Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 246808, 13 pages
doi:10.1155/2010/246808

Research Article

Strong Convergence Theorem for a New General System of Variational Inequalities in Banach Spaces

S. Imnang[1,2] and S. Suantai[2,3]

[1] *Department of Mathematics, Faculty of Science, Thaksin University, Phatthalung Campus, Phatthalung 93110, Thailand*
[2] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*
[3] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*

Correspondence should be addressed to S. Suantai, scmti005@chiangmai.ac.th

Received 26 July 2010; Revised 7 December 2010; Accepted 30 December 2010

Academic Editor: S. Reich

We introduce a new system of general variational inequalities in Banach spaces. The equivalence between this system of variational inequalities and fixed point problems concerning the nonexpansive mapping is established. By using this equivalent formulation, we introduce an iterative scheme for finding a solution of the system of variational inequalities in Banach spaces. Our main result extends a recent result acheived by Yao, Noor, Noor, Liou, and Yaqoob.

1. Introduction

Let X be a real Banach space, and X^* be its dual space. Let $U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. X is said to be *uniformly convex* if for each $\epsilon \in (0,2]$ there exists a constant $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \text{ implies } \left\|\frac{x+y}{2}\right\| \leq 1 - \delta. \tag{1.1}$$

The norm on X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for each $x, y \in U$ and in this case X is said to have a *uniformly Frechet differentiable norm* if the limit (1.2) is attained uniformly for $x, y \in U$ and in this case X is said to be *uniformly smooth*. We define a function $\rho : [0,\infty) \to [0,\infty)$, called the *modulus of smoothness* of X, as follows:

$$\rho(\tau) = \sup\left\{ \frac{1}{2(\|x+y\| + \|x-y\|)} - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| = \tau \right\}. \tag{1.3}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau\to 0}\rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. For $q > 1$, the generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f \in X^* : \langle x, f\rangle = \|x\|^q,\ \|f\| = \|x\|^{q-1} \right\}, \quad \forall x \in X. \tag{1.4}$$

In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and usually, we write $J_2 = J$. If X is a Hilbert space, then $J = I$. Further, we have the following properties of the generalized duality mapping J_q:

(1) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in X$ with $x \neq 0$,

(2) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in X$ and $t \in [0,\infty)$,

(3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

It is known that if X is smooth, then J is single-valued, which is denoted by j. Recall that the duality mapping j is said to be *weakly sequentially continuous* if for each $\{x_n\} \subset X$ with $x_n \to x$ weakly, we have $j(x_n) \to j(x)$ weakly-$*$. We know that if X admits a weakly sequentially continuous duality mapping, then X is smooth. For the details, see the work of Gossez and Lami Dozo in [1].

Let C be a nonempty closed convex subset of a smooth Banach space X. Recall that a mapping $A : C \to X$ is said to be *accretive* if

$$\langle Ax - Ay, j(x-y)\rangle \geq 0 \tag{1.5}$$

for all $x, y \in C$. A mapping $A : C \to X$ is said to be *α-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|^2 \tag{1.6}$$

for all $x, y \in C$. A mapping $A : C \to X$ is said to be *α-inverse strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|Ax-Ay\|^2 \tag{1.7}$$

for all $x, y \in C$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

Let D be a nonempty subset of C. A mapping $Q : C \to D$ is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx, \tag{1.8}$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q : C \to D$ is called a *retraction* if $Qx = x$ for all $x \in D$. Furthermore, Q is a *sunny nonexpansive retraction* from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C.

Conveying an idea of the *classical variational inequality*, denoted by $\mathrm{VI}(C, A)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.9}$$

where $X = H$ is a Hilbert space and A is a mapping from C into H. The variational inequality has been widely studied in the literature; see, for example, the work of Chang et al. in [2], Zhao and He [3], Plubtieng and Punpaeng [4], Yao et al. [5] and the references therein.

Let $A, B : C \to H$ be two mappings. In 2008, Ceng et al. [6] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.10}$$

which is called *a general system of variational inequalities*, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.10) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.11}$$

which is defined by Verma [7] and is called *the new system of variational inequalities*. Further, if we add up the requirement that $x^* = y^*$, then problem (1.11) reduces to the classical variational inequality $\mathrm{VI}(C, A)$.

