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*Research Article*

# On the Fixed-Point Property of Unital Uniformly Closed Subalgebras of $C(X)$

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Let $X$ be a compact Hausdorff topological space and let $C(X)$ and $C_{\mathbb{R}}(X)$ denote the complex and real Banach algebras of all continuous complex-valued and continuous real-valued functions on $X$ under the uniform norm on $X$, respectively. Recently, Fupinwong and Dhompongsa (2010) obtained a general condition for infinite dimensional unital commutative real and complex Banach algebras to fail the fixed-point property and showed that $C_{\mathbb{R}}(X)$ and $C(X)$ are examples of such algebras. At the same time Dhompongsa et al. (2011) showed that a complex $C^*$-algebra $A$ has the fixed-point property if and only if $A$ is finite dimensional. In this paper we show that some complex and real unital uniformly closed subalgebras of $C(X)$ do not have the fixed-point property by using the results given by them and by applying the concept of peak points for those subalgebras.

## 1. Introduction and Preliminaries

We let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in C : |z| \leq 1\}$ denote the fields of complex, real numbers, the set of natural numbers, the unit circle, the open unit disc, and the closed unit disc, respectively. The symbol $\mathbb{F}$ denotes a field that can be either $\mathbb{C}$ or $\mathbb{R}$. The elements of $\mathbb{F}$ are called scalars.

Let $X$ be a compact topological space. We denote by $C_{\mathbb{F}}(X)$ the unital commutative Banach algebra (over $\mathbb{F}$) of continuous functions from $X$ to $\mathbb{F}$ with pointwise addition, scalar multiplication, and product with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \quad (f \in C_{\mathbb{F}}(X)). \tag{1.1}$$

For applying the usual notation, we write $C(X)$ instead of $C_{\mathbb{C}}(X)$.

Let $T : E \to E$ be a self-map on the nonempty set $E$. We denote $\{x \in E : T(x) = x\}$ by $\mathrm{Fix}(T)$ and call the *fixed-points set* of $T$.

Let $\mathfrak{X}$ be a normed space over the field $\mathbb{F}$. A mapping $T : E \subseteq \mathfrak{X} \to \mathfrak{X}$ is *nonexpansive* if $\|T(f) - T(g)\| \leq \|f - g\|$ for all $f, g \in E$. We say that the normed space $\mathfrak{X}$ has the *fixed-point property* if for every nonempty bounded closed convex subset $E$ of $\mathfrak{X}$ and every *nonexpansive* mapping $T : E \to E$ we have $\operatorname{Fix}(T) \neq \emptyset$. One of the central goals in fixed point theory is to find which Banach spaces have the fixed-point property.

Let $A$ be a unital algebra (over $\mathbb{F}$) with unit 1 and let $G(A)$ denote the set of all invertible elements of $A$. We define the *spectrum* of an element $f$ of $A$ to be the set $\{\lambda \in \mathbb{F} : \lambda 1 - f \notin G(A)\}$ and denote it by $\sigma(f)$. The *spectral radius* of $f$, denoted by $r(f)$, is defined to be $\sup\{|\lambda| : \lambda \in \sigma(f)\}$. Note that if $A$ is a unital complex Banach algebra, then $r(f) = \lim_{n\to\infty}\|f^n\|^{1/n} = \inf\{\|f^n\|^{1/n} : n \in \mathbb{N}\}$ (see [1, Theorem 10.13]).

A *character* on a unital algebra $A$ over $\mathbb{F}$ is a nonzero homomorphism $\varphi : A \to \mathbb{F}$. We denote by $\Omega(A)$ the set of all characters on $A$. If $A$ is a unital commutative complex Banach algebra, $\Omega(A) \neq \emptyset$ and $\sigma(f) = \{\varphi(f) : \varphi \in \Omega(A)\}$ for all $f \in A$ (see [2, 3]). Note that if $A$ is real algebra, it may be the case that $\Omega(A) = \emptyset$ (see [4, Example 2.4] and Example 3.9 below) or $\Omega(A) \neq \emptyset$ and $\sigma(f) \neq \{\varphi(f) : \varphi \in \Omega(A)\}$(see Example 3.8 below).

Let $A$ be a unital commutative real Banach algebra. A *complex character* on $A$ is a nonzero homomorphism $\varphi : A \to \mathbb{C}$, regarded as a real algebra. The set of all complex character on $A$ is called the *carrier space* of $A$ and denoted by $\operatorname{Car}(A)$. Obviously, $\Omega(A) \subseteq \operatorname{Car}(A)$.

