ_2-condition.

Definition 1.5. A function modular is said to satisfy the Δ_2-type condition if there exists $K > 0$ such that for any $f \in L_\rho$ we have $\rho(2f) \leq K\rho(f)$.

In general, Δ_2-type condition and Δ_2-condition are not equivalent, even though it is obvious that Δ_2-type condition implies Δ_2-condition on the modular space L_ρ.

Definition 1.6. Let $Ł_\rho$ be a modular space.

(1) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(2) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-a.e. convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega; f_n(\omega) \nrightarrow f(\omega)\}$ is ρ-null.

(3) The sequence $\{f_n\} \subset L_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m go to ∞.

(4) A subset C of L_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.

(5) A subset C of L_ρ is called ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of C always belongs to C.

(6) A subset C of L_ρ is called ρ-a.e. compact if every sequence in C has a ρ-a.e. convergent subsequence in C.

(7) A subset C of L_ρ is called ρ-bounded if

$$\delta_\rho(C) = \sup\{\rho(f-g); f, g \in C\} < \infty. \tag{1.6}$$

We recall two basic results (see [15]) in the theory of modular spaces.

(i) If there exists a number $\alpha > 0$ such that $\rho(\alpha(f_n - f)) \to 0$, then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to f\,\rho$-a.e.

(ii) (Lebesgue's Theorem) If $f_n, f \in \mathcal{M}$, $f_n \to f\,\rho$-a.e. and there exists a function $g \in E_\rho$ such that $|f_n| \leq |g|\rho$-a.e. for all n, then $\|f_n - f\|_p \to 0$.

We know, by [15, 16] that under Δ_2-condition the norm convergence and modular convergence are equivalent, which implies that the norm and modular convergence are also the same when we deal with the Δ_2-type condition. In the sequel we will assume that the modular function ρ is convex and satisfies the Δ_2-type condition.

Definition 1.7. Let ρ be as aforementioned. We define a growth function ω by

$$\omega(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)},\ f \in L_\rho \setminus \{0\}\right\} \quad \forall 0 \leq t < \infty. \tag{1.7}$$

We have the following:

Lemma 1.8 (see [19]). *Let ρ be as aforementioned. Then the growth function ω has the following properties:*

(1) $\omega(t) < \infty$,$\forall t \in [0,\infty)$,

(2) $\omega : [0,\infty) \to [0,\infty)$ *is a convex, strictly increasing function. So, it is continuous,*

(3) $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0,\infty)$,

(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta); \forall \alpha, \beta \in [0,\infty)$, *where ω^{-1} is the function inverse of ω.*

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

Lemma 1.9 (see [19]). *Let ρ be a convex function modular satisfying the Δ_2-type condition. Then*

$$\|f\|_p \leq \frac{1}{\omega^{-1}(1/\rho(f))} \quad \textit{whenever} \quad f \in L_\rho. \tag{1.8}$$

The next lemma will be of major interest throughout this work.

Lemma 1.10 (see [16]). *Let ρ be a function modular satisfying the Δ_2-condition and let $\{f_n\}$ be a sequence in L_ρ such that $f_n \overset{\rho-a.e}{\to} f \in L_\rho$, and there exists $k > 1$ such that $\sup_n \rho(k(f_n - f)) < \infty$. Then,*

$$\liminf_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f - g) \quad \forall g \in L_\rho. \tag{1.9}$$

Moreover, one has

$$\rho(f) \le \liminf_{n\to\infty} \rho(f_n). \tag{1.10}$$

2. Fixed Points of Contractive-Type and Nonexpansive-Type Maps

In the sequel we assume that ρ is a convex, σ-finite modular function satisfying the Δ_2-type condition, and C is a nonempty ρ-bounded subset of the modular function space L_ρ. We denote that $\mathcal{C}(C)$ is a collection of all nonempty ρ-closed subsets of C, and $\mathcal{K}(C)$ is a collection of all nonempty ρ-compact subsets of C.

We say that a multivalued map $T : C \to 2^C$ is ρ-contractive-type if there exists $k \in (0,1)$ such that for any $f, g \in C$ and for any $F \in T(f)$, there exists $G \in T(g)$ such that

$$\rho(F - G) \le k\rho(f - g), \tag{2.1}$$

and ρ-nonexpansive-type if for any $f, g \in C$ and for any $F \in T(f)$, there exists $G \in T(g)$ such that

$$\rho(F - G) \le \rho(f - g). \tag{2.2}$$

We have the following fixed point theorem (for which a similar result may be found in [17]).