In 2006, Aoyama et al. [8] first considered the following generalized variational inequality problem in Banach spaces. Let $A : C \to X$ be an accretive operator. Find a point $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{1.12}$$

The problem (1.12) is very interesting as it is connected with the fixed point problem for nonlinear mapping and the problem of finding a zero point of an accretive operator in Banach spaces, see [9–11] and the references therein.

Aoyama et al. [8] introduced the following iterative algorithm in Banach spaces:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= Q_C(x_n - \lambda_n A)x_n, \\ x_{n+1} &= a_n x_n + (1 - a_n) y_n, \quad n \geq 0, \end{aligned} \tag{1.13}$$

where Q_C is a sunny nonexpansive retraction from X onto C. Then they proved a weak convergence theorem which is generalized simultaneously theorems of Browder and Petryshyn [12] and Gol'shteĭn and Tret'yakov [13]. In 2008, Hao [14] obtained a strong convergence theorem by using the following iterative algorithm:

$$\begin{aligned} x_0 &\in C, \\ y_n &= b_n x_n + (1 - b_n) Q_C(I - \lambda_n A x_n), \\ x_{n+1} &= a_n u + (1 - a_n) y_n, \quad n \geq 0, \end{aligned} \tag{1.14}$$

where $\{a_n\}$, $\{b_n\}$ are two sequences in $(0,1)$ and $u \in C$.

Very recently, in 2009, Yao et al. [5] introduced the following system of general variational inequalities in Banach spaces. For given two operators $A, B : C \to X$, they considered the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \left\langle Ay^* + x^* - y^*, j\left(x - x^*\right)\right\rangle &\geq 0, \quad \forall x \in C, \\ \left\langle Bx^* + y^* - x^*, j\left(x - y^*\right)\right\rangle &\geq 0, \quad \forall x \in C, \end{aligned} \tag{1.15}$$

which is called *the system of general variational inequalities in a real Banach space*. They proved a strong convergence theorem by using the following iterative algorithm:

$$\begin{aligned} x_0 &\in C, \\ y_n &= Q_C(x_n - Bx_n), \\ x_{n+1} &= a_n u + b_n x_n + c_n Q_C\left(y_n - Ay_n\right), \quad n \geq 0, \end{aligned} \tag{1.16}$$

where $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are three sequences in $(0,1)$ and $u \in C$.

In this paper, motivated and inspired by the idea of Yao et al. [5] and Cheng et al. [6]. First, we introduce the following system of variational inequalities in Banach spaces.

Let C be a nonempty closed convex subset of a real Banach space X. Let $A_i : C \to X$ for all $i = 1, 2, 3$ be three mappings. We consider the following problem of finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{aligned} \left\langle \lambda_1 A_1 y^* + x^* - y^*, j\left(x - x^*\right)\right\rangle &\geq 0, \quad \forall x \in C, \\ \left\langle \lambda_2 A_2 z^* + y^* - z^*, j\left(x - y^*\right)\right\rangle &\geq 0, \quad \forall x \in C, \\ \left\langle \lambda_3 A_3 x^* + z^* - x^*, j\left(x - z^*\right)\right\rangle &\geq 0, \quad \forall x \in C, \end{aligned} \tag{1.17}$$

which is called *a new general system of variational inequalities in Banach spaces*, where $\lambda_i > 0$ for all $i = 1, 2, 3$. In particular, if $A_3 = 0$, $z^* = x^*$, and $\lambda_i = 1$ for $i = 1, 2, 3$, then problem (1.17) reduces to problem (1.15). Further, if $A_3 = 0$, $z^* = x^*$, then problem (1.17) reduces to the problem (1.10) in a real Hilbert space. Second, we introduce iteration process for finding a solution of a new general system of variational inequalities in a real Banach space. Starting with arbitrary points $v, x_1 \in C$ and let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{aligned} z_n &= Q_C(x_n - \lambda_3 A_3 x_n), \\ y_n &= Q_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} &= a_n v + b_n x_n + (1 - a_n - b_n) Q_C(y_n - \lambda_1 A_1 y_n), \quad n \geq 1, \end{aligned} \tag{1.18}$$

where $\lambda_i > 0$ for all $i = 1, 2, 3$ and $\{a_n\}$, $\{b_n\}$ are two sequences in $(0, 1)$. Using the demiclosedness principle for nonexpansive mapping, we will show that the sequence $\{x_n\}$ converges strongly to a solution of a new general system of variational inequalities in Banach spaces under some control conditions.