Let $X$ be a compact topological space and let $A$ be a unital uniformly closed subalgebra of $C_{\mathbb{F}}(X)$. For each $x \in X$, the map $\varepsilon_x : A \to \mathbb{F}$ defined by $\varepsilon_x(f) = f(x)$, belongs to $\Omega(A)$ which is called the *evaluation character* on $A$ at $x$. It is known that $\Omega(C(X)) = \{\varepsilon_x : x \in X\}$.

Let $\mathfrak{F}$ be a collection of complex-valued functions on a nonempty set $X$. We say that:

(i) $\mathfrak{F}$ separates the points of X if for each $x, y \in X$ with $x \neq y$, there is a function $f$ in $\mathfrak{F}$ such that $f(x) \neq f(y)$;

(ii) $\mathfrak{F}$ is *self-adjoint* if $f \in \mathfrak{F}$ implies that $\overline{f} \in \mathfrak{F}$;

(iii) $\mathfrak{F}$ is *inverse-closed* if $1/f \in \mathfrak{F}$ whenever $f \in \mathfrak{F}$ and $f(x) \neq 0$ for all $x \in X$.

Let $A$ be a unital commutative complex Banach algebra. It is known that each $\varphi \in \Omega(A)$ is continuous and $\|\varphi\| = 1$. For each $f \in A$, we define the map $\widehat{f} : \Omega(A) \to \mathbb{C}$ by $\widehat{f}(\varphi) = \varphi(f)$ $(\varphi \in \Omega(A))$ and say that $\widehat{f}$ is the *Gelfand transform* of $f$. We denote the set $\{\widehat{f} : f \in A\}$ by $\widehat{A}$. It is easy to see that $\widehat{A}$ separates the points of $\Omega(A)$. The *Gelfand topology* of $\Omega(A)$ is the weakest topology on $\Omega(A)$ for which every $\widehat{f} \in \widehat{A}$ is continuous. In fact, the Gelfand topology of $\Omega(A)$ coincides with the relative topology on $\Omega(A)$ which is given by weak* topology of $A^*$, the dual space of $A$. We know that $\Omega(A)$ with the Gelfand topology is a compact Hausdorff topological space and $\widehat{A}$ is a complex subalgebra of $C(\Omega(A))$ (see [1, 3]). Clearly, the following statements are equivalent.

(i) $\widehat{A}$ is self-adjoint.

(ii) For each $f \in A$, there exists an element $g \in A$ such that $\varphi(g) = \overline{\varphi(f)}$ for all $\varphi \in \Omega(A)$.

Let $X$ be a topological space. A self-map $\tau : X \to X$ is called a *topological involution* on $X$ if $\tau$ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Let $X$ be a compact Hausdorff topological space and $\tau$ be a topological involution on $X$. We denote by $C(X, \tau)$ the set of all $f \in C(X)$ for which $\overline{f} \circ \tau = f$. Then $C(X, \tau)$ is a unital uniformly closed real subalgebra of $C(X)$ which separates the points of $X$, does not contain the constant function $i$ and we have $C(X) = C(X, \tau) \oplus iC(X, \tau)$. Moreover, $C(X, \tau) = C_{\mathbb{R}}(X)$ if and only if $\tau$ is the identity map on $X$. Let $A$ be a unital uniformly closed real subalgebra of $C(X,\tau)$. For each $x \in X$ the map $e_x : A \to \mathbb{C}$ defined by $e_x(f) = f(x)$, is a complex character on $A$ which is called the *evaluation complex character* on $A$ at $x$. We know that $\mathrm{Car}(C(X,\tau)) = \{e_x : x \in X\}$ (see [5]). The algebra $C(X,\tau)$ was first introduced by Kulkarni and Limaye in [6]. We denote by $C_{\mathbb{R}}(X,\tau)$ the set of all $f \in C(X,\tau)$ for which $f$ is real-valued on $X$. Then $C_{\mathbb{R}}(X,\tau)$ is a unital uniformly closed real subalgebra of $C(X,\tau)$.

