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Research Article

Some Fixed Point Properties of Self-Maps Constructed by Switched Sets of Primary Self-Maps on Normed Linear Spaces

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This paper is devoted to the investigation of the existence of fixed points in a normed linear space X endowed with a norm $\|\cdot\|$ for self-maps f from $T\times X$ to X which are constructed from a given class of so-called primary self- maps being also from $T\times X$ to X. The construction of the self-maps of interest is performed via a so-called switching rule which is a piecewise-constant map from a set T to some finite subset of the positive integers or a sequence map which domain in some discrete subset of T.

1. Introduction

This paper is devoted to the investigation of the existence of fixed points in a normed linear space X with norm $\|\cdot\|$ for self-maps from $T\times X$ to X which are constructed from a given class of so-called primary self-maps from $T\times X$ to X. The construction of the maps $f: T\times X\to X$ of interest is performed via a so-called switching rule $\sigma: T\to\overline{N}\subset\mathbf{Z}_+$ which is a piecewise-constant map from a set T to some finite subset of the positive integers. The potential discontinuity points of such a self-map in a discrete subset $\mathrm{ST}\subset S$ are the so-called switching points at which a new primary self-map in a class P_M is activated to construct the self-map $f: T\times X\to X$ of interest, each of those self-maps depends also on some given switching rule $\sigma:\mathrm{ST}:=\{t_i\}\subset T\to\overline{N}:=\{1,2,\dots,N\}\subset\mathbf{Z}_+$.

In particular, $f(t)=f_i(t)\equiv f_{\sigma(t)}(t)\in P_M$, for all $t\in[t_j,t_{j+1})$ where $t_j,t_{j+1}>t_j$ are two consecutive elements in the sequence ST generated by the switching rule $\sigma: T\to\overline{N}$ such that $\sigma(t)=i\in\overline{N}$, for all $t\in[t_j,t_{j+1})$, for all $t_j,t_{j+1}(>t_j)\in\mathrm{ST}$ provided that there is no $\mathrm{ST}\ni t\in(t_j,t_{j+1})$.

The class of primary self-maps P_M used to generate the self-map f from $T \times X$ to X might contain itself, in the most general case, a class of contractive primary self-maps from $T \times X$ to X, a class of large contractive self-maps from $T \times X$ to X, a class of nonexpansive self-maps from $T \times X$ to X, as well as a class of expansive self-maps from $T \times X$ to X. The problem is easily extendable to the case when the switching rule is a discrete sequence of domain in a discrete set of T and of codomain again being the set of nonnegative integers. The study is of particular interest for its potential application to the study of eventual fixed points in the state-trajectory solution of either continuous-time or discrete-time switched dynamic systems [1–6], constructed from a fixed set of primary parameterizations.

It is well known that contractive self-maps in normed linear spaces and in metric spaces possess a fixed point which is unique in Banach spaces $(X, \|\cdot\|)$ and in complete metric spaces (X, d), [7–10]. Under additional boundedness-type conditions, a large contractive self-map f from $T \times X$ to X which generates uniformly bounded iterates for any number of iterations still possesses a unique fixed point in a complete metric space (X, d) (or in a Banach space $(X, \|\cdot\|)$ [6]. Some nonexpansive self-maps as well as certain expansive self-maps also possess fixed points (see, e.g., [11, 12]). On the other hand, pseudocontractive self-maps and semicontinuous compact maps in Banach spaces can also possess fixed points [13, 14]. *Those features motivate in this paper the choice of the given class of primary self-maps for this investigation*. It is also taken into account as motivation that unforced linear time-invariant dynamic systems are exponentially stable to the origin if the matrix of dynamics is a stability matrix. In the case that such a matrix has some pair of complex conjugate eigenvalues then the solution is bounded and the solution trajectory may oscillate, and if there is some eigenvalue with positive real part (i.e., within the instability region), then the state trajectory solution is unbounded. The two first situations can be discussed using the fixed pointy formalism, [1, 2, 6]. *Thus, it is of interest to have some extended formalism to investigate time-varying switched dynamic systems obtained under switched linear primary parameterizations not all of them being necessarily stable and then associated with asymptotic contractive self-maps. It is also of interest the basic investigation on the existence of fixed points of discontinuous self-maps which are identical to some self-maps in a prescribed class over each connected subset of T being generated from a switching rule $\sigma : T \to \overline{N}$ of switching points in the sequence* ST $\subset T$. The switching rule which governs the definition of the self-map f from $T \times X$ to X from the primary class of self-maps from $T \times X$ to X is shown to be crucial for $f : T \times X \to X$ to possess a fixed point. X is assumed to be a normed linear space which is not necessarily either a Banach space or assumed to have some uniform structure [15].

Three examples of the formalism are provided two of them being referred to the use of arbitrary primary self-maps on X while the third one refers to linear time-varying dynamic systems subject to simultaneous parameterization switching and impulsive controls.

2. Notation

$\mathbf{R}_{0+} := \mathbf{R}_{+} \cup \{0\}$ where $\mathbf{R}_{+} := \{z \in \mathbf{R} : z > 0\}$.
$\mathbf{Z}_{0+} := \mathbf{Z}_{+} \cup \{0\}$ where $\mathbf{Z}_{+} := \{z \in \mathbf{Z} : z > 0\}$.

$C^{(q)}(\mathbf{R}_{0+}, \mathbf{R}^n)$, respectively, $C^{(q)}(\mathbf{R}_{0+}, \mathbf{C}^n)$ are the sets of n-real, respectively, n-complex vector functions of domain $\mathbf{R}_{0+}$ of class q, that is, q times n-real, respectively, n-complex continuously differentiable everywhere in its definition domain $\mathbf{R}_{0+}$.

$\mathrm{PC}^{(q)}(\mathbf{R}_{0+}, \mathbf{R}^n)$, respectively, $PC^{(q)}(\mathbf{R}_{0+}, \mathbf{C}^n)$ are the sets of n-real or, respectively, n-complex vector functions of domain $\mathbf{R}_{0+}$ of class $(q-1)$ with its qth time derivative being necessarily everywhere piecewise-continuous in $\mathbf{R}_{0+}$. Thus, $\mathrm{PC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^n)$, respectively, $\mathrm{PC}^{(0)}(\mathbf{R}_{0+}, \mathbf{C}^n)$ are the sets of all n-real, respectively, n-complex vector functions being piecewise—continuous everywhere in $\mathbf{R}_{0+}$.

If $t \in T$ is a first-class discontinuity point of $f : T \times X \to X$, then $f(t, x(t))$, simplifying the customary notation $f(t^-, x(t))$, denotes the lower limit of f at t and $f(t^+, x(t))$ denotes the right limit of f at t.

$\mu(A)$ denotes the Lebesgue measure of a subset A of $\mathbf{R}$.

$\mu(D) = \sum_i d_i \delta_i$ is the discrete measure of the sequence $D := \{\delta_i \in A\} \subset A$ (A being Lebesgue-measurable) defined via the Kronecker-delta defined by $\delta_i = 1$ if $d_i \in D$ and $\delta_i = 0$, otherwise.

The symbols $\wedge$, $\vee$, and $\neg$ mean logic conjunction, disjunction, and negation, respectively, and $\overline{N} := \{1, 2, \ldots, N\}$ is a subset of the positive integers numbers.

To establish the general framework for the formulation, consider a set T which is a proper or improper subset of $\mathbf{R}_{0+}$ (it could also be a subset of $\mathbf{Z}_{0+}$) and X being a linear normed space endowed with the norm $\|\cdot\|$.

Consider also a class of so-called primary (i.e., auxiliary) self-maps $f_i : T \times X \to X$, for all $i \in \overline{N}$ to be used to build the class of maps under study through Fixed Point Theory which is defined by

$$M = M(\sigma, P_M)$$
$$:= \left\{ f : T \times X \longrightarrow X : f(t, x(t)) = f_{\sigma(t)}(t, x(t));\ \forall t \in T,\ \forall x \in X,\ \sigma : T \longrightarrow \overline{N},\ f_i \in P_M,\ i \in \overline{N} \right\}. \tag{2.1}$$

The class P_M of primary self-maps of M, which generate the class M, is defined by $P_M := \{f_i : T_i(\subseteq T) \times X \to X : i \in \overline{N}\}$ for some proper or improper subset T_i of T, for all $i \in \overline{N}$. In many applications $T_i = T$; for all $i \in \overline{N}$. However, the possibility of taking different subsets T_i of T remains open within this formulation, for instance, for cases when the various N parameterizations (or some of them) are not defined or, simply, not allowed to switch arbitrarily but each with its own switching restrictions. The map $\sigma : T \to \overline{N} := \{1, 2, \ldots, N\}$ ($N \geq 1$) is the so-called *switching rule* being an integer-valued switching function if T is a subset of $\mathbf{R}_{0+}$ of nonzero measure and a nonnegative integer-valued switching sequence of real domain if T is a subset of $\mathbf{Z}_{0+}$. Each element of T for which the initial condition is fixed is also axiomatically considered a switching point for any arbitrary switching rule. That intuitively means that a switching rule involves a switch $\sigma(0) \equiv \sigma(t_1) \in \overline{N}$. The discrete *switching sequence* $\mathrm{ST} = \mathrm{ST}(\sigma, P_M)$ of switching points $T_i \in \mathrm{ST}$, indexed for $i \in c_{\mathrm{ST}} \subset \mathbf{Z}_{0+}$ and which is generated by the switching rule $\sigma : T \to \overline{N}$ [3–5], is defined as follows,

$$\mathrm{ST} := \left\{ t_i = t_i(\sigma) \text{ fulfill } S_{p\sigma};\ \forall i \in \overline{c_{\mathrm{ST}}} \right\}_{i=1}^{c_{\mathrm{ST}}} \subset T \tag{2.2}$$

$c_{\mathrm{ST}} := \mathrm{card}(\mathrm{ST}) \leq \chi_0$ is the number of switching points which is either finite or infinity numerable. $S_{p\sigma}$ (the so-called *Switching Property* of the switching rule σ): the switching sequence ST is defined according to the $S_{p\sigma}$-Property:

$$t_1 \in \mathrm{ST} \wedge t_i = \min\Big(T \ni t > t_{i-1}(\in \mathrm{ST}) : \sigma(t) \neq \sigma(t_{i-1}) = j(t_{i-1}) \in \overline{N} \Big); \quad \forall i(> 1) \in \mathbf{Z}_+. \tag{2.3}$$

Note that each switching-dependent integer $j(t_i)$ is some integer in the set $\overline{N} = \{1, 2, \ldots, N\}$ which defines the configuration within the set of N configurations which remains active within the interval $[t_i, t_{i+1})$ for two consecutive switching points $t_i, t_{i+1} \in \mathrm{ST}$ so that for any three consecutive $t_{i-1}, t_i, t_{i+1} \in \mathrm{ST}$,

$$\sigma(t_{i-1}) \neq \sigma(t) = \sigma(t_i) = j(t_i) \neq \sigma(t_{i+1}); \quad \forall t \in [t_i, t_{i+1}). \tag{2.4}$$

The discrete switching sequence $\mathrm{ST} = \mathrm{ST}(\sigma, P_M)$ may also be viewed as a discrete strictly ordered set of real or integer nonnegative numbers of first element (i.e., first switching point) t_1 in the sense that $\neg\, \exists t \in \mathrm{ST}$ such that $t < t_1$. Also, $\neg\, \exists t \in \mathrm{ST}$ such that $t \in (t_j, t_{j+1})$; for all $t_j, t_{j+1} \in \mathrm{ST}$. Note that the switching sequence is a strictly ordered sequence of real numbers which depends on the switching rule $\sigma : T \to \overline{N}$. If the switching rule is a piecewise continuous function, that is, T is a countable union of real intervals then its discontinuity point happen at points of ST since the function $f \in M$ changes to another primary function being distinct from the previous one. It is being supposed through the manuscript that $T_i \equiv T$ (being either $\mathbf{R}_{0+}$ or $\mathbf{Z}_{0+}$) and $\mathrm{ST} \subset \mathbf{R}_{0+}$.

The above framework is useful for examples of composed functions, multi-parameterizations of dynamic systems, and so forth involving mappings with some kind of switching. In particular, it is useful to investigate the stability and asymptotic stability of certain dynamic systems which switched parameterizations. Examples of problems situations adjusting to the above description are [3–5].

(a) X, a subset of $\mathbf{R}^n$, is the state space of a continuous-time linear time-varying unforced dynamic system described for $T = \mathbf{R}_{0+}$ by

$$\dot{x}(t) = A(t)x(t), \quad x(0) \in \mathbf{R}^n. \tag{2.5}$$

$A : \mathbf{R}_{0+} \to \mathbf{R}^{n\times n}$ is a real matrix function of piecewise constant entries whose image is $A(t) = A_{\sigma(t)} \in \{A_1, A_2, \ldots, A_N\}$, where $A_i \in \mathbf{R}^{n\times n}$, for all $i \in \overline{N}$ with $A_i \neq A_j$ if $i \neq j$.