Theorem 2.1. *Let C be a nonempty ρ-closed subset of the modular function space L_ρ. Then any $T : C \to \mathcal{C}(C)$ ρ-contractive-type map has a fixed point, that is, there exists $f \in C$ such that $f \in T(f)$.*

Proof. Let $f_0 \in C$. Without loss of generality, assume that f_0 is not a fixed point of T. Then there exists $f_1 \in T(f_0)$ such that $f_1 \ne f_0$. Hence $\rho(f_0, f_1) > 0$. Since T is ρ-contractive-type, then there exists $f_2 \in T(f_1)$ such that

$$\rho(f_1 - f_2) \le k\rho(f_0 - f_1). \tag{2.3}$$

By induction, one can easily construct a sequence $\{f_n\} \in C$ such that $f_{n+1} \in T(f_n)$ and

$$\rho(f_{n+1} - f_n) \le k\rho(f_n - f_{n-1}), \tag{2.4}$$

for any $n \geq 1$. In particular we have

$$\rho(f_{n+1} - f_n) \leq k^n \rho(f_1 - f_0). \tag{2.5}$$

Without loss of generality, we may assume $\rho(f_{n+1}, f_n) \neq 0$, otherwise f_n is a fixed point of T. Hence

$$\frac{1}{k^n \rho(f_1 - f_0)} \leq \frac{1}{\rho(f_{n+1} - f_n)} \tag{2.6}$$

Using Lemma 1.9, we get

$$\|f_{n+1} - f_n\|_\rho \leq \frac{1}{\omega^{-1}(1/\rho(f_{n+1} - f_n))}. \tag{2.7}$$

Using the properties of $\omega(t)$, we get

$$\omega^{-1}\left(\frac{1}{k^n \rho(f_1 - f_0)}\right) \leq \omega^{-1}\left(\frac{1}{\rho(f_{n+1} - f_n)}\right). \tag{2.8}$$

So

$$\omega^{-1}\left(\frac{1}{k}\right)^n \omega^{-1}\left(\frac{1}{\rho(f_1 - f_0)}\right) \leq \omega^{-1}\left(\frac{1}{\rho(f_{n+1} - f_n)}\right), \tag{2.9}$$

which implies

$$\|f_{n+1} - f_n\|_\rho \leq \frac{1}{\omega^{-1}(1/k)^n \omega^{-1}(1/\rho(f_1 - f_0))}. \tag{2.10}$$

Since $\omega(1) = 1$ and $k < 1$, then $1 < \omega^{-1}(1/k)$. This forces $\{f_n\}$ to be $\|\cdot\|_\rho$-Cauchy. Hence the sequence $\{f_n\}\|\cdot\|_\rho$-converges to some $f \in L_\rho$. Since ρ satisfies the Δ_2-condition, then $\{f_n\}\rho$-converges to f. Since C is ρ-closed, then $f \in C$. Let us prove that f is indeed a fixed point of T. Since T is a ρ-contractive-type mapping, then for any $n \geq 1$, there exists $F_n \in T(f)$ such that

$$\rho(f_{n+1} - F_n) \leq k\rho(f_n - f). \tag{2.11}$$

Hence $\{\rho(f_{n+1} - F_n)\}$ converges to 0. Since ρ satisfies the Δ_2-condition, we have $\{\|f_{n+1} - F_n\|_\rho\}$ converges to 0. Since $\{f_n\}\|\cdot\|_\rho$-converges to f, then $\{F_n\}\|\cdot\|_\rho$-converges to f. Hence $\{F_n\}\rho$-converges to f. Since $T(f)$ is ρ-closed and $\{F_n\} \in T(f)$, we get $f \in T(f)$. □

Remark 2.2. Consider the multivalued map $T_A(f) = A$, where A is a nonempty ρ-closed subset of C. Then it is easy to show that T_A is a ρ-contractive-type map. The set of all fixed point of T_A is exactly the set A. In particular, ρ-contractive-type maps may not have a unique fixed point.

As an application of the above theorem, we have the following result.