2. Preliminaries

In this section, we recall the well known results and give some useful lemmas that will be used in the next section.

Lemma 2.1 (see [15]). *Let X be a q-uniformly smooth Banach space with $1 \leq q \leq 2$. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q \tag{2.1}$$

for all $x, y \in X$, where K is the q-uniformly smooth constant of X.

The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.2 (see [16, 17]). *Let C be a closed convex subset of a smooth Banach space X. Let D be a nonempty subset of C and $Q : C \to D$ be a retraction. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0, \tag{2.2}$$

for all $u \in C$ and $y \in D$.

The first result regarding the existence of sunny nonexpansive retractions on the fixed point set of a nonexpansive mapping is due to Bruck [18].

Remark 2.3. If X is strictly convex and uniformly smooth and if $T : C \to C$ is a nonexpansive mapping having a nonempty fixed point set $F(T)$, then there exists a sunny nonexpansive retraction of C onto $F(T)$.

Lemma 2.4 (see [19]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad n \geq 1, \tag{2.3}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 (see [20]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$. Suppose $x_{n+1} = (1-b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.6 (see [21]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ be an nonexpansive mapping. Then $I - T$ is demiclosed at 0, that is, if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$.*

3. Main Results

In this section, we establish the equivalence between the new general system of variational inequalities (1.17) and some fixed point problem involving a nonexpansive mapping. Using the demiclosedness principle for nonexpansive mapping, we prove that the iterative scheme (1.18) converges strongly to a solution of a new general system of variational inequalities (1.17) in a Banach space under some control conditions. In order to prove our main result, the following lemmas are needed.

The next lemmas are crucial for proving the main theorem.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $A : C \to X$ be α-inverse strongly accretive. Then, we have*

$$\|(I-\lambda A)x - (I-\lambda A)y\|^2 \leq \|x-y\|^2 + 2\lambda\left(\lambda K^2 - \alpha\right)\|Ax - Ay\|^2, \tag{3.1}$$

where K is the 2-uniformly smooth constant of X. In particular, if $\alpha \geq \lambda K^2$, then $I - \lambda A$ is a nonexpansive mapping.

Proof. Indeed, for all $x, y \in C$, from Lemma 2.1, we have

$$\begin{aligned}
\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|(x-y) - \lambda(Ax - Ay)\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\langle (Ax - Ay), j(x-y)\rangle \\
&\quad + 2K^2\lambda^2\|Ax - Ay\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + 2K^2\lambda^2\|Ax - Ay\|^2 \\
&= \|x-y\|^2 + 2\lambda\left(\lambda K^2 - \alpha\right)\|Ax - Ay\|^2.
\end{aligned} \tag{3.2}$$

It is clear that, if $\alpha \geq \lambda K^2$, then $I - \lambda A$ is a nonexpansive mapping. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Q_C be the sunny nonexpansive retraction from X onto C. Let $A_i : C \to X$ be an α_i-inverse strongly accretive mapping for $i = 1,2,3$. Let $G : C \to C$ be a mapping defined by*

$$\begin{aligned} G(x) = Q_C[&Q_C(Q_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(x - \lambda_3 A_3 x)) \\ &-\lambda_1 A_1 Q_C(Q_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(x - \lambda_3 A_3 x))], \quad \forall x \in C. \end{aligned} \tag{3.3}$$

If $\alpha_i \geq \lambda_i K^2$ for all $i = 1,2,3$, then $G : C \to C$ is nonexpansive.

Proof. For all $x, y \in C$, we have

$$\begin{aligned} \|G(x) - G(y)\| &= \|Q_C[Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\qquad -\lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x)] \\ &\qquad - Q_C\left[Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y)\right. \\ &\qquad\qquad \left.- \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y)\right]\| \\ &\leq \|Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\qquad - \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x) \\ &\qquad - \left[Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y)\right. \\ &\qquad\qquad \left.- \lambda_1 A_1 Q_C(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y)\right]\| \\ &= \|(I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)x \\ &\qquad -(I - \lambda_1 A_1)Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_3 A_3)y\|. \end{aligned} \tag{3.4}$$

