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Research Article

A Note on Geodesically Bounded $\mathbb{R}$-Trees

W. A. Kirk

Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

Correspondence should be addressed to W. A. Kirk, kirk@math.uiowa.edu

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It is proved that a complete geodesically bounded $\mathbb{R}$-tree is the closed convex hull of the set of its extreme points. It is also noted that if X is a closed convex geodesically bounded subset of a complete $\mathbb{R}$-tree Y, and if a nonexpansive mapping $T : X \to Y$ satisfies $\inf\{d(x, T(x)) : x \in X\} = 0$, then T has a fixed point. The latter result fails if T is only continuous.

1. Introduction

Recall that for a metric space (X, d), a geodesic path (or metric segment) joining x and y in X is a mapping c of a closed interval $[0, l]$ into X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for each $t, t' \in [0, l]$. Thus c is an isometry and $d(x, y) = l$. An $\mathbb{R}$-tree (or metric tree) is a metric space X such that:

(i) there is a unique geodesic path (denoted by $[x, y]$) joining each pair of points $x, y \in X$;

(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

From (i) and (ii), it is easy to deduce that

(iii) if $x, y, z \in X$, then $[x, y] \cap [x, z] = [x, w]$ for some $w \in X$.

The concept of an $\mathbb{R}$-tree goes back to a 1977 article of Tits [1]. Complete $\mathbb{R}$-trees posses fascinating geometric and topological properties. Standard examples of $\mathbb{R}$-trees include the "radial" and "river" metrics on $\mathbb{R}^2$. For the radial metric, consider all rays emanating from the origin in $\mathbb{R}^2$. Define the radial distance d_r between $x, y \in \mathbb{R}^2$ to be the usual distance if they are on the same ray; otherwise take

$$d_r(x, y) = d(x, 0) + d(0, y). \tag{1.1}$$

(Here d denotes the usual Euclidean distance and 0 denotes the origin.) For the river metric ρ on $\mathbb{R}^2$, if two points x, and y are on the same vertical line, define $\rho(x,y) = d(x,y)$. Otherwise define $\rho(x,y) = |x_2|+|y_2|+|x_1-y_1|$, where $x = (x_1,x_2)$ and $y = (y_1,y_2)$. More subtle examples of $\mathbb{R}$-trees also exist, for example, the real tree of Dress and Terhalle [2].

It is shown in [3] that $\mathbb{R}$-trees complete are hyperconvex metric spaces (a fact that also follows from Theorem B of [4] and the characterization of [5]). They are also CAT(0) spaces in the sense of Gromov (see, e.g., [6, page 167]). Moreover, complete and geodesically bounded $\mathbb{R}$-trees have the fixed point property for continuous maps. This fact is a consequences of a result of Young [7] (see also [8]), and it suggests that complete geodesically bounded $\mathbb{R}$-trees have properties that one often associates with compactness. The two observations below serve to affirm this.

2. A Krein-Milman Theorem

In [9] Niculescu proved that a nonempty compact convex subset X of a complete CAT(0) space (called a global NPC space in [9]) is the convex hull of the set of all its extreme points. Subsequently, in [10], Borkowski et al. proved (among other things) that compactness is not needed in the special case when X is a complete and bounded $\mathbb{R}$-tree. Here we show that in complete $\mathbb{R}$-trees even the boundedness assumption may be relaxed.

Theorem 2.1. *Let X be a complete and geodesically bounded $\mathbb{R}$-tree. Then X is the convex hull of its set E of extreme points.*

Proof. Let $x \in E$, and let $z \in X \setminus E$. We will show that z lies on a segment joining x to some other element of E. We proceed by transfinite induction. Let Ω denote the set of all countable ordinals, let $z_0 = z$, let $\alpha \in \Omega$, and assume that for all $\beta \in \Omega$ with $\beta < \alpha$, z_β has been defined so that the following condition holds:

(i) $\mu < \gamma < \alpha \Rightarrow z_\mu \in [x, z_\gamma]$, and $z_\gamma \notin E \Rightarrow z_\mu \neq z_\gamma$.

There are two cases.

(1) $\alpha = \beta + 1$. If $z_\beta \in E$, there is nothing to prove because $z = z_0 \in [x, z_\beta]$. Otherwise, there are elements $a, b \in X$ such that z_β lies on the segment $[a,b]$ and $a \neq z_\beta \neq b$. At least one of these points, say a, does not lie on the segment $[z_\beta, x]$. Set $z_\alpha = a$, and observe that z_β lies on the segment $[z_\alpha, x]$.

(2) α is a limit ordinal. Since X is geodesically bounded, it must be the case that $\sum_{\beta<\alpha} d(z_\beta, z_{\beta+1}) < \infty$. This implies that $(z_\beta)_{\beta<\alpha}$ is a Cauchy net. Since X is complete, it must converge to some $z_\alpha \in X$.