Proposition 2.3. *Let C be a ρ-closed convex subset of the modular function space L_ρ. Let $T : C \to \mathcal{C}(C)$ be ρ-nonexpansive-type map. Then there exists an approximate fixed points sequence $\{f_n\}$ in C, that is, for any $n \geq 1$ there exists $F_n \in T(f_n)$ such that*

$$\lim_{n\to\infty} \rho(f_n - F_n) = 0. \tag{2.12}$$

In particular one has $\lim_{n\to\infty}\text{dist}_\rho(f_n, T(f_n)) = 0$, where

$$\text{dist}_\rho(f_n, T(f_n)) = \inf\{\rho(f_n - g);\ g \in T(f_n)\}. \tag{2.13}$$

Proof. Let $\lambda \in (0,1)$ and let f_0 be a fixed point in C. For each $f \in C$, define a map

$$T_\lambda(f) = \lambda f_0 + (1-\lambda)T(f) = \{\lambda f_0 + (1-\lambda)g;\ g \in T(f)\}. \tag{2.14}$$

Note that $T_\lambda(f)$ is nonempty and ρ-closed subset of C because $T(f)$ is ρ-closed and C is convex. Since T is a ρ-nonexpansive-type map, for each $f, g \in C$ and for any $F \in T(f)$, there exists $G \in T(g)$ such that

$$\rho(F - G) \leq \rho(f - g). \tag{2.15}$$

Since ρ is convex we get

$$\rho\big((\lambda f_0 + (1-\lambda)F) - (\lambda f_0 + (1-\lambda)G)\big) = \rho((1-\lambda)(F-G)) \leq (1-\lambda)\rho(F-G), \tag{2.16}$$

which implies

$$\rho\big((\lambda f_0 + (1-\lambda)F) - (\lambda f_0 + (1-\lambda)G)\big) \leq (1-\lambda)\rho(f-g). \tag{2.17}$$

In other words, the map T_λ is a ρ-contractive-type. Theorem 2.1 implies the existence of a fixed point f_λ of T_λ, thus there exists $F_\lambda \in T(f_\lambda)$ such that

$$f_\lambda = \lambda f_0 + (1-\lambda)F_\lambda. \tag{2.18}$$

In particular, we have

$$\rho(f_\lambda - F_\lambda) = \rho\lambda(f_0 - F_\lambda) \leq \lambda\rho(f_0 - F_\lambda) \leq \lambda\delta_\rho(C), \tag{2.19}$$

where $\delta_\rho(C) = \sup_{f,g\in C}\rho(f-g)$ is the ρ-diameter of C. Note that since C is ρ-bounded, then $\delta_\rho(C) < \infty$. If we choose $\lambda = 1/n$, for $n \geq 1$ and write $f_n = f_{\lambda_n}$ and $F_n = F_{\lambda_n}$, we get

$$\rho(f_n - F_n) \leq \frac{\delta_\rho(C)}{n}, \tag{2.20}$$

for any $n \geq 1$, which implies $\lim_{n\to\infty}\rho(f_n - F_n) = 0$. □

Using the above result, we are now ready to prove the main fixed point result for ρ-nonexpansive-type multivalued maps.

Theorem 2.4. *Let C be a nonempty ρ-closed convex subset of the modular function space L_ρ. Assume that C is ρ-a.e. compact. Then each ρ-nonexpansive-type map $T : C \to \mathcal{K}(C)$ has a fixed point.*

Proof. Proposition 2.3 ensures the existence of a sequence $\{f_n\}$ in C and a sequence $\{F_n\}$ such that $F_n \in T(f_n)$ and $\lim_{n\to\infty}\rho(f_n - F_n) = 0$. Without loss of generality we may assume that $\{f_n\}\rho$-a.e. converges to $f \in C$ and $\{F_n\}\rho$-a.e. converges to $F \in C$. Lemma 1.10 implies

$$\rho(f - F) \leq \liminf_{n\to\infty}\rho(f_n - F_n) = 0. \tag{2.21}$$

Hence $f = F$. Since T is a ρ-nonexpansive-type map, then there exists a sequence $\{G_n\} \in T(f)$ such that

$$\rho(F_n - G_n) \leq \rho(f_n - f), \tag{2.22}$$

for all $n \geq 1$. Since $T(f)$ is ρ-compact, we may assume that $\{G_n\}$ is ρ-convergent to some $h \in T(f)$. Lemma 1.10 implies

$$\liminf_{n\to\infty}\rho(f_n - f) + \rho(f - h) = \liminf_{n\to\infty}\rho(f_n - h). \tag{2.23}$$