Let $X$ be a compact Hausdorff topological space and let $A$ be a unital real or complex subspace of $C(X)$. A nonempty subset $P$ of $X$ called a *peak set* for A if there exists a function $f$ in $A$ such that $P = \{x \in X : f(x) = 1\}$ and $|f(y)| < 1$ for all $y \in X \setminus P$, the function $f$ is said to peak on $P$. If the peak set $P$ for $A$ is the singleton $\{x\}$, we call $x$ a *peak point* for $A$. The set of all peak points for $A$ is denoted by $S_0(A,X)$. A nonempty subset $E$ of $X$ is called a *boundary* for $A$, if for each $f \in A$ there is an element $x$ of $E$ such that $\|f\|_X = |f(x)|$. Clearly, $S_0(A,X) \subseteq E$ whenever $E$ is a boundary for $A$. It is known that, if $X$ is a first countable compact Hausdorff topological space then $S_0(C(X),X) = X$ (see [7]).

Let $\tau$ be a topological involution on a compact Hausdorff topological space $X$ and let $A$ be a unital uniformly closed real subspace of $C(X,\tau)$. If $P \subseteq X$ is a peak set for $A$, then $\tau(P) = P$.

*Definition 1.1.* Let $\tau$ be a topological involution on a compact Hausdorff topological space $X$ and $A$ be a unital uniformly closed real subspace of $C(X,\tau)$. We say that $x \in X$ is a *$\tau$-peak point* for $A$ if $\{x,\tau(x)\}$ is a peak set for $A$. We denote by $T_0(A,X,\tau)$ the set of all $\tau$-peak points for $A$.

Let $X$ be a compact Hausdorff topological space and $\tau$ be a topological involution on $X$. Let $B$ be a unital uniformly closed subalgebra of $C(X)$ such that $\overline{f} \circ \tau \in B$ for all $f \in B$ and define $A = \{f \in B : \overline{f} \circ \tau = f\}$. Then $A$ is a unital uniformly closed real subalgebra of $(C(X,\tau))$, $B = A \oplus iA$, $S_0(A,X) = S_0(B,X) \cap \mathrm{Fix}(\tau)$ and $T_0(A,X,\tau) = S_0(B,X)$ (see [5]).

Fupinwong and Dhompongsa studied the fixed-point property of unital commutative Banach algebras over field $\mathbb{F}$ in [4]. In the case $\mathbb{F} = \mathbb{R}$, they obtained the following results.

**Theorem 1.2** (see [4, Theorem 3.1]). *Let $A$ be an infinite dimensional unital commutative real Banach algebra satisfying each of the following:*

(i) $\Omega(A) \neq \emptyset$ *and* $\sigma(f) = \{\varphi(f) : f \in \Omega(A)\}$,

(ii) *if* $f, g \in A$ *such that* $|\varphi(f)| \le |\varphi(g)|$ *for each* $\varphi \in \Omega(A)$, *then* $\|f\| \le \|g\|$,

(iii) $\inf\{r(f) : f \in A,\ \|f\| = 1\} > 0$.

*Then $A$ does not have the fixed-point property.*

**Theorem 1.3** (see [4, Corollary 3.2]). *Let $X$ be a compact Hausdorff topological space. If $C_{\mathbb{R}}(X)$ is infinite dimensional, then $C_{\mathbb{R}}(X)$ fails to have the fixed-point property.*

In the case $\mathbb{F} = \mathbb{C}$, they obtained the following result.

**Theorem 1.4** (see [4, Theorem 4.3]). *Let $A$ be an infinite dimensional unital commutative complex Banach algebra satisfying each of the following:*

(i) $\widehat{A}$ *is self-adjoint,*

(ii) *if* $f, g \in A$ *such that* $|\varphi(f)| \le |\varphi(g)|$ *for each* $\varphi \in \Omega(A)$, *then* $\|f\| \le \|g\|$,

(iii) $\inf\{r(f) : f \in A,\ \|f\| = 1\} > 0$.

*Then $A$ does not have the fixed-point property.*

By using the above theorem, we obtain the following result.

**Theorem 1.5.** *Let $X$ be a compact Hausdorff topological space. If $C(X)$ is infinite dimensional, then $C(X)$ fails to have the fixed-point property.*

Dhompongsa et al. studied the fixed-point property of complex $C^*$-algebras in [8] and obtained the following result.

**Theorem 1.6** (see [8, Theorem 1.4]). *The following properties for a complex $C^*$-algebras $A$ are equivalent:*

(i) *$A$ has the fixed-point property;*

(ii) *$A$ has finite dimension.*

In this paper, we give a general condition for some infinite dimensional unital uniformly closed subalgebras of $C(X)$ to fail the fixed-point property by applying Theorems 1.4 and 1.6. By using the concept of peak points for unital uniformly closed subalgebras of $C(X)$, we show that some of these algebras do not have the fixed-point property. We also prove that $C_{\mathbb{R}}(X,\tau)$ and $C(X,\tau)$ fail to have the fixed-point property. By using the concept of $\tau$-peak points for unital uniformly closed real subalgebras of $C(X,\tau)$, we show that some of these algebras do not have the fixed-point property.