From Lemma 3.1, we have $(I-\lambda_1 A_1)Q_C(I-\lambda_2 A_2)Q_C(I-\lambda_3 A_3)$ is nonexpansive which implies by (3.4) that G is nonexpansive. □

Lemma 3.3. *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let Q_C be the sunny nonexpansive retraction from X onto C. Let $A_i : C \to X$ be three nonlinear mappings. For given $(x^*, y^*, z^*) \in C \times C \times C$, (x^*, y^*, z^*) is a solution of problem* (1.17) *if and only if $x^* \in F(G)$, $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$, where G is the mapping defined as in Lemma 3.2.*

Proof. Note that we can rewrite (1.17) as

$$\begin{aligned} &\langle x^* - (y^* - \lambda_1 A_1 y^*), j(t - x^*)\rangle \geq 0, \quad \forall t \in C, \\ &\langle y^* - (z^* - \lambda_2 A_2 z^*), j(t - y^*)\rangle \geq 0, \quad \forall t \in C, \\ &\langle z^* - (x^* - \lambda_3 A_3 x^*), j(t - z^*)\rangle \geq 0, \quad \forall t \in C. \end{aligned} \tag{3.5}$$

From Lemma 2.2, we can deduce that (3.5) is equivalent to

$$\begin{aligned} x^* &= Q_C(y^* - \lambda_1 A_1 y^*), \\ y^* &= Q_C(z^* - \lambda_2 A_2 z^*), \\ z^* &= Q_C(x^* - \lambda_3 A_3 x^*). \end{aligned} \tag{3.6}$$

It is easy to see that (3.6) is equivalent to $x^* = Gx^*$, $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$. □

From now on we denote by Ω^* the set of all fixed points of the mapping G. Now we prove the strong convergence theorem of algorithm (1.18) for solving problem (1.17).

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X onto C. Let the mappings $A_i : C \to X$ be α_i-inverse strongly accretive with $\alpha_i \geq \lambda_i K^2$, for all $i = 1, 2, 3$ and $\Omega^* \neq \emptyset$. For given $x_1, v \in C$, let the sequence $\{x_n\}$ be generated iteratively by* (1.18). *Suppose the sequences $\{a_n\}$ and $\{b_n\}$ are two sequences in* $(0, 1)$ *such that*

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$.

Then $\{x_n\}$ converges strongly to $Q'v$ where Q' is the sunny nonexpansive retraction of C onto Ω^*.

Proof. Let $x^* \in \Omega^*$ and $t_n = Q_C(y_n - \lambda_1 A_1 y_n)$, it follows from Lemma 3.3 that

$$\begin{aligned} x^* = Q_C[&Q_C(Q_C(x^* - \lambda_3 A_3 x^*) - \lambda_2 A_2 Q_C(x^* - \lambda_3 A_3 x^*)) \\ &-\lambda_1 A_1 Q_C(Q_C(x^* - \lambda_3 A_3 x^*) - \lambda_2 A_2 Q_C(x^* - \lambda_3 A_3 x^*))]. \end{aligned} \tag{3.7}$$

Put $y^* = Q_C(z^* - \lambda_2 A_2 z^*)$ and $z^* = Q_C(x^* - \lambda_3 A_3 x^*)$. Then $x^* = Q_C(y^* - \lambda_1 A_1 y^*)$ and

$$x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) t_n. \tag{3.8}$$

From Lemma 3.1, we have $I - \lambda_i A_i$ $(i = 1, 2, 3)$ is nonexpansive. Therefore

$$\begin{aligned} \|t_n - x^*\| &= \|Q_C(y_n - \lambda_1 A_1 y_n) - Q_C(y^* - \lambda_1 A_1 y^*)\| \\ &\leq \|y_n - y^*\| \\ &= \|Q_C(z_n - \lambda_2 A_2 z_n) - Q_C(z^* - \lambda_2 A_2 z^*)\| \\ &\leq \|z_n - z^*\| \\ &= \|Q_C(x_n - \lambda_3 A_3 x_n) - Q_C(x^* - \lambda_3 A_3 x^*)\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|a_n v + b_n x_n + (1-a_n-b_n)t_n - x^*\| \\ &\le a_n\|v-x^*\| + b_n\|x_n-x^*\| + (1-a_n-b_n)\|t_n-x^*\| \\ &\le a_n\|v-x^*\| + b_n\|x_n-x^*\| + (1-a_n-b_n)\|x_n-x^*\| \\ &= a_n\|v-x^*\| + (1-a_n)\|x_n-x^*\|.\end{aligned} \tag{3.10}$$