Then, $f(t, x(t)) = f_{\sigma(t)}(t, x(t)) = e^{A_{\sigma(t)}(t-t_j)}x(t_j)$, where $e^{A_{\sigma(t)}(t-t_j)}$ is a fundamental matrix of (2.5), for all $t \in T$ on the interval $[t_j, t_{j+1})$, for all $t_j, t_{j+1}(> t_j) \in \mathrm{ST} \subset T$ and σ is a piecewise constant function taking values in the integer set $\overline{N}$ which changes value at each $t \in \mathrm{ST}$ so that $\sigma(t) = i \in \overline{N}$, for all $t \in [t_j, t_{j+1})$. The unique state trajectory-solution for each given initial conditions $x(0) \in X$ is $x : \mathbf{R}_{0+} \to \mathbf{R}^n$ with $x \in C^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^n)$ being differentiable everywhere in S with $\dot{x} \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^n)$ having first-class discontinuities in ST.

The eventual discontinuity points of the piecewise continuous switching function $\sigma : T \to \overline{N}$, that is,

$$D_T := \left\{ t \in T : \overline{N} \ni j = \sigma(t^+) \neq \sigma(t) = i(\neq j) \in \overline{N} \right\} \tag{2.6}$$

are the discontinuity points of the state trajectory time-derivative $\dot{x} : \mathbf{R}_{0+} \to \mathbf{R}^n$ since they generate discontinuities in at least one entry of $A(t)$, for all $t \in D_T$.

(b) X, a subset of $\mathbf{R}^n$, is the state space of a discrete-time linear time-varying unforced dynamic system described for $T = \mathbf{Z}_{0+}$ by

$$x_{j+1} = A(j)x_j, \quad x_0 \in \mathbf{R}^n;\ \forall j \in \mathbf{Z}_{0+}. \tag{2.7}$$

$A : \mathbf{Z}_{0+} \to \mathbf{R}^{n\times n}$ is a real matrix sequence whose image is $A(j) = A_{\sigma(j)} \in \{A_1, A_2, \ldots, A_N\}$, where $A_i \in \mathbf{R}^{n\times n}$, for all $i \in \overline{N}$.

The changes of value of the discrete switching function $\sigma : T \to \overline{N}$ at a sample $j \in \mathbf{Z}_{0+}$ imply changes of values in at least one entry of $A(j)$.

The class P_M of primary self-maps of M is a union $P_M = P_{Mc} \cup P_{M\ell c} \cup P_{Mne} \cup P_{Me} \cup P_{Mr}$ of disjoint sets, of respective disjoint indexing sets $I_{Mc},\ I_{M\ell c},\ I_{Mne},\ I_{Me}$, and I_{Mr} whose sum of respective cardinals equalizes $\overline{N}$ that is

$$\overline{N} = \text{card } I_{Mc} + \text{card } I_{M\ell c} + \text{card } I_{Mne} + \text{card } I_{Me} + \text{card } I_{Mr} \tag{2.8}$$

with at least one of them being nonempty. Note from (2.1) that any function in M is constructed by taking a function in its primary class P_M for each interval $[t_i, t_{i+1})$; for all $t_i,\ t_{i+1} \in \text{ST}(\sigma)$ and some $\sigma : T \to \overline{N}$. In this way, $f \in M$, with $M = M(\sigma, P_M)$, is identical to some $f_{j_i} \in PM$, for some $j_i \in \overline{N}$ within each real interval $[t_i, t_{i+1})$ for each pair $t_i, t_{i+1} \in \text{ST}(\sigma)$ with the switching rule $\sigma : T \to \overline{N}$ being defined in such a way that $\sigma(t_i) = j_i \in \overline{N}$ and $\sigma(t_{i+1}) = j_{i+1}(\neq j_i) \in \overline{N}$. Thus, $f(t) = f_{j_i}(t)$; for all $t \in [t_i, t_{i+1})$. The sets $P_{Mc},\ P_{M\ell c},\ P_{Mne},\ P_{Me}$ and P_{Mr} considered in this section of the manuscript are defined in the sequel.

(1) The class P_{Mc} of *strictly contractive primary self-maps* from $T \times X$ to X is defined as follows.

Definition 2.1. $f : T \times X \to X$ belongs to the class P_{Mc} of strictly contractive primary self-maps from $T \times X$ to X if for any $x, y \in X$, $t(> t_1) \in T$, for all $t_1 \in \text{ST}$, the following inequality holds:

$$\|f(t, x(t)) - f(t, y(t))\| \leq k(t, t_1)\|x(t_1) - y(t_1)\|, \tag{2.9}$$

where $k \in \text{PC}^{(0)}(\mathbf{R}_{0+}, [0, \overline{k}) \cap \mathbf{R}_{0+})$ for some real constants $\overline{k} \in [0, 1)$ where $x(t)$ denotes the value in X of $x : T \times X \to X$ at $t \in T$.

Note from Definition 2.1 that for each $f_i : T \times X \to X$ in P_{Mc} it exists a function $k_i \in \text{PC}^{(0)}(\mathbf{R}_{0+}, [0, \overline{k}_i) \cap \mathbf{R}_{0+})$ for some real constant $\overline{k}_i \in [0, 1)$, for all $i \in I_{Mc} \subset \overline{N}$. It is assumed that t_1 is the first element in ST. The terminology "strictly contractive self-map" is used for the standard contractions referred to in the Banach contraction principle [6, 16] as a counterpart of the alternative terminology used for large contractions [6], here introduced below. Since strict contractions are also large contractions, since contractive self-maps are also nonexpansive ones, and since the sets P_M and $\overline{N}$ are investigated as corresponding unions of disjoint components, the class of large contractions (resp., that of nonexpansive self-maps) are characterized as members of a set which excludes the strict contractions (resp., as members of a class which excludes any contractive self-map).

Remark 2.2. Note that the dependence of the functions $k_i : \mathbf{R}_{0+} \to \mathbf{R}$ on the switching rule $\sigma : T \to \overline{N}$ is a generalization which often occurs in practical cases. For instance, if a dynamic system (2.5) changes its parameterization at time $t = t_\ell \in$ ST from a stability matrix A_j to another one A_i, $j, i(\neq j) \in \overline{N}$, then $A(t) = A_{\sigma(t)} = A_i$; some $i \in \overline{N}$ so that $\sigma(t) = i$, for all $t \in [t_\ell - \varepsilon, t_\ell)$ what leads to

$$x(t, x(t)) \equiv f(t, x(t)) \equiv f_i(t, x(t)) = e^{A_i(t-t_\ell)}x(t_\ell), \quad \forall t \in [t_\ell, t_\ell + \varepsilon) \tag{2.10}$$

for some sufficiently small real constant $\varepsilon > 0$ if $t_\ell \in \mathrm{ST}$. Since $e^{A_i(t-t_\ell)}$ is also a fundamental matrix of a time-invariant linear dynamic system (i.e., the active system parameterization starting at time t_ℓ) for initial vector state $x(t_i)$, then it is of exponential order so that $\|e^{A_i(t-t_\ell)}\| \leq k_i(t, t_\ell(\sigma)) := K_i e^{-\rho_i(t-t_\ell)} \leq \overline{k}_i < 1$ for $t \geq t_\ell + \mu$ for some real constant μ and some real constants $K_i \geq 1$ (being norm-dependent) and $\rho_i \in \mathbf{R}_+$; $i \in \overline{N}$.

(2) The class $P_{M\ell c}$ of *nonstrictly contractive large-contractive primary self-maps* from $T \times X$ to X is now defined as follows.

Definition 2.3. $f : T \times (X \setminus P_{Mc}) \to X$ belongs to the class $P_{M\ell c}$ of nonstrictly contractive large-contractive primary self-maps from $T \times X$ to X if it fulfils the joint condition $C_1 \wedge C_2$ for any $x, y \in X$, $t(> t_1) \in T$, for all $t_1 \in \mathrm{ST}$, where Conditions C_1 and C_2 are defined as follows.

(a) $f : T \times (X \setminus P_{Mc}) \to X$ satisfies Condition $C_1 \Longleftrightarrow$

$$\|f(t, x(t)) - f(t, y(t))\| < \|x(t_1) - y(t_1)\|; \quad \forall x, y(\neq x) \in X,\ \forall t(> t_1),\ \forall t_1 \in T. \tag{2.11}$$

(b) $f : T \times (X \setminus P_{Mc}) \to X$ satisfies Condition $C_2 \Leftrightarrow$ for all $\varepsilon \in \mathbf{R}_+$, $\exists \delta = \delta(\varepsilon)(\in \mathbf{R}_{0+}) < 1$ such that

$$\begin{aligned}&[(x, y) \in X \wedge \|x - y\| > \varepsilon] \\ &\quad \Longrightarrow \|f(t, x(t)) - f(t, y(t))\| \leq \delta(t, t_1)\|x(t_1) - y(t_1)\|; \quad \forall t(> t_1),\ \forall t_1 \in \mathrm{ST},\end{aligned} \tag{2.12}$$

where $\delta \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, [0, \overline{\delta}) \cap \mathbf{R}_{0+})$ for some real constant $\overline{\delta} \in [0, 1)$.

Definition 2.3 applies to all f_i in $P_M \setminus P_{Mc}$ with $\delta_i \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, [0, \overline{\delta}_i) \cap \mathbf{R}_{0+})$ for some real constant $\overline{\delta}_i \in [0, 1)$, for all $i \in I_{M\ell c} \subset \overline{N}$.

(3) The class P_{Mne} of *noncontractive nonexpansive primary self-maps* from $T \times X$ to X is defined as follows.

Definition 2.4. $f : T \times (X \setminus (P_{Mc} \cup P_{M\ell c})) \to X$ belongs to the class P_{Mne} of noncontractive nonexpansive primary self-maps from $T \times X$ to X if for any $x, y \in X$, $t(> t_1) \in T$, $t_1 \in \mathrm{ST}$, the following inequality holds:

$$\|f(t, x(t)) - f(t, y(t))\| \leq \|x(t_1) - y(t_1)\|; \quad \forall x, y \in X,\ \forall t(> t_1),\ t_1 \in \mathrm{ST}. \tag{2.13}$$

Note that the above inequality is fulfilled by any $f_i \in P_M \setminus (P_{Mc} \cup P_{M\ell c})$, for all $i \in I_{Mne} \subset \overline{N}$.

(4) The class P_{Me} of *expansive upper-bounded primary self-maps* from $T \times X$ to X *satisfying a global Lipschitz condition and an additional bounding property* is now defined as follows.

Definition 2.5. $f : T \times (X \setminus (P_{Mc} \cup P_{M\ell c})) \to X$ belongs to the class P_{Me} of expansive upper-bounded primary self-maps from $T \times X$ to X satisfying a global Lipschitz condition and an additional bounding property if for any $x, y \in X$, $t(> t_1) \in T$, $t_1 \in \mathrm{ST}$, the following inequalities hold:

$$\begin{aligned} \left\| f(t, x(t)) - f(t, y(t)) \right\| &\geq \beta(t, t_1) \left\| x(t_1) - y(t_1) \right\| \\ \left\| f(t, x(t)) - f(t, y(t)) \right\| &\leq k(t, t_1) \left\| x(t_1) - y(t_1) \right\| \end{aligned} \quad \forall x, y \in X,\ \forall t(> t_1),\ \forall t_1 \in T, \tag{2.14}$$

where $\beta \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, [\overline{\beta}, \infty) \cap \mathbf{R}_+)$, and $k \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, [\overline{\beta}, \overline{k}) \cap \mathbf{R}_+)$ for some real constants $\overline{\beta} \in (1, \infty)$ and $\overline{k} \in [\overline{\beta}, \infty) \subset \mathbf{R}_+$.

The above upper-bounding condition has been assumed to facilitate the subsequent exposition. Note that there exists $\beta_i \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, [\overline{\beta}_i, \infty) \cap \mathbf{R}_+)$ and $k_i \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, [\overline{\beta}_i, \overline{k}_i) \cap \mathbf{R}_+)$ for some finite real constants $\overline{\beta}_i \in (1, \infty)$ and $\overline{k}_i \in [\overline{\beta}_i, \infty) \subset \mathbf{R}_+$ for any f_i in P_{Me}, for all $i \in I_{Me} \subset \overline{N}$ and that $(P_{Mc} \cup P_{M\ell c} \cup P_{Mne}) \not\subset P_{Me}$. Note also that this requirement is not very restrictive since it is fulfilled, for instance, by compact self-maps from $T \times X$ to X, also by bounded self-maps from $T \times X$ to X and, even, by unbounded piecewise-continuous maps of positive exponential order. That means that any state-trajectory solution generated from bounded initial conditions in globally exponentially stable continuous-time linear dynamic systems fulfil such a property for finite time in T. Many nonlinear dynamic systems whose state-trajectory solutions do not exhibit finite escape times also possess this property.