## 2. FPP of Complex Subalgebras of $C(X)$

We first obtain a general condition for infinite dimensional unital uniformly closed subalgebra of $C(X)$ to fail the fixed-point property and give an infinite collection of these algebras.

**Theorem 2.1.** *Let $X$ be a compact topological space. If $A$ is a infinite dimensional self-adjoint uniformly closed subalgebras of $C(X)$, then $A$ does not have the fixed-point property.*

*Proof.* By hypothesises, $A$ is an infinite dimensional complex $C^*$-algebra under the natural involution $f \hookrightarrow \overline{f} : A \to A$. Then, $A$ does not have the fixed-point property by Theorem 1.6. □

*Example 2.2.* Let $m \in \mathbb{N}$ and let $A_m$ be the uniformly closed subalgebra of $C(\mathbb{T})$ generated by $1, Z^{2m}$ and $\overline{Z}^{2m}$, where $Z$ is the coordinate function on $\mathbb{T}$. Then $A_m$ is an infinite dimensional self-adjoint uniformly closed subalgebra of $C(\mathbb{T})$ and so $A_m$ does not have the fixed-point property.

*Proof.* It is easy to see that $A_m$ is self-adjoint. To complete the proof, it is enough to show that $A_m$ is infinite dimensional. We define the sequence $\{f_{m,\,n}\}_{n=0}^{\infty}$ of elements of $A_m$ by

$$f_{m,0} = 1, \qquad f_{m,n} = Z^{2^n m} - 1 \quad (n \in \mathbb{N}). \tag{2.1}$$

We can prove that for each $n \in \mathbb{N}$ the set $\{f_{m,0}, f_{m,1}, \ldots, f_{m,n}\}$ is a linearly independent set of elements of $A_m$. Therefore, $A_m$ is infinite dimensional. □

We now show that some of the unital uniformly closed subalgebras of $C(X)$ fail to have the fixed-point property by using the concept of peak points for these algebras.

**Theorem 2.3.** *Let $X$ be a compact Hausdorff topological space and let $A$ be a unital uniformly closed complex subalgebra of $C(X)$. If $S_0(A, X)$ contains a limit point of $X$, then $A$ does not have the fixed-point property.*

*Proof.* Let $x_0 \in S_0(A, X)$ be a limit point of $X$. Then there exists a function $f_0 \in A$ with $f_0(x_0) = 0$ and $|f_0(x)| < 1$ for all $x \in X \setminus \{x_0\}$, and there exists a net $\{x_\alpha\}_\alpha$ in $X \setminus \{x_0\}$ such that $\lim_\alpha x_\alpha = x$ in $X$. We define $E = \{f \in A : \|f\|_X = f(x_0) = 1\}$. Then $E$ is a nonempty bounded closed convex subset of $A$ and $f_0 f \in E$ for all $f \in E$. We define the map $T : E \to E$ by $T(f) = f_0 f$. It is easy to see that $T$ is a nonexpansive mapping on $E$.

We claim that $\mathrm{Fix}(T) = \emptyset$. Suppose $f_1 \in \mathrm{Fix}(T)$. Then $f_0 f_1 = f_1$ and so $f_1(x) = 0$ for all $x \in X \setminus \{x_0\}$. The continuity of $f_1$ in $x_0$ implies that $\lim_\alpha f_1(x_\alpha) = f_1(x_0)$. Therefore, $f_1(x_0) = 0$, contradicting to $f_1 \in E$. Hence, our claim is justified. Consequently, $A$ does not have the fixed-point property. □

**Corollary 2.4.** *Let $X$ be a perfect compact Hausdorff topological space. If $A$ is a unital uniformly closed subalgebras of $C(X)$ with $S_0(A, X) \neq \emptyset$, then $A$ does not have the fixed-point property.*

*Example 2.5.* Let $A(\overline{\mathbb{D}})$ denote the disk algebra, the complex Banach algebra of all continuous complex-valued functions on $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$ under the uniform norm $\|f\|_{\overline{\mathbb{D}}} = \sup\{|f(z)| : z \in \overline{\mathbb{D}}\}$ $(f \in A(\overline{\mathbb{D}}))$. Then $A(\overline{\mathbb{D}})$ does not have the fixed-point property.