By induction, we have

$$\|x_{n+1}-x^*\| \le \max\{\|v-x^*\|, \|x_1-x^*\|\}. \tag{3.11}$$

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, $\{A_1y_n\}$, $\{A_2z_n\}$, and $\{A_3x_n\}$ are also bounded. By nonexpansiveness of Q_C and $I-\lambda_iA_i$ $(i=1,2,3)$, we have

$$\begin{aligned}\|t_{n+1}-t_n\| &= \left\|Q_C\left(y_{n+1}-\lambda_1A_1y_{n+1}\right) - Q_C\left(y_n-\lambda_1A_1y_n\right)\right\| \\ &\le \left\|y_{n+1}-y_n\right\| \\ &= \|Q_C(z_{n+1}-\lambda_2A_2z_{n+1}) - Q_C(z_n-\lambda_2A_2z_n)\| \\ &\le \|z_{n+1}-z_n\| \\ &= \|Q_C(x_{n+1}-\lambda_3A_3x_{n+1}) - Q_C(x_n-\lambda_3A_3x_n)\| \\ &\le \|x_{n+1}-x_n\|.\end{aligned} \tag{3.12}$$

Let $w_n = (x_{n+1}-b_nx_n)/(1-b_n)$, $n\in\mathbb{N}$. Then $x_{n+1} = b_nx_n + (1-b_n)w_n$ for all $n\in\mathbb{N}$ and

$$\begin{aligned}w_{n+1}-w_n &= \frac{x_{n+2}-b_{n+1}x_{n+1}}{1-b_{n+1}} - \frac{x_{n+1}-b_nx_n}{1-b_n} \\ &= \frac{a_{n+1}v + (1-a_{n+1}-b_{n+1})t_{n+1}}{1-b_{n+1}} - \frac{a_nv + (1-a_n-b_n)t_n}{1-b_n} \\ &= \frac{a_{n+1}}{1-b_{n+1}}(v-t_{n+1}) + \frac{a_n}{1-b_n}(t_n-v) + t_{n+1}-t_n.\end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we have

$$\begin{aligned}\|w_{n+1}-w_n\| - \|x_{n+1}-x_n\| &\le \frac{a_{n+1}}{1-b_{n+1}}\|v-t_{n+1}\| + \frac{a_n}{1-b_n}\|t_n-v\| \\ &\quad + \|t_{n+1}-t_n\| - \|x_{n+1}-x_n\| \\ &\le \frac{a_{n+1}}{1-b_{n+1}}\|v-t_{n+1}\| + \frac{a_n}{1-b_n}\|t_n-v\|.\end{aligned} \tag{3.14}$$

This together with (C1) and (C2), we obtain that

$$\limsup_{n\to\infty} \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \le 0. \tag{3.15}$$

Hence, by Lemma 2.5, we get $\|x_n - w_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - b_n)\|w_n - x_n\| = 0. \tag{3.16}$$

Since

$$x_{n+1} - x_n = a_n(v - x_n) + (1 - a_n - b_n)(t_n - x_n), \tag{3.17}$$

therefore

$$\|t_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.18}$$

Furthermore, by Lemma 3.2, we have $G : C \to C$ is nonexpansive. Thus, we have

$$\begin{aligned}
\|t_n - G(t_n)\| &= \left\|Q_C(y_n - \lambda_1 A_1 y_n) - G(t_n)\right\| \\
&= \|Q_C[Q_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 Q_C(z_n - \lambda_2 A_2 z_n)] - G(t_n)\| \\
&= \|Q_C[Q_C(Q_C(x_n - \lambda_3 A_3 x_n) - \lambda_2 A_2 Q_C(x_n - \lambda_3 A_3 x_n)) \\
&\qquad - \lambda_1 A_1 Q_C(Q_C(x_n - \lambda_3 A_3 x_n) - \lambda_2 A_2 Q_C(x_n - \lambda_3 A_3 x_n))] - G(t_n)\| \\
&= \|G(x_n) - G(t_n)\| \le \|x_n - t_n\|,
\end{aligned} \tag{3.19}$$

which implies $\|t_n - G(t_n)\| \to 0$ as $n \to \infty$.