(5) The class P_{Mr} of *neither nonexpansive nor expansive primary self-maps* from $T \times X$ to X *satisfying a global Lipschitz condition* is defined in the sequel. This class includes, for instance, primary self-maps which are expansive and globally Lipschitzian and nonexpansive over alternate subsets of T of finite measure and primary functions which are, for instance, asymptotically nonexpansive while being expansive for proper (then finite) subsets $T_f \subset T = \mathbf{R}_{0+}$.

Definition 2.6. $f : T \times X \to X$ belongs to the class P_{Mr} of neither nonexpansive nor expansive primary self-maps from $T \times X$ to X satisfying a global Lipschitz condition if it satisfies the following inequality:

$$\left\| f(t, x(t)) - f(t, y(t)) \right\| \leq k(t, t_1) \left\| x(t_1) - y(t_1) \right\|; \quad \forall x, y \in X,\ \forall t(> t_1),\ \forall t_1 \in \mathrm{ST}, \tag{2.15}$$

where $k \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+} \cap \mathbf{R})$ is uniformly upper-bounded by some finite real constant $\overline{k}\ (> 1) \in \mathbf{R}_+$.

There exist $k_i \in \mathrm{PC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+} \cap \mathbf{R})$ being uniformly upper-bounded by some finite real constants $\overline{k}_i(> 1) \in \mathbf{R}_+$, for each f_i in P_{Mr}, for all $i \in I_{Mr} \subset \overline{N}$. Note that $P_{Mr} := P_M \setminus (P_{Mc} \cup P_{M\ell c} \cup P_{Mne} \cup P_{Me})$.

Since $P_M = P_{Mc} \cup P_{M\ell c} \cup P_{Mne} \cup P_{Me} \cup P_{Mr}$, and all the self-mappings f_i from $T \times X$ to X in all the component subsets satisfy a global Lipschitz condition, the following result is direct via recursion for each pair of consecutive elements of $\mathrm{ST}(\sigma)$ from the definition of the class of primary functions P_M for M as a union of disjoint classes and Definition 2.1 and Definitions 2.3–2.6.

Lemma 2.7. *$f(\in M): T\times X \to X$, being generated from a class of primary self-maps P_M of M in X by any switching rule $\sigma : T \to \overline{N}$, possesses the two following properties:*

(i)

$$\begin{aligned}\left\| f\left(t_j, x(t_j)\right) - f\left(t_j, y(t_j)\right)\right\| &\leq k(t,t_1)\left\|x(t_1)-y(t_1)\right\| \\ &= \prod_{i=1}^{j-1}\left[k_{\sigma(t_i)}(t_{i+1},t_i)\right]\left\|x(t_1)-y(t_1)\right\|; \quad \forall x,y\in X,\ \forall t_i,t_j(>t_i)\in \mathrm{ST}(\sigma)(i\in \mathbf{Z}_+),\end{aligned} \tag{2.16}$$

(ii)

$$\begin{aligned}\left\| f\left(t, x(t)\right) - f\left(t, y(t)\right)\right\| &\leq k(t,t_1)\left\|x(t_1)-y(t_1)\right\| \\ &= k_{\sigma(t_j)}(t,t_j)\prod_{i=1}^{j-1}\left[k_{\sigma(t_i)}(t_{i+1},t_i)\right]\left\|x(t_1)-y(t_1)\right\|; \quad \forall x,y\in X,\ \forall t_i,t_j(>t_i)\in \mathrm{ST}(\sigma)\ (i\in \mathbf{Z}_+), \\ &\left[\forall (T\ni) t\in \left[t_j,t_{j+1}\right) \ \textit{if}\ t_{j+1}\in \mathrm{ST}\right] \vee \left[\forall (T\ni t)\in \left(t_j,\infty\right)\cap T \ \textit{if}\ \neg\exists\, t(>t_j)\in \mathrm{ST}(\sigma)\right].\end{aligned} \tag{2.17}$$

Proof. Any function $f: T\times X \to X$ in $M = M(\sigma, P_M)$ is constructed for each interval $[t_i, t_{i+1})$ by taking a function in its primary class P_M, for all $t_i, t_{i+1} \in \mathrm{ST}(\sigma)$ and some $\sigma : T \to \overline{N}$. In this way, $f \in M$, with $M = M(\sigma, P_M)$, is identical to some $f_{j_i} \in PM$, for some $j_i \in \overline{N}$ within each real interval $[t_i, t_{i+1})$ for each pair $t_i, t_{i+1} \in \mathrm{ST}(\sigma)$ with the switching rule $\sigma : T \to \overline{N}$ being defined in such a way that $\sigma(t_i) = j_i \in \overline{N}$ and $\sigma(t_{i+1}) = j_{i+1}(\neq j_i) \in \overline{N}$. Thus, $f(t) = f_{j_i}(t)$, for all $t \in [t_i, t_{i+1})$. The primary class consists of a disjoint union of classes defined in Definition 2.1 and Definitions 2.3–2.6 which all have upper-bounding functions of the form of the first inequality in (2.16). The function $k(t, t_1)$ can be directly expanded for any $t \in [t_j, t_{j+1})$ and $t_j \in \mathrm{ST}(\sigma)$, provided that some next consecutive $t_{j+1} \in \mathrm{ST}(\sigma)$ exists, and for any $t \in [t_j, \infty)$, otherwise (i.e., if switching ends, i.e., $\mathrm{ST}(\sigma)$ has finite cardinal, and the last switching point is t_j) via recursion from the preceding interswitching intervals $[t_j, t_{i+1})$; $t_i \in \mathrm{ST}(\sigma)$, for all $i \in \overline{j}$. □

Remark 2.8. The last logic proposition for the validity of Property (ii) of Lemma 2.7 means that $k_{\sigma(t_j)}(\cdot, t_j) \in \mathrm{PC}^{(0)}((t_{j+1}, t_j) \cap T, \mathbf{R}_{0+})$ if $\exists t_{j+1} \in \mathrm{ST}$ and, otherwise, that is, if $\{t \in \mathrm{ST} : t > t_j\} = \emptyset$ so that t_j is the last element in ST (with the physical sense that the switching process generated by the switch rule $\sigma : T \to \overline{N}$ stops in finite time), then $k_{\sigma(t_j)}(\cdot, t_j) \in \mathrm{PC}^{(0)}(S \setminus [t_1, t_j] \cap S, \mathbf{R}_{0+})$.

Lemma 2.7(ii) leads to the following direct consequent result.

Lemma 2.9. *Assume that T is Lebesgue-mesasurable with $\mu(T) = \infty$. Then, the self-map $f(\in M) : X \to X$, being generated from a class of primary self-maps P_M from $T \times X$ to X, which satisfies the given assumptions, from the switching rule $\sigma : T \to \overline{N}$, is strictly contractive if there exists some finite $T_0 \in T$ with $[t_1, t_1 + T_0) \subset T$ such that*

$$\left(k_{\sigma(t_j)}(t_1 + T_0, t_j) \prod_{i=1}^{j-1} \left[k_{\sigma(t_i)}(t_{i+1}, t_i) \right] \right) \leq \gamma < 1 \tag{2.18}$$

with $t_i \in \mathrm{ST}(\sigma)$, for all $i \in \overline{j}$, $j := \{\max i \in \mathbf{Z}_+ : \ t_i(\leq t_1 + T_0) \in \mathrm{ST}(\sigma)\}$. The condition (2.18) *implies*

$$\exists \lim_{\mathbf{Z}_+ \ni \ell \to \infty} \left(k_{\sigma(t_j)}(t_1 + \ell T_0, t_j) \prod_{i=1}^{j_\ell - 1} \left[k_{\sigma(t_i)}(t_{i+1}, t_i) \right] \right) = 0, \tag{2.19}$$

where $t_i \in \mathrm{ST}(\sigma)$, for all $i \in \overline{j}$, $j_\ell := \{\max i \in \mathbf{Z}_+ : \ t_i(\leq t_1 + \ell T_0) \in \mathrm{ST}(\sigma)\}$, and also

$$\exists \lim_{T \ni t \to \infty} \left(k_{\sigma(t_j)}(t, t_j) \prod_{i=1}^{j(t)-1} \left[k_{\sigma(t_i)}(t_{i+1}, t_i) \right] \right) = 0, \tag{2.20}$$

where $t_i \in \mathrm{ST}(\sigma)$, for all $i \in \overline{j}$, $j(t) := \{\max i \in \mathbf{Z}_+ : t_i(\leq t_1 + t) \in \mathrm{ST}(\sigma)\}$.

The self-map $f(\in M) : X \to X$ is still strictly contractive if (2.18) *holds by replacing $t_1 \to t^*$ for some finite $t^* \in T$ even if $t^* \notin ST(\sigma)$.*

Proof. It follows that if (2.18) holds, then proceeding inductively

$$\begin{aligned} &\| f(t_1 + \ell T_0, x) - f(t_1 + \ell T_0, y) \| \\ &\quad \leq \gamma \| f(t_1 + (\ell - 1)T_0, x(t_1 + (\ell - 1)T_0)) - f(t_1 + (\ell - 1)T_0, y(t_1 + (\ell - 1)T_0)) \| \\ &\quad \leq \gamma^\ell \| x(t_1) - y(t_1) \|; \quad \forall x, y \in X, \ \forall \ell \in \mathbf{Z}_+ \end{aligned} \tag{2.21}$$

so that one gets

$$\begin{aligned} &\exists \lim_{\mathbf{Z}_+ \ni \ell \to \infty} \| f(t_1 + \ell T_0, x(t_1 + \ell T_0)) - f(t_1 + \ell T_0, y(t_1 + \ell T_0)) \| \\ &\quad \leq \left(\lim_{\mathbf{Z}_{0+} \ni \ell \to \infty} \gamma^\ell \right) \| x(t_1) - y(t_1) \| = 0; \quad \forall x, y \in X, \\ &\exists \lim_{\mathbf{R}_+ \ni t \to \infty} \| f(t, x(t)) - f(t, y(t)) \| = \limsup_{\mathbf{R}_+ \ni t \to \infty} \| f(t, x(t)) - f(t, y(t)) \| \\ &\quad = \lim_{\mathbf{Z}_+ \ni \ell \to \infty} \| f(t_1 + \ell T_0, x(t_1 + \ell T_0)) - f(t_1 + \ell T_0, y(t_1 + \ell T_0)) \| = 0; \quad \forall x, y \in X, \end{aligned} \tag{2.22}$$

so that $f(\in M) : X \to X$, being generated from a classof primary self-maps P_M from $T \times X$ to X from the switching rule $\sigma : T \to \overline{N}$, is strictly contractive from Banach's contraction principle. Equations (2.19) and (2.20) are a direct consequence of the fact that (2.18) implies directly

$$\begin{aligned} &\limsup_{T\ni t\to\infty}\left(k_{\sigma(t_j)}(t,t_j)\prod_{i=1}^{j(t)-1}\left[k_{\sigma(t_i)}(t_{i+1},t_i)\right]\right) \\ &\quad= \liminf_{T\ni t\to\infty}\left(k_{\sigma(t_j)}(t,t_j)\prod_{i=1}^{j(t)-1}\left[k_{\sigma(t_i)}(t_{i+1},t_i)\right]\right) = 0. \end{aligned} \tag{2.23}$$

The second part of the result is obvious since the finite interval $[t_1, t^*)$ of T may be removed from the discussion by still keeping the strict contraction property. □