Since ρ satisfies the Δ_2-condition, then

$$\begin{aligned}\liminf_{n\to\infty}\rho(f_n - h) &= \liminf_{n\to\infty}\rho(f_n - F_n + F_n - G_n + G_n - h)\\ &= \liminf_{n\to\infty}\rho(F_n - G_n)\end{aligned} \tag{2.24}$$

(see, [20]). Since $\rho(F_n - G_n) \leq \rho(f_n - f)$, we get

$$\liminf_{n\to\infty}\rho(f_n - h) \leq \liminf_{n\to\infty}\rho(f_n - f), \tag{2.25}$$

which implies

$$\liminf_{n\to\infty}\rho(f_n - f) + \rho(f - h) \leq \liminf_{n\to\infty}\rho(f_n - f). \tag{2.26}$$

Hence $\rho(f-h)=0$ or $f=h$. Hence $f\in T(f)$; that is, f is a fixed point of T. □

Proposition 2.3 and Theorem 2.4 are also hold if we assume that C is starshaped instead of Convex. (A set C is called starshaped if there exists $f_0\in C$ such that $\lambda f_0-(1-\lambda)f\in C$ provided $f\in C$ and $\lambda\in[0,1]$.)

3. Fixed Points of w-Contractive-Type Maps

In [21] the authors introduced the concept of w-distance in metric spaces which they connected to the existence of fixed point of single and multivalued maps (see also [22]). Similarly we extend their definition and results to modular spaces. Indeed let ρ be a convex, σ-finite modular function. A function $p: L_\rho\times L_\rho\to[0,\infty)$ is called w-modular on the modular function space L_ρ if the following are satisfied:

(1) $p(f,g)\le p(f,h)+p(h,g)$ for any $f,g,h\in L_\rho$;

(2) for any $f\in L_\rho$, $p(f,\cdot): L_\rho\to[0,\infty)$ is lower semicontinuous; that is, if $\{g_n\}\rho$-converges to g, then

$$p(f,g)\le\liminf_{n\to\infty}p(f,g_n),\tag{3.1}$$

(3) for any $\varepsilon>0$, there exists $\delta>0$ such that $p(f,g)\le\delta$ and $p(f,h)\le\delta$ imply $\rho(g,h)\le\varepsilon$.

As it was done in [21], we need the following technical lemma.

Lemma 3.1. *Let $p(\cdot,\cdot)$ be w-modular on the modular function space L_ρ. Let $\{f_n\}$ and $\{g_n\}$ be sequences in L_ρ, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and $f,g,h\in L_\rho$. Then the following hold:*

(1) *if $p(f_n,g)\le\alpha_n$ and $p(f_n,h)\le\beta_n$, for all $n\ge1$, then $g=h$; in particular if $p(f,g)=0$ and $p(f,h)=0$, then $g=h$;*

(2) *if $p(f_n,g_n)\le\alpha_n$ and $p(f_n,h)\le\beta_n$, for any $n\ge1$, then $\{g_n\}\rho$-converges to h;*

(3) *if $p(f_n,f_m)\le\alpha_n$ for any $n,m\ge1$ with $m>n$, then $\{f_n\}$ is a ρ-Cauchy sequence;*

(4) *if $p(g,f_n)\le\alpha_n$ for any $n\ge1$, then $\{f_n\}$ is a ρ-Cauchy sequence.*

The proof is easy and similar to the one given in [21]. Now we are ready to give the first fixed point result in this setting. Let C be a nonempty ρ-closed subset of the modular function space L_ρ. We say that a multivalued map $T: C\to\mathcal{C}(C)$ is weakly ρ-contractive-type map if there exists w-modular $p(\cdot,\cdot)$ on L_ρ and $k\in[0,1)$ such that for any $f,g\in C$ and any $F\in T(f)$, there exists $G\in T(g)$ such that $p(F,G)\le kp(f,g)$.