Since

$$\begin{aligned}
\|x_n - G(x_n)\| &\le \|x_n - t_n\| + \|t_n - G(t_n)\| + \|G(t_n) - G(x_n)\| \\
&\le \|x_n - t_n\| + \|t_n - G(t_n)\| + \|t_n - x_n\|,
\end{aligned} \tag{3.20}$$

therefore

$$\lim_{n\to\infty} \|x_n - G(x_n)\| = 0. \tag{3.21}$$

Let Q' be the sunny nonexpansive retraction of C onto Ω^*. Now we show that

$$\limsup_{n\to\infty} \langle v - Q'v, j(x_n - Q'v)\rangle \le 0. \tag{3.22}$$

To prove (3.22), since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\overline{x}$ and

$$\limsup_{n\to\infty}\langle v-Q'v, j(x_n-Q'v)\rangle = \lim_{i\to\infty}\langle v-Q'v, j(x_{n_i}-Q'v)\rangle. \tag{3.23}$$

From Lemma 2.6 and (3.21), we obtain $\overline{x}\in\Omega^*$. Now, from Lemma 2.2, (3.23), and the weakly sequential continuity of the duality mapping j, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle v-Q'v, j(x_n-Q'v)\rangle &= \lim_{i\to\infty}\langle v-Q'v, j(x_{n_i}-Q'v)\rangle \\ &= \langle v-Q'v, j(\overline{x}-Q'v)\rangle \le 0.\end{aligned} \tag{3.24}$$

From (3.9), we have

$$\begin{aligned}\|x_{n+1}-Q'v\|^2 &= \langle a_n v + b_n x_n + (1-a_n-b_n)t_n - Q'v, j(x_{n+1}-Q'v)\rangle \\ &= a_n\langle v-Q'v, j(x_{n+1}-Q'v)\rangle + b_n\langle x_n-Q'v, j(x_{n+1}-Q'v)\rangle \\ &\quad + (1-a_n-b_n)\langle t_n-Q'v, j(x_{n+1}-Q'v)\rangle \\ &\le a_n\langle v-Q'v, j(x_{n+1}-Q'v)\rangle + b_n(\|x_n-Q'v\|\|j(x_{n+1}-Q'v)\|) \\ &\quad + (1-a_n-b_n)(\|t_n-Q'v\|\|j(x_{n+1}-Q'v)\|) \\ &= a_n\langle v-Q'v, j(x_{n+1}-Q'v)\rangle + b_n(\|x_n-Q'v\|\|x_{n+1}-Q'v\|) \\ &\quad + (1-a_n-b_n)(\|t_n-Q'v\|\|x_{n+1}-Q'v\|) \\ &\le a_n\langle v-Q'v, j(x_{n+1}-Q'v)\rangle + \frac{1}{2}b_n\left(\|x_n-Q'v\|^2+\|x_{n+1}-Q'v\|^2\right) \\ &\quad + \frac{1}{2}(1-a_n-b_n)\left(\|t_n-Q'v\|^2+\|x_{n+1}-Q'v\|^2\right) \\ &\le a_n\langle v-Q'v, j(x_{n+1}-Q'v)\rangle + \frac{1}{2}b_n\left(\|x_n-Q'v\|^2+\|x_{n+1}-Q'v\|^2\right) \\ &\quad + \frac{1}{2}(1-a_n-b_n)\left(\|x_n-Q'v\|^2+\|x_{n+1}-Q'v\|^2\right) \\ &= a_n\langle v-Q'v, j(x_{n+1}-Q'v)\rangle + \frac{1}{2}(1-a_n)\left(\|x_n-Q'v\|^2+\|x_{n+1}-Q'v\|^2\right),\end{aligned} \tag{3.25}$$

which implies that

$$\|x_{n+1}-Q'v\|^2 \le (1-a_n)\|x_n-Q'v\|^2 + 2a_n\langle v-Q'v, j(x_{n+1}-Q'v)\rangle. \tag{3.26}$$

It follows from Lemma 2.4, (3.24), and (3.26) that $\{x_n\}$ converges strongly to $Q'v$. This completes the proof. □

Letting $A_3 = 0$ and $\lambda_i = 1$ for $i = 1, 2, 3$ in Theorem 3.4, we obtain the following result.