To discuss some practical situations that guarantee the fulfilment of the condition (2.18), let us define the following subsets $\mathrm{ST}(a,b,\sigma,P_M)$ of $\mathrm{ST}(\sigma,P_M)$, as a union of disjoint components associated with some proper or improper subset of the class of primary functions being active to build $f(\in M) : X \to X$ each on some nonempty subset of the subset $[a,b)$ of T:

$$\mathrm{ST}(a,b,\sigma,P_M) = \bigcup_{i\in\overline{N}(a,\,b)} \mathrm{ST}_i(a,b,\sigma,P_M) = \bigcup_{i\in\overline{N}} \mathrm{ST}_i(a,b,\sigma,P_M) = \mathrm{ST}(\sigma,P_M) \setminus \left(\overline{[a,b)} \cap T\right) \tag{2.24}$$

with

$$\begin{aligned} \overline{N}(a,b,\sigma,P_M) &= \bigcup_{i\in\overline{N}} N_i(a,b,\sigma,P_M) \\ &= N_{Mc}(a,b,\sigma,P_M) \cup N_{M\ell c}(a,b,\sigma,P_M) \\ &\quad \cup N_{Mne}(a,b,\sigma,P_M) \cup N_{Me}((a,b,\sigma,P_M)\sigma) \cup N_{Mr}(a,b,\sigma,P_M) \\ &\subseteq \overline{N} = \bigcup_{i\in\overline{N}} N_i = N_{Mc} \cup N_{M\ell c} \cup N_{Mne} \cup N_{Me} \cup N_{Mr}, \end{aligned} \tag{2.25}$$

being such that $\sigma : T \setminus [a,b) \to \overline{N}(a,b)$:

$$\left[\emptyset \neq \mathrm{ST}_i(a,b,\sigma,P_M) \Longleftrightarrow \left[\exists t_j \in \mathrm{ST}(\sigma,P_M) : \sigma(t) = \sigma(t_j) = i \in \overline{N} \text{ for some } [t_j, t_j+\varepsilon) \subset [a,b) \subset T\right]\right] \tag{2.26}$$

$$\begin{aligned} &\Longrightarrow N_i(a,b) \neq \emptyset \quad \text{for any given } i \in \overline{N}, \\ &\overline{N}(a,b) := \left\{i \in \overline{N} : \mathrm{ST}_i(a,b,\sigma,P_M) \neq \emptyset\right\} \\ &= \text{card } I_{Mc}(a,b,\sigma,P_M) + \text{card } I_{M\ell c}(a,b,\sigma,P_M) + \text{card } I_{Mne}(a,b,\sigma,P_M) \\ &\quad + \text{card } I_{Me}(a,b,\sigma,P_M) + \text{card } I_{Mr}(a,b,\sigma,P_M), \end{aligned} \tag{2.27}$$

where $\mathrm{ST}_i(a,b,\sigma,P_M)\neq\emptyset$ for any $i\in\overline{N}$ such that $f(t)=f_i(t)$ for any proper or improper subset of (a,b) and

$$\begin{aligned}
N &= \text{card } \overline{N} = \text{card } I_{Mc} + \text{card } I_{M\ell c} + \text{card } I_{Mne} + \text{card } I_{Me} + \text{card } I_{Mr},\\
\overline{N}_\sigma &= I_{Mc\sigma}\cup I_{M\ell c\sigma}\cup I_{Mne\sigma}\cup I_{Me\sigma}\cup I_{Mr\sigma}\subseteq\overline{N},\\
N_\sigma &= \text{card } I_{Mc\sigma} + \text{card } I_{M\ell c\sigma} + \text{card } I_{Mne\sigma} + \text{card } I_{Me\sigma} + \text{card } I_{Mr\sigma}\leq\overline{N},\\
T &= T_{Mc}(\sigma,P_M)\cup T_{M\ell c}(\sigma,P_M)\cup T_{Mne}(\sigma,P_M)\cup T_{Me}(\sigma,P_M)\cup T_{Mr}(\sigma,P_M)\\
&= \left(\bigcup_{i\in I_{Mc\sigma}} T_{Mc_i}(\sigma,P_M)\right)\cup\left(\bigcup_{i\in I_{M\ell c}} T_{M\ell c_i}(\sigma,P_M)\right)\cup\left(\bigcup_{i\in I_{Mne\sigma}} T_{Mne_i}(\sigma,P_M)\right)\\
&\quad\cup\left(\bigcup_{i\in I_{Me\sigma}} T_{Me_i}(\sigma,P_M)\right)\cup\left(\bigcup_{i\in I_{Mr\sigma}} T_{Mr_i}(\sigma,P_M)\right),\\
\mathrm{ST}(\sigma,P_M) &= \mathrm{ST}_{Mc}(\sigma,P_M)\cup\mathrm{ST}_{M\ell c}(\sigma,P_M)\cup\mathrm{ST}_{Mne}(\sigma,P_M)\cup\mathrm{ST}_{Me}(\sigma,P_M)\cup\mathrm{ST}_{Mr}(\sigma,P_M)\\
&= \left(\bigcup_{i\in I_{Mc\sigma}} \mathrm{ST}_{Mc_i}(\sigma,P_M)\right)\cup\left(\bigcup_{i\in I_{M\ell c\sigma}} \mathrm{ST}_{M\ell c_i}(\sigma,P_M)\right)\\
&\quad\cup\left(\bigcup_{i\in I_{Mne\sigma}} \mathrm{ST}_{Mne_i}((\sigma,P_M))\right)\cup\left(\bigcup_{i\in I_{Me\sigma}} \mathrm{ST}_{Me_i}(\sigma,P_M)\right)\cup\left(\bigcup_{i\in I_{Mr\sigma}} \mathrm{ST}_{Mr_i}(\sigma,P_M)\right)\\
&\quad\subset\left(\bigcup_{i\in I_{Mc\sigma}} T_{Mc_i}(\sigma,P_M)\right)\cup\left(\bigcup_{i\in I_{M\ell c\sigma}} T_{M\ell c_i}(\sigma,P_M)\right)\\
&\quad\cup\left(\bigcup_{i\in I_{Mne\sigma}} T_{Mne_i}(\sigma,P_M)\right)\cup\left(\bigcup_{i\in I_{Me\sigma}} T_{Me_i}(\sigma,P_M)\right)\cup\left(\bigcup_{i\in I_{Mr\sigma}} T_{Mr_i}(\sigma,P_M)\right)\\
&= T_{Mc}(\sigma,P_M)\cup T_{M\ell c}(\sigma,P_M)\cup T_{Mne}(\sigma,P_M)\cup T_{Me}(\sigma,P_M)\cup T_{Mr}(\sigma,P_M)\subset T.
\end{aligned}\tag{2.28}$$

Note that N depends on P_M, but not on the σ, since a particular switching rule can remove some primary self-maps from generating a particular self-map from $T\times X$ to X. Note also that the above decompositions are also extendable "mutatis-mutandis" to any subsets $T(a,b)\subset T$ and $\mathrm{ST}(a,b)\subset\mathrm{ST}$.

The interpretation of card $I_{Mc}(a,b,\sigma,P_M)\leq$ card $I_{Mc}\leq N$ is the number of strictly contractive primary self-maps from $T\times X$ to X; that is, members of P_M, being active on any of the subsets of finite measure of $(a,b)\subset T$. Note that the sets $\mathrm{ST}(a,b,\sigma,P_M)$ and $\overline{N}(a,b,\sigma,P_M)$ are, respectively, the set of switching points used to build $f: X\to X$ from the primary class of functions P_M on $[a,b)\subset T$ by the switching rule and $\sigma: T\setminus[a,b)\to\overline{N}$ which restrict its image (since its domain is restricted) to some $\overline{N}(a,b,\sigma,P_M)\subseteq\overline{N}$ which is the subset of active primary functions for some nonempty subset of $[a,b)\subset T$. The interpretations of the disjoint decompositions of, in general, nonconnected subsets, of the sets T and ST in (2.28) are related to. Note that for any given $\sigma : T \to \overline{N}$, one has by construction

$$\begin{aligned} T &= \bigcup_{i\in\chi}[a_i, b_i) = \bigcup_{t_i\in \mathrm{ST}}[t_i, t_{i+1}) = \bigcup_i\left(\bigcup_{j\in\overline{N}}\left(\bigcup_{\ell\in N_j(a_i,b_i)}[a_{ij\ell}, b_{ij\ell})\right)\right) \\ &= \bigcup_i\left(\left(\bigcup_{j\in N_{Mc}}[a_{ij}, b_{ij})\right)\cup\left(\bigcup_{j\in N_{Mc}}[a_{ij}, b_{ij})\right)\right. \\ &\qquad \left.\cup\left(\bigcup_{j\in N_{Mc}}[a_{ij}, b_{ij})\right)\cup\left(\bigcup_{j\in N_{Mc}}[a_{ij}, b_{ij})\right)\cup\left(\bigcup_{j\in N_{Mc}}[a_{ij}, b_{ij})\right)\right), \end{aligned} \tag{2.29}$$

$$\supseteq \bigcup_i\left(\bigcup_{j\in N_{Mc}}[a_{ij}, b_{ij})\right) = \bigcup_{i\in \mathrm{ST}_j}\left(\bigcup_{j\in N_{Mc}}[t_{ij}, t_{i+1j})\right), \tag{2.30}$$

where the sets $[a_i, b_i)$ are connected disjoint subsets of T since $a_i = t_i(\sigma) \in \mathrm{ST}(\sigma)$ and $a_{i+1} = b_i = t_{i+1}(\sigma) \in \mathrm{ST}(\sigma)$ are consecutive switching points under the switching rule $\sigma : T \to \overline{N}$, $\chi = \mathbf{Z}_+$ if card ST $= \chi_0$ (i.e., infinity numerable if $\sigma : T \to \overline{N}$ generates infinitely many switching points), and $\chi =$ card ST $< \chi_0$ with $a_{\text{card ST}}(\in \mathbf{Z}_+) < \infty$ and $b_{\text{card ST}}(\in \mathbf{Z}_+) = \infty$, otherwise. However, the subsets $[a_{ij\ell}, b_{ij\ell})$ and $[a_{ij}, b_{ij})$ of $[a_i, b_i)$ are not necessarily connected for any given switching rule $\sigma : T \to \overline{N}$ since it can make a particular primary function at disjoint subsets of T active to build $f \in M(\sigma)$.

Note that the set T also includes any subset being obtained by replacing N_{Mc} in (2.30) with any other of the disjoint components of $\overline{N}$. Note also that Condition (2.18) needs the presence of a strictly contractive self-map as a member of the primary functions for the given switching rule as it is discussed in the subsequent result.

Lemma 2.10. *A necessary condition for the strict contractive condition* (2.18) *to hold is that switching rule* $\sigma : T \to \overline{N}$ *which generates* $f(\in M) : X \to X$ *has at least one self-map* $f_i \in P_{Mc} \neq \emptyset$ *being a member of the primary self-maps* P_M.

Proof. Proceed by contradiction by assuming that the generalized condition obtained from (2.18) for any finite $t_1 \in T$ holds with $P_{Mc} = \emptyset$. If $P_{Mc} = P_{M\ell c} = \emptyset$ then $P_M = P_{Mne} \cup P_{Me} \cup P_{Mr}$ and then (2.18) does not hold and, instead, we have,

$$\left(k_{\sigma(t_j)}(t_1 + T_0, t_j)\prod_{i=1}^{j-1}\left[k_{\sigma(t_i)}(t_{i+1}, t_i)\right]\right) \geq 1; \quad \forall T_0 \in T \tag{2.31}$$

so that

$$\liminf_{T\ni t\to\infty}\left(k_{\sigma(t_j)}(t, t_j)\prod_{i=1}^{j(t)-1}\left[k_{\sigma(t_i)}(t_{i+1}, t_i)\right]\right) \geq 1 > 0, \tag{2.32}$$

and, then, either

$$\lim_{T\ni t\to\infty}\left(k_{\sigma(t_j)}(t,t_j)\prod_{i=1}^{j(t)-1}\left[k_{\sigma(t_i)}(t_{i+1},t_i)\right]\right)\geq 1>0 \tag{2.33}$$

provided that it exists, or if the above limit does not exist, then $f(\in M):\ X\to X$ is not strictly contractive. A second possibility is $P_{Mc}=\emptyset$ and $P_{M\ell c}\neq\emptyset$ so that $P_M=P_{M_{\ell c}}\cup P_{Mne}\cup P_{Me}\cup P_{Mr}$. Then, for any such a $\sigma:T\to\overline{N}$

$$\lim_{T\ni t\to\infty}\inf\left(k_{\sigma(t_j)}(t,t_j)\prod_{i=1}^{j(t)-1}\left[k_{\sigma(t_i)}(t_{i+1},t_i)\right]\right)\geq 0. \tag{2.34}$$

Now, if $\lim_{T\ni t\to\infty}\inf(k_{\sigma(t_j)}(t,t_j)^j\prod_{i=1}^{(t)-1}[k_{\sigma(t_i)}(t_{i+1},t_i)])>0$, then the same contradiction as above follows. Otherwise, if $\lim_{T\ni t\to\infty}\inf(k_{\sigma(t_j)}(t,t_j)^j\prod_{i=1}^{(t)-1}[k_{\sigma(t_i)}(t_{i+1},t_i)])=0$, then either $(k_{\sigma(t_j)}(t_1+T_0,t_j)\prod_{i=1}^{j-1}[k_{\sigma(t_i)}(t_{i+1},t_i)])<1$ fails, for all $T_0\in T$, or it holds for some $T_0\in T$ but there is no $0\leq\gamma<1$ such that (2.18) holds since it has been assumed that P_M satisfies $P_{Mc}\cap P_{M\ell c}=\emptyset$. □

Remark 2.11. Concerning Lemma 2.10, note that fixed points can still exist for $f(\in M):X\to X$ even if $P_{Mc}=\emptyset$. A such a situation can happen, for instance, if a self-map $f\in M$ of X is built with a switching rule $\sigma:T\to\overline{N}$ involving above primary functions in a class $P_M=P_{M\ell c}\neq\emptyset$. It is well known that a large contraction self-map in a Banach space can possess fixed points (see, e.g., [6]). However, Lemma 2.10 proves that if $P_{Mc}=\emptyset$, the strict contraction condition (2.18) does not hold for any $T_0\in T$.