*Proof.* Clearly $\overline{\mathbb{D}}$ is a perfect compact Hausdorff topological space and $A(\overline{\mathbb{D}})$ is a unital uniformly closed complex subalgebra of $C(\overline{\mathbb{D}})$. By the principle of maximum modulus, $S_0(A(\overline{\mathbb{D}}), \overline{\mathbb{D}}) \subseteq \mathbb{T}$. Now let $\lambda \in \mathbb{T}$. It is easy to see that the function $f : \overline{\mathbb{D}} \to \mathbb{C}$, defined by $f(z) = (1/2)(1 + \overline{\lambda} z)$, belongs to $A(\overline{\mathbb{D}})$ and peaks at $\lambda$. Therefore, $S_0(A(\overline{\mathbb{D}}), \overline{\mathbb{D}}) = \mathbb{T}$. It follows that $A(\overline{\mathbb{D}})$ does not have the fixed-point property by Corollary 2.4. □

Now by giving an example we show that the converse of Theorem 2.3 is not necessarily true, in general.

*Example 2.6.* Let $J$ be an uncountable set and let $X_\alpha$ be the unit closed interval $[0, 1]$ with the standard topology for each $\alpha \in J$. Suppose $X = \prod_{\alpha \in J} X_\alpha$ with the product topology. Then $C(X)$ fails to have the fixed-point property but $S_0(C(X), X) = \emptyset$ and so $S_0(C(X), X)$ does not contain any limit points of $X$.

*Proof.* Clearly, $X$ is an infinite compact Hausdorff topological space. Choose a sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ such that $x_j \neq x_k$, where $j, k \in \mathbb{N}$ and $j \neq k$. By Urysohn's lemma, there exists a sequence $\{h_n\}_{n=1}^{\infty}$ in $C(X)$ such that $h_1 = 1$ and $h_n(x_1) = \cdots = h_n(x_{n-1}) = 0, h_n(x_n) = 1$ for all $n \geq 2$. It is easy to see that the set $\{h_1, \ldots, h_n\}$ is a linearly independent set in $C(X)$ for all $n \in \mathbb{N}$. Thus, $C(X)$ is an infinite dimensional complex vector space. Therefore, $C(X)$ does not have the fixed-point property by Theorem 1.5.

We now show that $S_0(C(X), X) = \emptyset$. We assume that $E$ is the set of all $\underline{x} = (x_\alpha)_{\alpha \in J} \in X$ for which there is a countable subset $I_{\underline{x}}$ of $J$ such that $x_\alpha = 0$ for all $\alpha \in J \setminus I_{\underline{x}}$ and $F$ is the set of all $\underline{x} = (x_\alpha)_{\alpha \in J} \in X$ for which there is a countable subset $J_{\underline{x}}$ of $J$ such that $x_\alpha = 1$ for all $\alpha \in J \setminus J_{\underline{x}}$. Clearly, $E \cap F = \emptyset$. It is easy to see that $E$ and $F$ are boundaries for $C(X)$. Therefore, $S_0(C(X), X) = \emptyset$. □

*Remark 2.7.* Let $X$ be an infinite first countable compact Hausdorff topological space. Then $S_0(C(X), X) = X$, and $X$ has at least one limit point. Hence $S_0(C(X), X)$ contains a limit point of $X$. Therefore, $C(X)$ fails to have the fixed-point property by Theorem 2.3.

## 3. FPP of Real Subalgebras of $C(X)$

We first give a sufficient condition for unital uniformly closed real subalgebras of $C_{\mathbb{R}}(X)$ to fail the fixed-point property.

**Lemma 3.1.** *If $A$ is a unital commutative real Banach algebra with $\Omega(A) \neq \emptyset$, then $\{\varphi(f) : \varphi \in \Omega(A)\} \subseteq \sigma(f)$ for all $f \in A$.*

*Proof.* Let $f \in A$. For each $\varphi \in \Omega(A)$, we define $g_\varphi = \varphi(f)1 - f$. Then $g_\varphi \in A$ and $\varphi(g_\varphi) = 0$. Therefore, $g_\varphi \notin G(A)$ and so $\varphi(f) \in \sigma(f)$. □

**Lemma 3.2.** *Let $X$ be a compact topological space. If $A$ is an inverse closed unital uniformly closed real subalgebra of $C_{\mathbb{R}}(X)$, then $\Omega(A) \neq \emptyset$, $\Omega(A) = \{\varepsilon_x : x \in X\}$ and $\sigma(f) = \{\varphi(f) : \varphi \in \Omega(A)\}$ for all $f \in A$.*

*Proof.* Since $A$ is a unital real subalgebra of $C_{\mathbb{R}}(X)$, $\varepsilon_x \in \Omega(A)$ for all $x \in X$. Therefore, $\Omega(A) \neq \emptyset$ and so $\{\varphi(f) : \varphi \in \Omega(A)\} \subseteq \sigma(f)$ for all $f \in A$ by Lemma 3.1.