Corollary 3.5 (see [5, Theorem 3.1]). *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X onto C. Let the mappings $A_i : C \to X$ be α_i-inverse strongly accretive with $\alpha_i \geq K^2$, for all $i = 1, 2$ and $\Omega^* \neq \emptyset$. For given $x_1, v \in C$, and let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{aligned} y_n &= Q_C(x_n - A_2 x_n), \\ x_{n+1} &= a_n v + b_n x_n + (1 - a_n - b_n) Q_C\left(y_n - A_1 y_n\right), \quad n \geq 1, \end{aligned} \tag{3.27}$$

where $\{a_n\}$, $\{b_n\}$ are two sequences in $(0, 1)$ such that

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$.

Then $\{x_n\}$ converges strongly to $Q'v$ where Q' is the sunny nonexpansive retraction of C onto Ω^*.

Acknowledgments

The authors wish to express their gratitude to the referees for careful reading of the manuscript and helpful suggestions. The authors would like to thank the Commission on Higher Education, the Thailand Research Fund, the Thaksin university, the Centre of Excellence in Mathematics, and the Graduate School of Chiang Mai University, Thailand for their financial support.

References

[1] J.-P. Gossez and E. Lami Dozo, "Some geometric properties related to the fixed point theory for nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 40, pp. 565–573, 1972.

[2] S.-S. Chang, H. W. J. Lee, and C. K. Chan, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 9, pp. 3307–3319, 2009.

[3] J. Zhao and S. He, "A new iterative method for equilibrium problems and fixed point problems of infinitely nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 670–680, 2009.

[4] S. Plubtieng and R. Punpaeng, "A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 548–558, 2008.

[5] Y. Yao, M. A. Noor, K. Inayat Noor, Y.-C. Liou, and H. Yaqoob, "Modified extragradient methods for a system of variational inequalities in Banach spaces," *Acta Applicandae Mathematicae*, vol. 110, no. 3, pp. 1211–1224, 2010.

[6] L.-C. Ceng, C.-Y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[7] Ram U. Verma, "On a new system of nonlinear variational inequalities and associated iterative algorithms," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 8, pp. 65–68, 1999.

[8] K. Aoyama, H. Iiduka, and W. Takahashi, "Weak convergence of an iterative sequence for accretive operators in Banach spaces," *Fixed Point Theory and Applications*, vol. 2006, Article ID 35390, 13 pages, 2006.

[9] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, vol. 83, Marcel Dekker, New York, NY, USA, 1984.

[10] S. Reich, “Extension problems for accretive sets in Banach spaces,” *Journal of Functional Analysis*, vol. 26, no. 4, pp. 378–395, 1977.

[11] S. Reich, “Product formulas, nonlinear semigroups, and accretive operators,” *Journal of Functional Analysis*, vol. 36, no. 2, pp. 147–168, 1980.

[12] F. E. Browder and W. V. Petryshyn, “Construction of fixed points of nonlinear mappings in Hilbert space,” *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[13] E. G. Gol’shteĭn and N. V. Tret’yakov, “Modified Lagrangians in convex programming and their generalizations,” *Mathematical Programming Study*, no. 10, pp. 86–97, 1979.

[14] Y. Hao, “Strong convergence of an iterative method for inverse strongly accretive operators,” *Journal of Inequalities and Applications*, vol. 2008, Article ID 420989, 9 pages, 2008.

[15] H. K. Xu, “Inequalities in Banach spaces with applications,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[16] R.d E. Bruck, Jr., “Nonexpansive retracts of Banach spaces,” *Bulletin of the American Mathematical Society*, vol. 76, pp. 384–386, 1970.

[17] S. Reich, “Asymptotic behavior of contractions in Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 44, pp. 57–70, 1973.

[18] R. E. Bruck, Jr., “Nonexpansive projections on subsets of Banach spaces,” *Pacific Journal of Mathematics*, vol. 47, pp. 341–355, 1973.

[19] H.-K. Xu, “Viscosity approximation methods for nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[20] T. Suzuki, “Strong convergence of Krasnoselskii and Mann’s type sequences for one-parameter nonexpansive semigroups without Bochner integrals,” *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[21] F. E. Browder, “Fixed-point theorems for noncompact mappings in Hilbert space,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, pp. 1272–1276, 1965.