The mains result of this section follows.

Theorem 2.12. *Assume that T is Lebesgue-mesasurable with $\mu(T)=\infty$ and consider a switching rule $\sigma:\ T\to\overline{N}$ which generates $f(\in M):T\times X\to X$, with $M=M(\sigma,P_M)$ defined by the class P_M of primary self-maps from $T\times X$ to X satisfying the given assumptions. The following properties hold.*

(i) *$f(\in M):\ T\times X\to X$ has a fixed point under Lemma 2.9 if $\mu(T_{Mc}(\sigma))=\infty$ and $\mu(T\setminus T_{Mc}(\sigma))<\infty$.*

(ii) *$f(\in M):T\times X\to X$ has a fixed point if $\mu(T_{M\ell c}(\sigma))=\infty$ and $\mu\ (T\setminus T_{M\ell c}(\sigma))<\infty$ and, furthermore, $\exists M(<\infty)\in\mathbf{R}_+$ such that the boundedness condition*

$$\left\|f(t^*+\ell T_0,x(t^*+\ell T_0))-x(t^*)\right\|\leq M;\quad \forall x\in X,\ \forall\ell\in\mathbf{Z}_{0+} \tag{2.35}$$

holds in the case that $f(\in M):\ T\times X\to X$ is not Lipschitzian.

(iii) *$f(\in M):\ T\times X\to X$ has a fixed point if $\mu(T_{Mc}(\sigma)\cup T_{M\ell c}(\sigma))=\infty$ and $\mu(T\setminus(T_{Mc}(\sigma)\cup T_{M\ell c}(\sigma)))<\infty$ and, furthermore, $\exists M(<\infty)\in\mathbf{R}_+$ such that the boundedness condition*

$$\left\|f(t^*+\ell T_0,x(t^*+\ell T_0))-x(t^*)\right\|\leq M;\quad \forall x\in X,\ \forall\ell\in\mathbf{Z}_{0+} \tag{2.36}$$

holds in the case that $f(\in M): X \to X$ *is not Lipschitzian; then* $f(\in M): T \times X \to X$ *has a fixed point.*

Proof. (i) Proceed by contradiction by taking into account the set inclusion properties (2.30) and by assuming that the extended form of (2.18) in Lemma 2.9 to any replacement $t_1 \to t^*$ (finite) does not hold. Since $\mu(T \setminus T_{Mc}(\sigma)) < \infty$ and $\mu(T_{Mc}(\sigma)) = \infty$, there exists a finite $t^* \in \mathrm{ST}(\sigma)$ such that $f(t, x(t)) = f_{\sigma(t)}(t, x(t))$ for $\sigma(t) \in \overline{N}_{\sigma(t)} \equiv I_{Mc\sigma(t)}$; for all $t(\geq t^* \in \mathrm{ST}(\sigma)) \in T$, and for $T_0 = t^* + t'$ (some finite $t' \in T$), it follows by complete induction that

$$\begin{aligned} &\left\| f(t^* + \ell T_0, x(t^* + \ell T_0)) - f(t^* + \ell T_0, y(t^* + \ell T_0))\right\| \\ &\quad \leq \gamma \left\| f(t^* + (\ell-1)T_0, x(t^* + (\ell-1)T_0)) - f(t^* + (\ell-1)T_0, y(t^* + (\ell-1)T_0))\right\| \\ &\quad \leq \gamma^{\ell} \left\| x(t^*) - y(t^*)\right\|; \quad \forall x, y \in X,\ \forall \ell \in \mathbf{Z}_+ \end{aligned} \tag{2.37}$$

for some $\gamma \in [0,1)$ since $\sigma(t) \in I_{Mc\sigma(t)}$, for all $t(\geq t^*) \in T$ which is a contradiction. Then, Lemma 2.9 holds with $T_0 = t^* + t'$ for the valid replacement $\mathrm{ST}(\sigma) \ni t_1 \to t^*(< \infty) \in T$ so that $f(\in M): T \times X \to X$ is strictly contractive and has a fixed point.

(ii) Since $f: X \to X$ is a large contraction, the real sequence below $\{g_\ell\}_{\ell \in \mathbf{Z}_+}$ is bounded monotonically strictly decreasing for any pair $x^* = x(t^*)$, $y^* = y(t^*) \neq x^* \in X$ since

$$\begin{aligned} g_\ell = g_\ell(t^*, T_0, x^*, y^*) &:= \left\| f(t^* + \ell T_0, x(t^* + \ell T_0)) - f(t^* + \ell T_0, y(t^* + \ell T_0))\right\| \\ &< g_{\ell-1} = \left\| f(t^* + (\ell-1)T_0, x(t^* + (\ell-1)T_0)) - f(t^* + (\ell-1)T_0, y(t^* + (\ell-1)T_0))\right\| \quad \forall \ell \in \mathbf{Z}_+. \end{aligned} \tag{2.38}$$

Thus, for any prefixed $\varepsilon_0 \in \mathbf{R}_+, \exists \delta_0 = \delta_0(\varepsilon_0) \in \mathbf{R}_+$ and $\ell_0 = \ell_0(\varepsilon_0) \in \mathbf{Z}_0$ such that

$$\begin{aligned} &\left\| f(t^* + \ell T_0, x(t^* + \ell T_0)) - f(t^* + \ell T_0, y(t^* + \ell T_0))\right\| \\ &\quad \leq L \left\| x(t^* + \ell T_0) - y(t^* + \ell T_0)\right\| \leq \varepsilon_0, \quad \forall \ell \geq \ell_0 \end{aligned} \tag{2.39}$$

provided that $g_\ell = \|x(t^* + \ell T_0) - y(t^* + \ell T_0)\|$ is sufficiently close to zero satisfying $g_\ell \leq g_{\ell_0} \leq \delta_0 \leq \varepsilon_0 / L$ where L is the Lipschitz constant of the self-map f from $T \times X$ to X, provided to be Lipschitzian, and any given $x^*, y^* \neq x^* \in X$. If $g_\ell \to 0$ as $\ell \to \infty$, then the self-map f from $T \times X$ to X has a fixed point in X and the result is proven. Furthermore, the error sequence $\{g_\ell(x^*, y^*) = \|g_\ell(x^*) - g_\ell(y^*)\|\}_{\ell \in \mathbf{Z}_{0+}}$ which maps $X \times X$ in $\mathbf{R}_{0+}$ is a Cauchy sequence and has a zero fixed point in $\mathbf{R}_{0+}$. Also, $\{\|g_\ell(x^*)\|\}_{\ell \in \mathbf{Z}_{0+}}$ is a Cauchy sequence with a limit in X which is a fixed point of $f(\in M): T \times X \to X$. Otherwise, assume that $\exists \varepsilon \in (0, \delta_0] \in \mathbf{R}_+$ such that $\{g_\ell\}_{\ell \in \mathbf{Z}_{0+}}$ is not a Cauchy sequence or, if so, it does not converge to zero while satisfying $g_\ell \geq \varepsilon$; for all $\ell \geq \ell_0$. Then, $\exists \delta = \delta(\varepsilon) \in \mathbf{R}_+$ such that

$$\varepsilon \leq g_{\ell+j} \leq \delta^j g_\ell \leq \frac{\delta^j \varepsilon_0}{L} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty;\ \forall \ell \geq \ell_0,\ \forall j \in \mathbf{Z}_{0+}. \tag{2.40}$$

Then, there is a finite sufficiently large $j_0 = j_0(\varepsilon) \in \mathbf{Z}_+$ such that the above result is a contradiction for all $j(\geq j_0) \in \mathbf{Z}_+$. Then, the self-map the Lipschitzian self-map f from

$T \times X \rightarrow X$ to X has a fixed point in X. Now, if $f(\in M) : T \times X \rightarrow X$ is not Lipschitzian, but it satisfies the given boundedness alternative condition, then

$$\varepsilon \leq g_{\ell+j} \leq \delta^j g_\ell \leq \delta^j M \longrightarrow 0 \quad \text{as } j \longrightarrow \infty; \ \forall \ell \geq \ell_0, \ \forall j \in \mathbf{Z}_{0+}, \tag{2.41}$$

which is again a contradiction for for all $j(\geq j_0) \in \mathbf{Z}_+$ such that $\varepsilon > \delta^{j_0} M$ concluding that $f(\in M) : X \rightarrow X$ has a fixed point.(iii)Since $\mu(T_{Mc}(\sigma) \cup T_{M\ell c}(\sigma)) = \infty$, then either

(1) $\mu(T_{Mc}(\sigma)) = \infty \wedge \mu(T_{M\ell c}(\sigma)) < \infty$ and then the proof follows from Property (i), or

(2) $\mu(T_{Mc}(\sigma)) < \infty \wedge \mu(T_{M\ell c}(\sigma)) = \infty$ and then the proof follows from Property (ii), or

(3) $\mu(T_{Mc}(\sigma)) = \infty \wedge \mu(T_{M\ell c}(\sigma)) = \infty$ and then the proof also follows from Property (ii).

□

Remark 2.13 (An interpretation of Theorem 2.12). Theorem 2.12 extends the Banach contraction principle of strictly contractive maps and the fixed point properties of large contractions to the case when the self-map is defined via switching-based combination of contractive primary self-maps as follows. If $\mu(T_{Mc}(\sigma)) = \infty$, then the self-map f from $T \times X$ to X is built with a set of strictly contractive self-maps from $T \times X$ to X on a subset of its domain of infinity Lebesgue measure. If, furthermore, $\mu(T \backslash T_{Mc}(\sigma)) < \infty$, then there is a finite $t^* \in T$ (e.g., a finite time instant if $T = \mathbf{R}_{0+}$) such that all the primary self-maps used to build the self-map f are strictly contractive for all $t \geq t^*$. A conclusion is that the self-map f from $T \times X$ to X is strictly contractive so that it has a fixed point. A close reasoning leads to the conclusion that the self-map f from $T \times X$ to X is a large contraction if $\mu(T_{M\ell c}(\sigma)) = \infty$ and $\mu(T \setminus T_{M\ell c}(\sigma)) < \infty$ or if $\mu(T_{Mc}(\sigma) \cup T_{M\ell c}(\sigma)) = \infty$ and $\mu(T \setminus (T_{Mc}(\sigma) \cup T_{M\ell c}(\sigma))) < \infty$. In all those cases, the subset of the domain of T where each primary self-map is activated by the switching rule $\sigma : T \rightarrow \overline{N}$ are not necessarily connected. In the case when $\mu(T_{Mc}(\sigma) \cup T_{M\ell c}(\sigma)) = \infty$ (Theorem 2.12(iii)) the joint subset of the domain of T where the primary self-maps building f are either strictly contractive or large contractive has infinite measure, what leads to the same conclusions about the existence of fixed points as in the two former cases, although it is not necessarily connected. A counterpart of Theorem 2.12 can be formulated for the case when T is discrete countable sequence (say, e.g., $\mathbf{Z}_{0+}$). In this case, the finite Lebesgue measures referred to in Theorem 2.12 are replaced by the cardinals of finite subsequences of T and the infinity Lebesgue measures are replaced by χ_0 (i.e., sequences of countable infinity many nonnegative integers). The usefulness of the extended results ofTheorem 2.12 relies on its use on the stability properties of switched dynamic systems with asymptotic convergence of their state trajectory solutions to a fixed point. They also rely, to a more basic level, on the fixed point properties of maps which are not necessarily Lipschitz-continuous but being built with Lipschitz-continuous functions through a switching process.

Remark 2.14. In the case that $\mu(T_{Mc}(\sigma) \cup T_{M\ell c}(\sigma)) = \mu(T \setminus (T_{Mc}(\sigma) \cup T_{M\ell c}(\sigma))) = \infty$ the existence of a fixed point is not guaranteed under the given conditions. Some "ad hoc" conditions for the existence of fixed points are given in the next section.

Remark 2.15. If a fixed point exists for a particular self-map $f(\in M) : T \times X \rightarrow X$ built with a class P_M of N primary self-maps switched according to a rule $\sigma : T \rightarrow \overline{N}$ under the sufficiency-type conditions of Theorem 2.12, then such a fixed point is unique in the Banach space X. The result is directly extendable to complete metric spaces (X, d) what allows to consider a parallel formulation for the case that the domain of $f(\in M) : T \times X \to X$, that is ST, is an infinite sequence on nonnegative real numbers. If the formalism is applied on a compact metric space, then it is not required for large contractions the fulfilment of the boundedness condition of Theorem 2.12(ii)-(iii) from Edelstein fixed point theorem [8] which can be proven using the Meir-Keeler theorem [9] as observed in [10].