Now, let $f \in A$ and let $\lambda \in \mathbb{C} \setminus \{\varphi(f) : \varphi \in \Omega(A)\}$. Then $\lambda - \varphi(f) \neq 0$ for each $\varphi \in \Omega(A)$, and so $(\lambda 1 - f)(x) \neq 0$ for all $x \in X$. Therefore, $\lambda 1 - f \in G(A)$ because $A$ is inverse-closed. It follows that $\lambda \in \mathbb{C} \setminus \sigma(f)$ and so $\sigma(f) \subseteq \{\varphi(f) : \varphi \in \Omega(A)\}$. We now show that $\Omega(A) \subseteq \{\varepsilon_x : x \in X\}$. Suppose $\varphi \in \Omega(A) \setminus \{\varepsilon_x : x \in X\}$. Let $x \in X$. Then there exists a function $f_x$ in $A$ such that $\varphi(f_x) \neq f_x(x)$. We define $g_x = f_x - \varphi(f_x)1$. Then $g_x \in A$, $\varphi(g_x) = 0$ and $g_x(x) \neq 0$. The continuity of $g_x$ on $X$ implies that there exists a neighborhood $U_x$ of $x$ in $X$ such that $g_x(y) \neq 0$ for all $y \in U_x$. By compactness of $X$, there exist finite elements $x_1, \ldots, x_m$ of $X$ such that $X = \bigcup_{j=1}^m U_{x_j}$. Define $g = \sum_{j=1}^m (g_{x_j})^2$. Clearly, $g \in A$ and $\varphi(g) = 0$. Moreover, $g(y) \neq 0$ for all $y \in X$. Since $A$ is inverse-closed, $1/g \in A$. It follows that $\varphi(g) \neq 0$. This contradiction implies that $\Omega(A) \subseteq \{\varepsilon_x : x \in X\}$. □

**Theorem 3.3.** *Let $X$ be a compact topological space. If $A$ is an infinite dimensional inverse-closed unital uniformly closed real subalgebra of $C_{\mathbb{R}}(X)$, then $A$ does not have the fixed-point property.*

*Proof.* Since $A$ is a unital uniformly closed real subalgebras of $C_{\mathbb{R}}(X)$, we have $\Omega(A) \neq \emptyset$, $\Omega(A) = \{\varepsilon_x : x \in X\}$ and $\sigma(f) = \{\varphi(f) : \varphi \in \Omega(A)\} = \{f(x) : x \in X\}$ for all $f \in A$ by Lemma 3.2. Therefore, $r(f) = \sup\{|f(x)| : x \in X\} = \|f\|_X$ for all $f \in A$. It follows that $\inf\{r(f) : f \in A, \|f\|_X = 1\} > 0$. Now, let $f, g \in A$ with $|\varphi(f)| \leq |\varphi(g)|$ for all $\varphi \in \Omega(A)$. Then, $|f(x)| \leq |g(x)|$ for each $x \in X$ and so $\|f\|_X \leq \|g\|_X$. Since $A$ is infinite dimensional, we conclude that $A$ does not have the fixed-point property by Theorem 1.2. □

**Proposition 3.4.** *Let $X$ be an infinite compact Hausdorff topological space and let $\tau$ be a topological involution on $X$. Then*

(i) *$C_{\mathbb{R}}(X, \tau)$ is infinite dimensional;*

(ii) *$C(X, \tau)$ is infinite dimensional.*

*Proof.* Choose a sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ such that $x_j \neq x_k$, where $j, k \in \mathbb{N}$ and $j \neq k$. By Urysohn's lemma, there exists a sequence $\{h_n\}_{n=1}^{\infty}$ in $C_{\mathbb{R}}(X)$ such that $h_1 = 1$ and $h_n(x_1) = h_n(\tau(x_1)) = \cdots = h_n(x_{n-1}) = h_n(\tau(x_{n-1})) = 0$, $h_n(x_n) = h_n(\tau(x_n))=1$ for all $n \geq 2$. We define the sequence $\{f_n\}_{n=1}^{\infty}$ in $C_{\mathbb{R}}(X,\tau)$ as the following:

$$f_1 = 1, \qquad f_n = (h_n \circ \tau)h_n \quad (n \in \mathbb{N}, n \geq 2). \tag{3.1}$$