Therefore, z_α is defined for all $\alpha \in \Omega$. Since X is geodesically bounded, $\sum_{\beta\in\Omega} d(z_\beta, z_{\beta+1}) < \infty$. But since Ω is uncountable, it is not possible that $d(z_\beta, z_{\beta+1}) > 0$ for each β. Hence this transfinite process must terminate, and $z_\beta = z_{\beta+1}$ for some $\beta \in \Omega$. It now follows from (i) that $z_\beta \in E$ and z lies on the segment $[z_\beta, x]$. □

Remark 2.2. The above proof shows that in fact each point of X is on a segment joining any given extreme point to some other extreme point.

3. A Fixed Point Theorem

It is known that if K is a bounded closed convex subset of a complete CAT(0) space Y, and if $f : K \to Y$ is a nonexpansive mapping for which

$$\inf\{d(x, f(x)) : x \in K\} = 0, \tag{3.1}$$

then f has a fixed point (see [11, Theorem 21]; also [12, Corollary 3.8]). This fact carries over to $\mathbb{R}$-trees since $\mathbb{R}$-trees are also CAT(0) spaces. However, we note here that if Y is an $\mathbb{R}$-tree, then again boundedness of K can be replaced by the assumption that K is merely geodesically bounded. In fact, we prove the following. (In the following theorem, we assume T is nonexpansive relative to the Hausdorff metric on the bounded nonempty closed subsets of Y.)

Theorem 3.1. *Suppose X is a closed convex and geodesically bounded subset of a complete $\mathbb{R}$-tree Y, and suppose $T : X \to 2^Y$ is a nonexpansive mapping taking values in the family of nonempty bounded closed convex subsets of Y. Suppose also that* $\inf\{\text{dist}(x, T(x)) : x \in X\} = 0$. *Then there is a point $x \in X$ for which $x \in T(x)$.*

We will need the following result in the proof of Theorem 3.1. (See [13, 14] for more general set-valued versions of this theorem.)

Theorem 3.2. *Suppose X is a closed convex geodesically bounded subset of a complete $\mathbb{R}$-tree Y and suppose $f : X \to Y$ is continuous. Then either f has a fixed point or there exists a point $z \in X$ such that*

$$0 < d(z, f(z)) = \inf\{d(x, f(z)) : x \in X\}. \tag{3.2}$$

Proof of Theorem 3.1. Since complete $\mathbb{R}$-trees are hyperconvex, by Corollary 1 of [15] the selection $f : X \to Y$ defined by taking $f(x)$ to be the point of $T(x)$ which is nearest to x for each $x \in X$ is a nonexpansive single-valued mapping. Now assume f does not have a fixed point. Then by Theorem 3.2 there exists $z \in X$ such that

$$0 < d(z, f(z)) = \inf\{d(x, f(z)) : x \in X\}. \tag{3.3}$$

We assert that $d(x, f(x)) \geq d(z, f(z))$ for each $x \in X$. Indeed let $x \in X$. By (iii) there exists $w \in Y$ such that $[z, f(z)] \cap [z, x] = [z, w]$. But since X is convex $[z, x] \subseteq X$, so $w \in [z, x]$ implies $w \in X$. Also $w \in [z, f(z)]$, so it follows from (3.3) that $w = z$. Thus $[z, f(z)] \cap [z, x] = \{z\}$, and the segment $[x, f(z)]$ must pass through z. Therefore,

$$\begin{aligned} d(x, z) + d(z, f(z)) &= d(x, f(z)) \\ &\leq d(x, f(x)) + d(f(x), f(z)) \\ &\leq d(x, f(x)) + d(x, z). \end{aligned} \tag{3.4}$$

Thus $\inf\{d(x, f(x)) : x \in X\} \geq d(z, f(z)) > 0$ – a contradiction. Therefore, there exists $x \in X$ such that $x = f(x) \in T(x)$. □

Corollary 3.3. *Suppose X is a closed convex and geodesically bounded subset of a complete $\mathbb{R}$-tree Y, and suppose $f : X \to Y$ is a nonexpansive mapping for which* $\inf\{d(x, f(x)) : x \in X\} = 0$. *Then f has a fixed point.*

Example 3.4. In view of the fact that continuous self-maps of $X \to X$ have fixed points, it is natural to ask whether Corollary 3.3 holds for continuous mappings. The answer is no, even when X is bounded. Let Y be the Euclidean plane $\mathbb{R}^2$ with the radial metric. Let $\{e_n\}$ be a sequence of distinct points on the unit circle, and let $X = \cup_{n=1}^{\infty}[e_n, 0]$. We now define a continuous fixed-point free map $f : X \to Y$ for which $\inf\{d(x, f(x)) : x \in X\} = 0$. First move each point of the segment $[0, e_1]$ to the right onto a segment $[e_1, b]$ where $b \neq e_1$ and $[e_1, b]$ is on the ray which extends $[0, e_1]$. (Thus $f([0, e_1]) = [e_1, b]$.) For each $n \geq 2$, let a_n denote the point on the segment $[e_n, 0]$ which has distance $1/n$ from e_n. It is now clearly possible to construct a continuous (even lipschitzian) fixed point-free map f (a shift) of the segment $[e_n, 0]$ onto the segment $[a_n, e_1], n \geq 2$, for which $f(e_n) = a_n$. Thus $d(e_n, f(e_n)) = 1/n$ for all n.

Remark 3.5. Corollary 3.3 for bounded X is also a consequence of Theorem 6 of [15].

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