The following property from $T \times X$ to X being a fixed point space relative to a set of maps [16] is obvious under the conditions which guarantee the existence of at least a fixed point in X for any f in the class $M = M(\sigma, P_M)$.

Assertion 2.16. The Banach space X is a fixed point space relative to the class of self-maps $M = M(\sigma, P_M)$ which satisfies any of the properties of Theorem 2.12.

Example 2.17. Consider $X = \mathbf{R}$ and a set $\overline{N}$ of $N \in \mathbf{Z}_+$ primary self-maps P_M all having the structure

$$f_i(t, x(t)) = k_{0i} - k_i(t - t_j)x(t_i), \quad x(t_j) \in X \tag{2.42}$$

for some given $k_{0i} \in \mathbf{R}$, $k_i : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ (being piecewise continuous and uniformly bounded on $\mathbf{R}_{0+}$); $i \in \overline{N}$, for all $t \in T$. Note that the primary self-maps depend on a switching rule $\sigma : \mathbf{R}_{0+} \to \overline{N}$ which is a piecewise constant real function defined as $\sigma(t) = \sigma(t_i)$, for all $t \in [t_i, t_{i+1}) \subset T$ where $\{t_j = t_j(\sigma)\} \equiv \mathrm{ST} \subset T \equiv \mathbf{R}_{0+}$ is the switching sequence. The class M consists of the piecewise continuous self-maps f on $\mathbf{R}$ built as follows:

$$f(t) \equiv x(t) = f_{\sigma(t)}(t, x(t)) = k_{0\sigma(t)} - k_{\sigma(t)}(t - t_j)x(t_i); \quad \forall t \in [t_j, t_{j+1}),\ \forall t_j \in \mathrm{ST}. \tag{2.43}$$

Note that $\sigma(t) = \sigma(t_j) \in \overline{N} = \{1, 2, \ldots, N\}$, for all $t \in [t_j, t_{j+1})$, for all $t_j \in \mathrm{ST} \subset \mathbf{R}_{0+}$.

If $\sigma(t) = i \in \overline{N}$, for all $t \in [t_j, t_{j+1})$, then

$$\begin{aligned} &\left|x(t_{j+1}) - y(t_{j+1})\right| \\ &\quad \le \left|k_{\sigma(t)}(t_{j+1} - t_j)\right|\left|x(t_j) - y(t_j)\right| \\ &\quad \le \prod_{\sigma(t_i)\in \mathrm{ST}(t_j)} \left|k_{\sigma(t_i)}(t_{i+1} - t_i)\right|\left|x(t_1) - y(t_1)\right|; \quad \forall t_j \in \mathrm{ST},\ \forall t_i \in \mathrm{ST}(t_j) := \mathrm{ST} \setminus \{t_1, \ldots, t_{j-1}\}. \end{aligned} \tag{2.44}$$

Theorem 2.12 is applied as follows. If all the primary self-maps are strictly contractive for the switching rule $\sigma : \mathbf{R}_{0+} \to \overline{N}$, that is, $k_i(t_{i+1} - t_i) < 1$, for all $i \in \overline{N}$, that is, a sufficiently fast switching cadence is used, it follows that the real self-map possesses a fixed point from Banach contraction principle for any switching rule. It is not difficult to see that the property also holds if the primary self-maps are large contractions or there are mixed large and strict contractions used by the switching rule to build the self-map f. Each particular fixed point may depend on the switching rule and $X = \mathbf{R}$ is a fixed point space for the class M of self-maps built in such a way. The property may be generalized by using also primary nonexpansive maps with associated $k(\cdot)$ having unity absolute upper-bound provided that the switching rule involves nonexpansive self-maps being always used on subsets of $\mathbf{R}_{0+}$ of finite Lebesgue measure or, otherwise, combined with contractive primary self-maps both (alternately) run on sets of infinity Lebesgue measure. If P_M has also nonexpansive self maps, then a fixed point does not exist for all the class M of constructed self-maps so that $X = \mathbf{R}$ is not a fixed point space. Finally, if expansive primary self-maps are also considered, then a fixed point still exists for switching rules satisfying a condition like that of Lemma 2.9 for some finite real T_0 according to the following constraint:

$$\begin{aligned} &\left|x(T_1+T_0)-y(T_1+T_0)\right| \\ &\quad\leq \prod_{\sigma(t_i)\in \mathrm{ST}^g(T_1,T_0)} \left|k_{\sigma(t_i)}(t_{i+1}-t_i)\right|\left|x(T_1)-y(T_1)\right| \leq \gamma\left|x(T_1)-y(T_1)\right| \end{aligned} \tag{2.45}$$

for some real constant $\gamma \in [0,1)$, where

$$\begin{gathered} T_1 = T_1(\ell, T_0) = t_1 + (\ell-1)T_0; \quad \forall \ell \in \mathbf{Z}_+, \\ \mathrm{ST}^g(T_1,T_0) := \left\{T_1, T_1(\ell,T_0), t_1^g(\ell\ ,T_0),\ t_2^g(\ell,T_0), t_m^g(\ell,T_0)\right\}, \\ t_1^g(\ell,T_0) = \{\min t_i : t_i \in \mathrm{ST}(\sigma),\ t_i > T_1(\ell,T_0)\}, \\ t_m^g(\ell,T_0) = \{\max\ t_i : t_i \in \mathrm{ST}(\sigma),\ t_i < T_1(\ell,T_0)+T_0\}. \end{gathered} \tag{2.46}$$

The above results are directly extendable to the linear space $X \equiv \mathbf{R}^n$ endowed with any Euclidean norm.

Example 2.18. If the replacements $k_i(t-t_i) \to \overline{\gamma}(T_i) := 1/k_i(t-t_i) \leq \gamma_i, i \in \overline{N}$, are performed in (2.42) to define the class of primary self-maps, that is, a sufficiently slow switching cadence is used, then, the residence interval taken for the next switch after the switch at t_i makes to strictly decrease the function $k_i(t-t_i)$. As a result, the existence of a fixed point is guaranteed by any switching rule involving at least a primary self-map generating $f : T \times \mathbf{R} \to \mathbf{R}$ or sufficiently large residence intervals compared with the times where the remaining primary self-maps are used to generate $f : T \times \mathbf{R} \to \mathbf{R}$.

3. Some Extensions

It is now assumed that the class $P_M = P_{Mc} \cup P_{M\ell c} \cup P_{Mne} \cup P_{Me} \cup P_{Mr}$ of primary self-maps from $T \times X$ to X still satisfies weaker assumptions than the given ones in the previous section as follows.

(1) The real constants $\gamma_i \in [0,1)$, $i \in I_{Mc} \subset \overline{N}$ are not necessarily upper-bounds for the primary self-maps f_i from $T \times X$ to X in P_{Mc}. Instead, the class P_{Mc} is redefined so that the upper-bounding functions $k_i(t,t_0)$, for all $t \in T$, for all $t_0 \in T$ are assumed to be nonnegative and uniformly upper-bounded by finite constants $C_i \in \mathbf{R}_+$ (possibly exceeding or being equal to unity within some subsets of their definition domains) on S. Furthermore, they are assumed to be *asymptotically strictly contractive (i.e., taking asymptotically values being less than unity)* in the precise sense that $\limsup_{t\to\infty} k_i(t,t_0) \leq \gamma_i$ for some $\gamma_i \in (0,1)$. Note that this condition implies that for any given $\varepsilon_i \in (0, 1-\gamma_i-\delta_i] \cap \mathbf{R}$ with arbitrary $\delta_i \in (0,1-\gamma_i) \cap$

$\mathbf{R}$, $\exists t_i^* = t_i^*(\varepsilon_i, \gamma_i) \in S$ such that $S \ni t \geq t_i^*$, the upper-bounding function $k_i(t, t_0)$ associated with $f_i \in P_{Mc}$ satisfies the limiting upper-bounding condition $k_i(t, t_0) \leq \gamma_i + \varepsilon_i \leq 1 - \delta_i < 1$.

Note that the condition $\limsup_{T\to\infty} k_i(t_0 + T, t_0) \leq \gamma_i$ is fulfilled if $k_i(t_0 + T, t_0)$ is uniformly bounded in $[t_0, t_0 + T]$ and it is also monotone strictly decreasing on some $[t_0 + T', t_0 + T] \subseteq [t_0, t_0 + T]$. In this case, there is also a subinterval $[t_0 + T' + T'', t_0 + T] \subseteq [t_0 + T', t_0 + T] \subseteq [t_0, t_0 + T]$ in which $k_i(t_0 + T, t) \leq \gamma_i < 1$, for all $t \in [t_0 + T' + T'', t_0 + T]$. This condition is important in practice since exponentially stable dynamics such as those in (2.5) systems fulfil it. Thus, it is possible to construct switching rules which respect a sufficiently large minimum residence time interval at least at one of their stable parameterizations to guarantee the existence of a fixed point and the exponential stability to the origin in the dynamic system is unforced as a result.

(2) The class $P_{M\ell c}$ is assumed to be the set of *asymptotically large contractive* primary self-maps from $T \times X$ to X in the sense that Condition C_1 for any primary self-map f_i from $T \times X$ to X in the class $P_{M\ell c}$ is replaced with its asymptotic counterpart:

$$\limsup_{t\to\infty} \| f_i(t, x(t)) - f_i(t, y(t)) \| < \| x(t_1) - y(t_1) \|; \quad \forall x, y(\neq x) \in X,\ \forall t(> t_1),\ t_1 \in \mathrm{ST} \tag{3.1}$$

so that $\| f_i(t, x(t)) - f_i(t, y(t)) \|$ fulfils the strict upper-bounding condition in some subset of T of infinite Lebesgue measure. The Condition C_2 is left unaltered.

(3) The class of noncontractive nonexpansive self-maps P_{Mne} from $T \times X$ to X is defined to fulfil a similar condition to the above one by using instead under nonstrict inequality.

(4) The remaining classes of primary functions are assumed to be as those given in Section 2.

Although three of the subsets of primary self-maps are redefined under weaker conditions, the notation of Section 2 is kept for them in order to facilitate the exposition. The subsequent notation $t \in \mathrm{ST}_j(\sigma, P_M, a, b)$, for $j \in \overline{N}$, stands for switching points of $f \in P_m$ the rule σ acting on the primary class P_M of self-maps from $T \times X$ to X being within the subset $[a, b) \subseteq T$. If the arguments and b are omitted, then $t \in \mathrm{ST}_j(\sigma, P_M)$ is understood to be within any subset of T.

The main result of this section extends Theorem 2.12 as follows.

Theorem 3.1. *Under the assumptions in this section, assume also that all the self-maps of the class P_M in X are also uniformly bounded, that T is Lebesgue-measurable with $\mu(T) = \infty$ and consider a switching rule $\sigma : T \to \overline{N}$ which generates $f(\in M) : T \times X \to X$, with $M = M(\sigma, P_M)$ defined by the set P_M of primary self-maps from $T \times X$ to X satisfying all the above assumptions. The following properties hold.*

(i) *$f(\in M) : T \times X \to X$ has a fixed point if $\mu(T \setminus T_{Mc}(\sigma)) < \infty$, $\mu(T_{Mc}(\sigma)) = \infty$ and a minimum finite residence interval T_{r_j} being sufficiently large compared to $\mu(T \setminus T_{Mc}(\sigma))$ is respected at any $t_i \in \mathrm{ST}_j(\sigma)$, for all $j \in I_{Mc} \subseteq \overline{N}$ before the next switching in the following precise sense:*

$$\begin{aligned} &\left[t_i \in \mathrm{ST}_j, \forall j \in I_{Mc} \subseteq \overline{N}\right] \\ &\quad \Longrightarrow \left[\left(\mathrm{ST}_\ell(\sigma) \ni t_{i+1} \geq t_i + T_{rj},\ \forall \ell(\neq j) \in I_{Mc} \subseteq \overline{N}\right) \wedge \left(\neg\exists t \in (t_i, t_{i+1}) \cap \mathrm{ST}_{Mc}(\sigma)\right)\right] \\ &\qquad \vee \left[\neg\exists t(\geq t_i)\mathrm{ST}(\sigma)\right], \end{aligned} \tag{3.2}$$

or there is a finite number of switches with the last switching point being to a primary self-map in P_{Mc}.

(ii) $f(\in M): T\times X \to X$ *has a fixed point if* $\mu(T_{M\ell c}(\sigma))=\infty$ *and* $\mu(T\setminus T_{M\ell c}(\sigma))<\infty$ *and, furthermore,* $\exists M(<\infty)\in \mathbf{R}_+$ *such that the boundedness condition*

$$\left\| f(t^*+\ell T_0, x(t^*+\ell T_0)) - x(t^*)\right\| \le M; \quad \forall x\in X,\ \forall l \in \mathbf{Z}_{(0+)} \tag{3.3}$$

holds in the case that $f(\in M): X\to X$ *is not Lipschitzian, then* $f(\in M): X\to X$ *has a fixed point and a minimum finite residence interval* T_{r_j} *being sufficiently large compared to* $\mu(T\setminus T_{M\ell c}(\sigma))$ *is respected at any* $t_i\in \mathrm{ST}_j(\sigma)$, *for all* $j\in I_{M\ell c}\subseteq \overline{N}$.