It is easy to see that the set $\{f_1, \ldots, f_n\}$ is a linearly independent set in $C_{\mathbb{R}}(X,\tau)$ for all $n \in \mathbb{N}$. Therefore, $C_{\mathbb{R}}(X,\tau)$ is an infinite dimensional real vector space. (ii) Since $C_{\mathbb{R}}(X,\tau)$ is a real linear subspace of $C(X,\tau)$, we conclude that $C(X,\tau)$ is infinite dimensional by (i). □

**Theorem 3.5.** *Let $X$ be an infinite compact Hausdorff topological space and let $\tau$ be a topological involution on $X$. Then $C_{\mathbb{R}}(X,\tau)$ does not have the fixed-point property.*

*Proof.* By part (i) of Proposition 3.4, $C_{\mathbb{R}}(X,\tau)$ is an infinite dimensional real vector space. On the other hand, $C_{\mathbb{R}}(X,\tau)$ is an inverse-closed unital uniformly closed real subalgebras of $C_{\mathbb{R}}(X)$. Therefore, $C_{\mathbb{R}}(X,\tau)$ does not have the fixed-point property by Theorem 3.3. □

**Corollary 3.6.** *Let $X$ be an infinite compact Hausdorff topological space and let $\tau$ be a topological involution on $X$. Then $C(X,\tau)$ does not have the fixed-point property.*

*Proof.* By Theorem 3.5, $C_{\mathbb{R}}(X,\tau)$ does not have the fixed-point property. Since $(C(X,\tau), \|\cdot\|_X)$ is a real Banach space and $C_{\mathbb{R}}(X,\tau)$ is a uniformly closed real subspace of $C(X,\tau)$, we conclude that $C(X,\tau)$ does not have the fixed-point property. □

We now give a characterization of $\Omega(C(X,\tau))$ as the following.

**Theorem 3.7.** *Let $X$ be an infinite compact Hausdorff topological space and let $\tau$ be a topological involution on $X$.*

(i) *If $x \in \mathrm{Fix}(\tau)$, then $\varepsilon_x \in \Omega(C(X,\tau))$, where $\varepsilon_x$ is evaluation character on $C(X,\tau)$ at $x$.*

(ii) *If $\varphi \in \Omega(C(X,\tau))$, there exists $x \in \mathrm{Fix}(\tau)$ such that $\varphi = \varepsilon_x$.*

(iii) *$\Omega(C(X,\tau)) = \emptyset$ if and only if $\mathrm{Fix}(\tau) = \emptyset$.*

*Proof.* (i) is obvious. To prove (ii), let $\varphi \in \Omega(C(X,\tau))$. Then $\varphi \in \mathrm{Car}(C(X,\tau))$ and so there exists $x \in X$ such that $\varphi = e_x$, where $e_x$ is the complex character on $C(X,\tau)$ at $x$. Since $\varphi(C(X,\tau)) \subseteq \mathbb{R}$, we conclude that $f(x) \in \mathbb{R}$ for all $f \in C(X,\tau)$. Therefore, $f(\tau(x)) = f(x)$ for all $f \in C(X,\tau)$. It follows that $x \in \mathrm{Fix}(\tau)$, because $C(X,\tau)$ separates the points of $X$. Thus $e_x = \varepsilon_x$ and so $\varphi = \varepsilon_x$.

(iii) This follows from (i) and (ii). □

Now by giving two examples, we show that there may be a unital commutative real Banach algebra that fails to have the fixed-point property without satisfying any of the conditions of Theorem 1.2.