Theorem 3.2. *Let C be a nonempty ρ-closed subset of the modular function space L_ρ. Then each weakly ρ-contractive-type map $T: C\to\mathcal{C}(C)$ has a fixed point $f\in C$, and $p(f,f)=0$.*

Proof. Let $p(\cdot,\cdot)$ be a w-modular and $k \in [0,1)$ associated to T, that is, for any $f, g \in C$ and any $F \in T(f)$, there exists $G \in T(g)$ such that $p(F,G) \leq kp(f,g)$. Fix $f_0 \in C$ and $f_1 \in T(f_0)$. By induction one can construct a sequence $\{f_n\}$ such that $f_{n+1} \in T(f_n)$ and

$$p(f_n, f_{n+1}) \leq kp(f_{n-1}, f_n), \tag{3.2}$$

for every $n \geq 1$. In particular we have $p(f_n, f_{n+1}) \leq k^n p(f_0, f_1)$, for every $n \geq 1$. Using the properties of $p(\cdot,\cdot)$, we get

$$p(f_n, f_{n+h}) \leq \frac{k^n}{1-k} p(f_0, f_1), \tag{3.3}$$

for any $n, h \geq 1$. Lemma 3.1 implies that the sequence $\{f_n\}$ is ρ-Cauchy. Hence $\{f_n\}\rho$-converges to some $f \in C$. Using the lower semicontinuity of p, we get

$$p(f_n, f) \leq \liminf_{n\to\infty} p(f_n, f_{n+h}) \leq \frac{k^n}{1-k} p(f_0, f_1), \tag{3.4}$$

for any $n \geq 1$. Since $f_n \in T(f_{n-1})$ and T is weakly ρ-contractive-type map, there exists $g_n \in T(f)$ such that

$$p(f_n, g_n) \leq kp(f_{n-1}, f) \leq \frac{k^n}{1-k} p(f_0, f_1), \tag{3.5}$$

for any $n \geq 2$. Lemma 3.1 implies that $\{g_n\}\rho$- converges to f as well. Since $T(f)$ is ρ-closed, then $f \in T(f)$, that is, f is a fixed point of T. Let us complete the proof by showing that $p(f,f) = 0$. Since $f \in T(f)$, there exists $h_1 \in T(f)$ such that $p(f,h_1) \leq kp(f,f)$. By induction we can construct a sequence $\{h_n\}$ in C such that $h_{n+1} \in T(h_n)$ and $p(f,h_{n+1}) \leq kp(f,h_n)$, for any $n \geq 1$. So we have $p(f,h_n) \leq k^n p(f,f)$, for any $n \geq 1$. Lemma 3.1 implies that $\{h_n\}$ is ρ-Cauchy. Hence $\{h_n\}\rho$- converges to some $h \in C$. Using the lower semicontinuity of $p(\cdot,\cdot)$ we get

$$p(f,h) \leq \liminf_{n\to\infty} p(f,h_n) \leq 0. \tag{3.6}$$

Hence $p(f,h) = 0$. Then for any $n \geq 1$, we have

$$p(f_n, h) \leq p(f_n, f) + p(f,h) \leq \frac{k^n}{1-k} p(f_0, f_1). \tag{3.7}$$

Lemma 3.1 implies $f = h$, or $p(f,f) = 0$. □

Note that in the proof above we did not use the Δ_2-condition. The reason behind is that $p(\cdot,\cdot)$ satisfies the triangle inequality. If T is single valued, then we have little more information about the fixed point. Indeed, let C be a nonempty ρ-closed subset of the modular function space L_ρ. The map $T : C \to C$ is called a weakly ρ-contractive type map if there exists w-modular $p(\cdot,\cdot)$ on L_ρ and $k \in [0,1)$ such that for any $f, g \in C$; $p(T(f), T(g)) \leq kp(f,g)$.

Theorem 3.3. *Let C be a nonempty ρ-closed subset of the modular function space L_ρ. Then each weakly ρ-contractive type map $T : C \to C$ has a unique fixed point $f \in C$, and $p(f, f) = 0$.*

Proof. Theorem 3.2 ensures the existence of a fixed point $f \in C$, that is, $T(f) = f$ and $p(f, f) = 0$. Let us show that f is the only fixed point of T. Assume that $h \in C$ is another fixed point of T. Then we must have $p(f, h) = 0$. Combining this with $p(f, f) = 0$, Lemma 3.1 implies $f = h$. □

Similar extensions of the results as found in [21–23] may be proved in our setting.

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