(iii) $f(\in M): T\times X \to X$ *has a fixed point if* $\mu(T_{Mc}(\sigma)\cup T_{M\ell c}(\sigma))=\infty$ *and* $\mu(T\setminus (T_{Mc}(\sigma)\cup T_{M\ell c}(\sigma)))<\infty$ *and, furthermore,* $\exists M(<\infty)\in \mathbf{R}_+$ *such that the boundedness condition*

$$\left\| f(t^*+\ell T_0, x(t^*+\ell T_0)) - x(t^*)\right\| \le M; \quad \forall x\in X,\ \forall \ell \in \mathbf{Z}_{0+} \tag{3.4}$$

holds in the case that $f(\in M): T\times X\to X$ *is not Lipschitzian, then* $f(\in M): T\times X\to X$ *has a fixed point providing that a minimum residence interval is respected for at least one of the asymptotically strictly contractive or asymptotically large contractive self-maps in* P_M.

Proof. (i) Assume $\mu(T\setminus T_{Mc}(\sigma))<\infty$, $\mu(T_{Mc}(\sigma))=\infty$. Since $f\in M(\sigma,P_M)$ is uniformly bounded piecewise-continuous since all the functions in P_M are also uniformly bounded, the corresponding rate over-bounding functions $k: T\times T\to \mathbf{R}_{0+}$ are also uniformly bounded. Then, since $\mu(T\setminus T_{Mc}(\sigma))<\infty$, then the following situations can occur.

(1) The last switching occurs at a finite point t_i in $\mathrm{ST}(\sigma,P_M)$ with switching of the self-map $f\in M(\sigma,P_M)$ from $T\times X$ to X to an asymptotically contractive primary self-map. Also, t_i is the left boundary of a connected interval of S being of infinity Lebesgue measure. Formally: $\exists t_i\in \mathrm{ST}_j(\sigma,P_M)\subseteq \mathrm{ST}_{Mc}(\sigma,P_M)\subseteq \mathrm{ST}(\sigma,P_M)$ such that $\mu(\mathrm{ST}_j(\sigma,P_M,t_i,\infty))=\infty$ for some $j\in I_{Mc}\subseteq \overline{N}$, that is, there is no switching point being larger than the largest switching point t_i in ST under the switching rule $\sigma: T\to \overline{N}$. Then,

$$\left\| f(t,x(t)) - f(t,y(t))\right\| \le k(t,t_i)\left\| f(t_i,x(t_i)) - f(t_i,y(t_i))\right\| \le 2Mk(t,t_i)\le \gamma_i+\varepsilon_i<1 \tag{3.5}$$

provided that $t\ge t_i+T_{rj}$ for any given positive real constant $\varepsilon_i<1-\gamma_i$ such that $\mathrm{ST}_j(\sigma,P_M,t_i,\infty)\ni t_i\ge t^*\in \mathrm{ST}(\sigma,P_M)$ is sufficiently large but finite since $\limsup_{t\to\infty} k_i(t,t_0)\le \gamma_i$. Thus, the self-map $f\in M(\sigma,P_M)$ has a fixed point.

(2) There is no last switching point but after a finite switching points all the switching points exceeding some sufficiently large finite one involve switches to asymptotically strictly contractive primary self-maps from $T\times X$ to X.

Then, $\mu(\mathrm{ST}_j(\sigma,P_M,t_i,t_{i+1}))<\infty$ such that $t_i\in \mathrm{ST}_j(\sigma,P_M)$ and $t_{i+1}=\min(t>t_i : t\in \mathrm{ST}(\sigma,P_M))\in \mathrm{ST}_\ell(\sigma,P_M)$ for some $\ell(\neq j)\in I_M\subseteq \overline{N}$. First, assume that $j\notin I_{Mc}$ and $\ell(\neq j)\in I_{Mc}$ generates the next switching point $t_{i+1}\in$ ST under a primary self-map in P_{Mc} with $\mu(\mathrm{ST}_\ell(\sigma,P_M,t_{i+1},t_{i+2}))<\infty$, and

$$\sum_{j\in I_{Mc},\ t_j\ge t_i} \mu\left(\mathrm{ST}_\ell(\sigma,P_M,t_j,t_{j+1})\right)=\infty \Longrightarrow \left[\mu(T\setminus T_{Mc}(\sigma))<\infty \wedge \mu(T_{Mc}(\sigma))=\infty\right]. \tag{3.6}$$

This situation can occur of a simply connected subinterval $[t_i, \infty)$of T_{Mc}. Using a parallel reasoning to that of case (1) involving complete induction, one gets that $\exists \lim_{t\to\infty}\| f(t,x(t)) - f(t,y(t))\| = 0$ since

$$\limsup_{T\ni t\to\infty}\left\| f(t,x(t)) - f(t,y(t))\right\| \le \lim_{Z_{0+}\ni j\to\infty} (2M\rho)^j = 0 \tag{3.7}$$

with $\rho := \max(\gamma_j + \varepsilon_j : \ j \in I_{Mc}) \le 1 - \delta$ provided that $t_{j+1} \ge t_j + T_{r\ell(j)} \ge t_j + T_r \ge t_i$ if $t_j \in \mathrm{ST}_\ell$ for some $\ell = \ell(j)(\neq \ell(j-1)) \in I_{Mc} \subseteq \overline{N}$ with $T_{r\ell(j)} \ge T_r > 0$ being sufficiently large but finite. It is again concluded that the self-map $f \in M(\sigma, P_M)$ has a fixed point.

(3) There is no last switching point but, after a finite switching point, the sequence of all the switching points exceeding some sufficiently large finite one contains an infinite sequence of switching points to primary self-maps from $T \times X$ to X which are not asymptotically contractive. This case cannot occur since then $\mu(T \setminus T_{Mc}(\sigma)) = \infty$ contradicting the given assumptions.

Property (i) has been fully proven.

Properties (ii)-(iii) are proven in a similar way to their stronger parallel properties in Theorem 2.12 by using the upper-bounding limiting property of (2.18) for the extended class of primary self-maps. The detailed proof is omitted. □

Corollary 3.2. *Theorem 3.1(i) is fulfilled for any switching rule such that the minimum residence intervals referred to are respected in only one of the asymptotically strictly contractive primary self-maps. Theorem 3.1(ii) is extendable to the fulfilment of a sufficiently large residence interval by one of the asymptotically large contractive primary self-maps. Theorem 3.1(iii) is extendable to the fulfilment of the above property by either one of the asymptotically strictly contractive or one of the asymptotically large contractive primary self-maps.*

Theorem 3.1 addresses the case when the subset of T, where $f \in M(\sigma, P_M)$ is defined via not asymptotically (strict or large) contractive primary self-maps in P_M, has a finite Lebesgue measure; *that is, switches in-between primary self-maps can involve no contractive self-maps over finite intervals*. It is furthermore interesting to make that assumption more powerful by considering that T is the countable union of infinitely many connected subsets of finite Lebesgue measure whose boundaries are each pair of consecutive switching points. Sets formed by unions of some finite number of those subsets for consecutive switching points are assumed to contain at least one asymptotically either strict or large contractive primary self-map generating $f \in M(\sigma, P_M)$. *The subsequent result extends Theorem 3.1 to the case when the conditions of Lemma 2.9 are modified to their asymptotic versions. It is admitted that the sets of primary self-maps which are not contractive may be asymptotically compensated by the contractive ones, so that the built $f(\in M) : T \times X \to X$ by the switching rule $\sigma : T \to \overline{N}$ is asymptotically contractive in some sense to guarantee the existence of a fixed point*. Its proof follows directly by combining a directly extended Lemma 2.9 to its asymptotic version with Theorem 3.1 since any $f(\in M) : T \times X \to X$ is uniformly bounded on its definition domain.

Corollary 3.3. *Assume that all the self-maps of the class P_M in X are also uniformly bounded, that T is Lebesgue-measurable with $\mu(T) = \infty$, and consider a switching rule $\sigma : T \to \overline{N}$ which generates*

$f(\in M) : T \times X \to X$, with $M = M(\sigma, P_M)$ defined by the class P_M of primary self-maps from $T \times X$ to X satisfying either

$$\limsup_{i\to\infty}\left(k_{\sigma(t_j)}(t_i + T_0, t_j)\prod_{i}^{i+j-1}\left[k_{\sigma(t_i)}(t_{i+1}, t_i)\right]\right) \le \gamma < 1 \tag{3.8}$$

for some sufficiently large $T_0 \in T$ or the Condition C_2 of Section 2 for large contractions, together with the (asymptotic) modified condition C_1 :

$$\limsup_{t\to\infty}\left\| f_i(t, x(t)) - f_i(t, y(t))\right\| < \left\|x(t_1) - y(t_1)\right\|; \quad \forall x, y(\neq x) \in X,\ \forall t(> t_1),\ t_1 \in \mathrm{ST}. \tag{3.9}$$

Then, $f(\in M) : T \times X \to X$ has a fixed point.

Remark 3.4. The results of Sections 2 and 3 are extendable directly to the discrete case for the sets P and P_M by replacing Lebesgue measures with discrete ones.

Example 3.5. Fixed point theory is a useful tool to investigate the stability of dynamic systems including standard linear continuous-time or discrete systems and time-delay systems [2, 6] as well as hybrid dynamic systems including coupled continuous-time and discrete-time subsystems [1]. *Now, it is discussed the case of a delay-free continuous-time system under a switching rule operating among a given set of parameterizations and subject to controlled and uncontrolled impulses*. Consider the linear dynamic unforced time-varying system:

$$\begin{gathered}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + B_{0\sigma(t)}u_0(t), \quad x(0) \in \mathbf{R}^n, \\ u(t) = \sum_{t_i \in \mathrm{Imp}_c} K_c(t_i)\delta(t - t_i)x(t_i); \qquad u_0(t) = \sum_{t_i \in \mathrm{Imp}_0} K_0(t_i)\delta(t - t_i),\end{gathered} \tag{3.10}$$

where Imp_c and Imp_0, which are not required to be disjoint, are the real sequences of impulsive time instants where feedback control impulses and open-loop (i.e., feedback-free) control impulses occur, respectively, with the control Dirac distributions being $u(t)$ and $u_0(t)$, respectively, of respective piecewise-constant function matrices of dynamics $A(t) = A_{\sigma(t)} : T \to \mathbf{R}^{n\times n}$ and control matrix functions $B(t) = B_{\sigma(t)} : T \to \mathbf{R}^{n\times m}$ and $B_0(t) = B_{0\sigma(t)} : T \to \mathbf{R}^{n\times m}$ being run by a switching rule $\sigma(t) = \sigma(t_j) \in \overline{N} = \{1, 2, \ldots, N\}$, for all $t \in [t_j, t_{j+1})$, for all $t_j \in \mathrm{ST} \subset \mathbf{R}_{0+}$ where ST is the strictly ordered sequence of switching time instants. The real impulsive amplitude sequences $\{K_c(t_i)\}_{t_i\in \mathrm{Imp}_c}$ and $\{K_0(t_i)\}_{t_i\in \mathrm{Imp}_0}$ of elements in $\mathbf{R}^{n\times m}$ are assumed to be uniformly bounded and can be finite or infinite. An empty or nonempty sequence of impulsive time instants can occur as follows for some $p_i \in \mathbf{Z}_+$:

$$\left\{t_i = t_{i0}, t_{i1}, \ldots, t_{ip_i-1}, t_{i+1}\right\} \subset [t_i, t_{i+1}); \quad \forall t_i, t_{i+1} \in \mathrm{ST}(\sigma), \tag{3.11}$$

either within any simply connected time interval $[t_i, t_{i+1})$, where $t_i, t_{i+1} \in \mathrm{ST}(\sigma)$ are two consecutive switching points, or within any interval $[t_i, \infty)$ if the switching rule $\sigma : \mathbf{R}_{0+} \equiv T \to \overline{N}$ generates a finite sequence ST of switching time instants of last element t_i. The following impulsive constraints are assumed.

(1) $\exists\varepsilon_{\text{Imp}} \in \mathbf{R}_+$ such that $t_{ij+1} - t_{ij} \geq \varepsilon_{\text{Imp}}$ for any two consecutive impulsive time instants $t_{ij}, t_{ij+1} \in \text{Imp}_c \cup \text{Imp}_0$, if any, within $[t_i,\ t_{i+1})$ with $t_i\ , t_{i+1} \in \text{ST}$ being any two consecutive switching time instants. Also, $\exists\varepsilon_s \in \mathbf{R}_+$ such that $t_{i+1} - t_i \geq \varepsilon_s$ for any two consecutive switching time instants. The interpretation is that there is no accumulation point either of switching time instants or of impulsive time instants.