*Example 3.8.* Let $X$ be the closed unit interval $[0,1]$ with the standard topology and let $\tau$ be the topological involution on $X$ defined by $\tau(x) = 1 - x$. Since $\mathrm{Fix}(\tau) = \{1/2\}$, we have $\Omega(C(X,\tau)) = \{\varepsilon_{1/2}\}$ by Theorem 3.7. Define the function $f : X \to \mathbb{C}$ by $f(x) = |1/2 - x|$. Clearly, $f \in C(X,\tau)$ and $f(X) = [0,1/2]$. If $\lambda \in (-\infty, 1/2) \cup (1,\infty)$, then $\lambda 1 - f \in G(C(X,\tau))$

and so $\lambda \notin \sigma(f)$. On the other hand, $\lambda 1 - f \notin G(C(X,\tau))$ for all $\lambda \in [1/2, 1]$. Therefore, $\sigma(f) = [1/2, 1]$. But

$$\{\varphi(f) : \varphi \in \Omega(C(X,\tau))\} = \{\varepsilon_{1/2}(f)\} = \left\{ f\left(\frac{1}{2}\right) \right\} = \{0\}. \tag{3.2}$$

Thus $\sigma(f) \neq \{\varphi(f) : \varphi \in \Omega(C(X,\tau))\}$. This shows that $C(X,\tau)$ does not satisfy in the condition (i) of Theorem 1.2, but $C(X,\tau)$ fail to have the fixed-point property by Corollary 3.6.

*Example 3.9.* Let $X = [-2,-1] \cup [1,2]$ with standard topology and let $\tau$ be the topological involution on $X$ defined by $\tau(x) = -x$. Since $\mathrm{Fix}(\tau) = \emptyset$, we have $\Omega(C(X,\tau)) = \emptyset$ by Theorem 3.7. It shows that $C(X,\tau)$ does not satisfy the condition (i) of Theorem 1.2, but $C(X,\tau)$ fails to have the fixed-point property by Corollary 3.6.

We now show that some of the unital closed real subalgebras of $C(X,\tau)$ fails to have the fixed-point property by applying the concept of $\tau$-peak points for these algebras.

**Theorem 3.10.** *Let $X$ be a compact Hausdorff topological space and let $\tau$ be a topological involution on $X$. Suppose $A$ is a unital uniformly closed real subalgebra of $C(X,\tau)$. If $T_0(A,X,\tau)$ contains a limit point of $X$, then $A$ does not have the fixed-point property.*

*Proof.* Let $x_0 \in T_o(A,X,\tau)$ be a limit point of $X$. Then there exists a function $f_0$ in $A$ with $f_0(x_0) = f_0(\tau(x)) = 1$ and $|f_0(x)| < 1$ for all $x \in X \setminus \{x_0, \tau(x_0)\}$, and there exists a net $\{x_\alpha\}_\alpha$ in $X \setminus \{x_0, \tau(x_0)\}$ such that $\lim_\alpha x_\alpha = x_0$ in $X$. We define $E = \{f \in A : \|f\|_X = f(x_0) = 1\}$. Then $E$ is a nonempty bounded closed convex subset of $A$ and $f_0 f \in E$ for all $f \in E$. We define the map $T : E \to E$ by $T(f) = f_0 f$. It is easy to see that $T$ is a nonexpansive mapping on $E$.

We claim that $\mathrm{Fix}(T) = \emptyset$. Suppose $f_1 \in \mathrm{Fix}(T)$. Then $f_0 f_1 = f_1$ and so $f_1(x) = 0$ for all $x \in X \setminus \{x_0, \tau(x_0)\}$. The continuity of $f_1$ in $x_0$ implies that $\lim_\alpha f_1(x_\alpha) = f_1(x_0)$. Therefore, $f_1(x_0) = 0$, contradicting to $f_1 \in E$. Hence, our claim is justified. Consequently, $A$ does not have the fixed-point property. □

*Example 3.11.* Let $\tau$ be the topological involution on $\overline{\mathbb{D}}$ defined by $\tau(z) = \overline{z}$. We denote by $A(\overline{\mathbb{D}},\tau)$ the set all $f \in A(\overline{\mathbb{D}})$ for which $\overline{f} \circ \tau = f$. Then $A(\overline{\mathbb{D}},\tau)$ is a unital uniformly closed real subalgebra of $C(\overline{\mathbb{D}})$ and $A(\overline{\mathbb{D}}) = A(\overline{\mathbb{D}},\tau) \oplus iA(\overline{\mathbb{D}},\tau)$. By Example 2.5,

$$T_0\left(A\left(\overline{\mathbb{D}},\tau\right),\overline{\mathbb{D}},\tau\right) = S_0\left(A\left(\overline{\mathbb{D}}\right),\overline{\mathbb{D}}\right) = \mathbb{T}. \tag{3.3}$$

Therefore, $A(\overline{\mathbb{D}},\tau)$ does not have the fixed-point property by Theorem 3.10.

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