(2) If $t_i \notin \text{Imp}_c$, then $K_c(t_i) = 0$, and if $t_i \notin \text{Imp}_0$, then $K_0(t_i) = 0$.

(3) If $p_i = 1$, then there is no $t \in (\text{Imp}_c \cup \text{Imp}_0) \cap (t_i, t_{i+1})$ and $\tau_{i0} = t_{i+1} - t_i$.

(4) If $p_i = 1$ and $t_i \notin \text{Imp}_c \cup \text{Imp}_0$, then there is no $t \in (\text{Imp}_c \cup \text{Imp}_0) \cap [t_i, t_{i+1})$.

(5) If $[t_i, t_{i+1})$ is of finite Lebesgue measure, then $1 \leq p_i < \infty$ (i.e., there is at most a finite number of impulsive instants in any finite time interval within two consecutive switching instants).

(6) If there is a finite number of switches generated by the switching law $\sigma : \mathbf{R}_{0+} \to \overline{N}$, so that a finite t_i is the last one, with $[t_i, \infty)$ being of infinite Lebesgue measure then $1 \leq p_i \leq \infty$ (i.e., an infinite or finite number of impulsive instants can occur within an infinite time interval).

(7) If $p_i = \infty$, then $[t_i, t_{i+1})$ is of infinity Lebesgue measure so that $t_{i+1} = \infty$.

The unique state trajectory solution of this dynamic system satisfies

$$\begin{aligned}
x(t_{ij+1}) &= e^{A_{\sigma(t_i)}\tau_{ij}} x\left(t_{ij}^+\right), \\
x\left(t_{ij+1}^+\right) &= \left(I_n + BK_c(t_{ij})\right)x(t_{ij+1}) + B_0K_0(t_{ij})n \\
&= \left(I_n + BK_c(t_{ij})\right)e^{A_{\sigma(t_i)}\tau_{ij}} x\left(t_{ij}^+\right) \\
&\quad + B_0K_0(t_{ij}); \quad \forall t_{ij}, t_{ij+1} \in \left(I_{mp_c} \cup I_{mp_0}\right) \subset [t_i, t_{i+1});\ \forall t_i, t_{i+1} \in \text{ST},
\end{aligned} \tag{3.12}$$

where $\tau_{ij} := t_{ij+1} - t_{ij}$ and I_n is the nth identity matrix. One gets by applying the above relations recursively for the sequence of tie instants $\{t_i = t_{i0}, t_{i1}, \ldots, t_{ip_i-1}, t_{i+1}\ \}$

$$x(t_{i+1}^+) = \prod_{j=0}^{p_i-1}\left[\left(I_n + BK_c(t_{ij})\right)\right]e^{A_{\sigma(t_i)}\tau_{ij}} x(t_i^+) + \sum_{j=0}^{p_i-1}\prod_{\ell=j+1}^{p_i-1}\left[\left(I_n + BK_c(t_{i\ell})\right)e^{A_{\sigma(t_i)}\tau_{i\ell}}\right]B_0K_0(t_{ij}) \tag{3.13}$$

so that

$$\left\|x(t_{i+1}^+)\right\| \leq \rho_{c\sigma(t_i)}\left\|x(t_i^+)\right\| + \rho_{0\sigma(t_i)}; \quad \forall t_i, t_{i+1} \in \text{ST} \tag{3.14}$$

with

$$\rho_{c\sigma(t_i)} := \left\|\prod_{j=0}^{p_i-1}\left[\left(I_n + BK_c(t_{ij})\right)e^{A_{\sigma(t_i)}\tau_{ij}}\right]\right\| \leq \overline{\rho}_{c\sigma(t_i)} := C_{\sigma(t_i)}e^{\lambda_{\sigma(t_i)}\tau_i}\left(\prod_{j=0}^{p_i-1}\left[\left\|I_n + BK_c(t_{ij})\right\|\right]\right), \tag{3.15}$$

where $\tau_i := t_{i+1} - t_i = \sum_{j=0}^{p_i-1} \tau_{ij}$ and $\lambda_{\sigma(t_i)}$ is the numerical radius $\lambda_{\max}(A_{\sigma(t_i)} + A_{\sigma(t_i)}^T)/2$ (or 2-matrix measure with respect to the spectral ℓ_2-norm) of $A_{\sigma(t_i)}$ and the fundamental matrix function is upper-bounded as follows $\|e^{A_i t}\| \le C_i e^{\lambda_i t}$ for any $A_i \in \mathbf{R}^{n\times n}$ and some real constants $C_i \ge 1$ and λ_i, for all $i \in \overline{N}$. Then,

$$\begin{aligned} \|x(t_{i+1}^+) - y(t_{i+1}^+)\| &\le \rho_{c\sigma(t_i)} \|x(t_i^+) - y(t_i^+)\| \le \overline{\rho}_{c\sigma(t_i)} \|x(t_i^+) - y(t_i^+)\| \\ &\le \prod_{j=1}^{i} \left[\rho_{c\sigma(t_j)}\right] \|x(t_1^+) - y(t_1^+)\| \\ &\le \prod_{j=1}^{i} \left[\overline{\rho}_{c\sigma(t_j)}\right] \|x(t_1^+) - y(t_1^+)\|; \quad \forall t_i, t_{i+1} \in \mathrm{ST},\ \forall x_1, \forall y_1 \in \mathrm{X}. \end{aligned} \tag{3.16}$$

If $t_i < \infty$ is the last switching time instant, then $\tau_i = t_{i+1} = \infty$ and

$$\begin{aligned} \|x(t) - y(t)\| &\le \rho_{c\sigma(t_i)} \|x(t_i^+) - y(t_i^+)\| \le \overline{\rho}_{c\sigma(t_i)} \|x(t_i^+) - y(t_i^+)\| \\ &\le \prod_{j=1}^{i} \left[\rho_{c\sigma(t_j)}\right] \|x(t_1^+) - y(t_1^+)\| \\ &\le C_{\sigma(t_i)} e^{\lambda_{\sigma(t_i)}(t-t_i)} \left(\prod_{j=0}^{p_i-1} [\|I_n + BK_c(t_{ij})\|] \right) \\ &\quad \times \prod_{j=1}^{i-1} \left[\overline{\rho}_{c\sigma(t_j)}\right] \|x(t_1^+) - y(t_1^+)\|; \quad \forall t \in [t_i, \infty), \end{aligned} \tag{3.17}$$

and furthermore,

$$\begin{aligned} &\limsup_{t\to\infty} \|x(t) - y(t)\| \\ &\quad \le \limsup_{t\to\infty} \left(C_{\sigma(t_i)} e^{\lambda_{\sigma(t_i)} t} \prod_{j=0}^{p_i-1} [\|I_n + BK_c(t_{ij})\|] \prod_{j=1}^{i-1} \left[\overline{\rho}_{c\sigma(t_j)}\right] \right) \|x(t_1^+) - y(t_1^+)\|. \end{aligned} \tag{3.18}$$

Also, one gets for any finite $T_0 \in \mathbf{R}_+$, any $p \in \mathbf{Z}_+$, for all $t_i, t_{i+1} \in \mathrm{ST}$

$$\begin{aligned} &\|x(t_1^+ + (p+1)T_0) - y(t_1^+ + (p+1)T_0)\| \\ &\quad \le C_{\sigma(t_\mu)} e^{\lambda_{\sigma(t_\mu)}(t-t_\mu)} \left(\prod_{j=0}^{p_\mu-1} [\|I_n + BK_c(t_j)\|] \right) \prod_{j=\nu}^{\mu-1} \left[\overline{\rho}_{c\sigma(t_j)}\right] \\ &\qquad \times C_{\sigma(t_1+T_0)} e^{\lambda_{\sigma(t_i)}(t_\nu - t_1 - T_0)} \|x(t_1^+ + pT_0) - y(t_1^+ + pT_0)\| \\ &\quad = k_p \|x(t_1^+ + pT_0) - y(t_1^+ + pT_0)\|, \end{aligned} \tag{3.19}$$

where $k_p = k_p(T_0, \sigma) \in \mathbf{R}_+$ is defined directly from the above expression, provided that there is at least one switching time instant within $[t_1 + pT_0, t_1 + (p+1)T_0]$, where

$$\begin{gathered} t_\nu = t_\nu(p, T_0) = \min\left(t \in \mathrm{ST} : t \geq t_1 + pT_0\right), \\ t_\mu = t_\mu(p, T_0) = \max\left(t \in \mathrm{ST} : t \leq t_1 + (p+1)T_0\right) \geq t_\nu, \\ t_1 + (p+1)T_0 > t_\nu \geq t_1 + pT_0. \end{gathered} \tag{3.20}$$

The case that there is no impulse but one switch within $[t_1 + pT_0, t_1 + (p+1)T_0]$ is included in the above formula by removing the norms $\|I_n + BK_c(t_{\mu j})\|$ since the involved $p_{(\cdot)}$ are zero. The case of no switch-no impulse occurring in $[t_1 + pT_0, t_1 + (p+1)T_0]$ is also particular case of the above formula (3.19) resulting to be for $t_\nu = t_1 + pT_0$

$$\begin{aligned} \left\|x\left(t_1^+ + (p+1)T_0\right) - y\left(t_1^+ + (p+1)T_0\right)\right\| &\leq C_{\sigma(t_\nu)} e^{\lambda_{\sigma(t_\nu)}(t - t_\nu)} \left\|x\left(t_1^+ + pT_0\right) - y\left(t_1^+ + pT_0\right)\right\| \\ &= k_p \left\|x\left(t_1^+ + pT_0\right) - y\left(t_1^+ + pT_0\right)\right\|. \end{aligned} \tag{3.21}$$

It follows directly from recursion in (3.21) that for any switching rule $\sigma : \mathbf{R}_{0+} \to \overline{N}$ if there exists a $T_0 = T_0(\sigma) \in \mathbf{R}_+$ such that $k_p = k_p(T_0, \sigma) < 1$,

$$\left\|x\left(t_1^+ + (p+1)T_0\right) - y\left(t_1^+ + (p+1)T_0\right)\right\| \leq \left(\prod_{i=1}^{p} [k_i]\right) \left\|x\left(t_1^+ + T_0\right) - y\left(t_1^+ + T_0\right)\right\| \tag{3.22}$$

so that the state-trajectory solution possesses a fixed point exists since

$$\begin{aligned} &\exists \lim_{\mathbf{Z}_+ \ni p \to \infty} \left\|x\left(t_1^+ + pT_0\right) - y\left(t_1^+ + T_0\right)\right\| \\ &\quad = \lim_{\mathbf{R}_+ \ni t \to +\infty} \left\|x\left(t_1^+ + pT_0\right) - y\left(t_1^+ + T_0\right)\right\| = 0; \quad \forall x(t_1), y(t_1) \in \mathbf{R}. \end{aligned} \tag{3.23}$$

It turns out that if there is at least one stability matrix $A_\ell \in \{A_i : i \in \overline{N}\}$, then there are always switching rules $\sigma : \mathbf{R}_{0+} \to \overline{N}$ which lead to a state-trajectory solution possessing a fixed point. Since $C_\ell e^{\lambda_\ell (t - t_i)} < 1$, since $\lambda_i < 0$, for any sufficiently large residence interval $(t - t_i)$ such that $\sigma(t_i) = \ell$ (i.e., for a sufficiently large time interval free of switches and impulses previous to the next switch after each switch to the stability matrix A_ℓ has happened), then the associated map for this switching is asymptotically strictly contractive. They can occur also switches to nonexpansive ($\lambda_i = 0$) or expansive ($\lambda_i > 0$) generated by the switching rule but a fixed point always exists for such a rule if for some $T_0 = T_0(\sigma) \in \mathbf{R}_+$, there is a dominance of the switching intervals associated with A_ℓ so that $k_p < 1$. In the presence of impulses, the result is still valid by increasing, if necessary, the residence interval before to the next switch after switches to the parameterizing matrix A_ℓ have happened. It is possible to achieve a constant $k_p = k_p(T_0, \sigma) < 1$ for some T_0 since the norm upper-bounding real function of time $C_\ell e^{\lambda_\ell (t - t_i)}$ is monotone strictly decreasing related to the residence interval $\tau_i = t_{i+1} - t_i$.

Remark 3.6. It is important to point out that it is obvious that the generalization of the given formalism to switching rules $\sigma : \mathrm{ST} \to \mathbf{Z}_+$ is direct; that is the codomain $\mathbf{Z}_+$ of σ coincides with the image of σ so that infinitely many distinct primary self-maps are used to construct $f : T \times X \to X$. This implies necessarily that the switching rule generates infinitely many switches so that the discrete measure of ST is infinity.

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