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Research Article

Eventually Periodic Points of Infra-Nil Endomorphisms

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Hyperbolic toral automorphisms provide important examples of chaotic dynamical systems. Generalizing automorphisms on tori, we study (infra-)nil endomorphisms defined on (infra-)nilmanifolds. In particular, we show that every infra-nil endomorphism has dense eventually periodic points.

1. Introduction

Let A be an $n \times n$ nonsingular integer matrix. Then A induces a map $L_A : T^n \to T^n$ on the n-torus $T^n = \mathbb{Z}^n \setminus \mathbb{R}^n$. If A is hyperbolic, we say that L_A is a hyperbolic toral endomorphism. If, in addition, $\det(A) = \pm 1$, then A is called a hyperbolic toral automorphism.

A hyperbolic toral automorphism provides an important example of a chaotic dynamical system. We review the most fundamental property about hyperbolic toral automorphisms, together with some definitions which are necessary to describe this property. See [1] for details.

A continuous surjection $f : X \to X$ of a topological space X is said to be *topologically transitive* if, for any pair of nonempty open sets U and V in X, there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$. Intuitively, a topologically transitive map has points which eventually move under iteration from one arbitrary small neighborhood to any other. The continuous map $f : X \to X$ of the metric space (X, d) is said to have *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood N of x, there exist $y \in N$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$. Intuitively, a map possesses sensitive dependence on initial conditions if there exist points arbitrarily close to x which eventually separate from x by at least δ under iteration of f.

The following proposition shows that a hyperbolic toral automorphism L_A is dynamically quite different from its linear counterpart.

Proposition 1.1 (see [1, Theorem II.4.8]). *A hyperbolic toral automorphism L_A is chaotic on T^n. That is,*

(1) *the set of periodic points of L_A is dense in T^n;*

(2) *L_A is topologically transitive;*

(3) *L_A has sensitive dependence on initial conditions.*

Anosov diffeomorphisms play an important role in dynamics. In [2], Smale raised the problem of classifying the closed manifolds (up to homeomorphism) which admit an Anosov diffeomorphism. Franks [3] and Manning [4] proved that every Anosov diffeomorphism on an infra-nilmanifold is topologically conjugate to a hyperbolic infra-nil automorphism. In [5], Gromov proved that every expanding map on a closed manifold is topologically conjugated to an expanding map on an infra-nilmanifold.

We will consider infra-nil *endomorphisms* in this paper. These include Anosov diffeomorphisms and expanding maps on infra-nilmanifolds up to topological conjugacy. The purpose of this paper is to show that the infra-nil endomorphisms have dense eventually periodic points. In the case of infra-nil automorphisms, this is already known (cf. [4, Lemma 3]).

2. Toral Endomorphisms

Now we show that every toral *endomorphism* has dense periodic points. This generalizes [1, Proposition II.4.2] in which it is shown that every toral automorphism has dense periodic points.

Definition 2.1. For a self-map $f : X \to X$, a point x of X is called an *eventually periodic point* of f if $f^{m+t}(x) = f^t(x)$ for some $m > 0,\ t \geq 0$. If $t = 0$, then it becomes a periodic point of f with period m.

Note that if $\{p_1, \ldots, p_t\}$ is a nonempty set of prime numbers, then the set $S = \{p_1^{n_1} \cdots p_t^{n_t} \mid n_i \in \mathbb{Z}, n_i \geq 0, i = 1, \ldots, t\}$ is a multiplicative subset of $\mathbb{Z}$. Let $S^{-1}\mathbb{Z}$ be the ring of quotients of $\mathbb{Z}$ by S. We denote $S^{-1}\mathbb{Z}$ by $\mathbb{Z}_{(p_1,\ldots,p_t)}$. Clearly, $\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p,q)}$ and $\mathbb{Z}_{(p)} + \mathbb{Z}_{(q)} = \mathbb{Z}_{(p,q)}$.

Lemma 2.2. *Let $L_A : T^d \to T^d$ be a toral endomorphism of the torus T^d induced by the automorphism $A : \mathbb{R}^d \to \mathbb{R}^d$ and let $\{p_1, \ldots, p_r\}$ be a nonempty set of prime numbers. Then every point with coordinates in $R = \mathbb{Z}_{(p_1,\ldots,p_r)}$ is an eventually periodic point of L_A. Moreover, if $(p_i, |\det(A)|) = 1$ for all i, then every point with coordinates in R is a periodic point of L_A.*

Proof. Let $\mathbf{x}$ be a point of $T^d = \mathbb{Z}^d \backslash \mathbb{R}^d$ with coordinates in R. Finding a common denominator, we may assume that $\mathbf{x}$ is of the form $(n_1/k, \ldots, n_d/k) \in \mathbb{R}^d$ where n_i and k are integers. Write $\mathbf{x} = [n_1/k, \ldots, n_d/k] \in T^d$. Then there are exactly k^d points in T^d of the form $[n_1/k, \ldots, n_d/k]$ with $0 \leq n_i < k$.

The image of any such point under L_A may also be written in this form, since the entries of A are integers. Thus $\mathbf{x}$ is an eventually periodic point of L_A. Moreover, if $(p_i, |\det(A)|) = 1$ for all $1 \leq i \leq r$, L_A is injective on these points and hence L_A is a permutation of k^d such points. In fact, if $L_A([n_1/k, \ldots, n_d/k]) = L_A([n'_1/k, \ldots, n'_d/k])$, then we see that

$A((n_1/k,\ldots,n_d/k)) - A((n'_1/k,\ldots,n'_d/k)) \in \mathbb{Z}^d$, or $((n_1-n'_1)/k,\ldots,(n_d-n'_d)/k) \in A^{-1}(\mathbb{Z}^d)$. Since $[A^{-1}(\mathbb{Z}^d):\mathbb{Z}^d] = [\mathbb{Z}^d : A(\mathbb{Z}^d)] = |\det(A)|$ and $(k,|\det(A)|) = 1$, we must have

$$\left(\frac{n_1-n'_1}{k},\ldots,\frac{n_d-n'_d}{k}\right) \in \mathbb{Z}^d. \tag{2.1}$$

Hence $[n_1/k,\ldots,n_d/k] = [n'_1/k,\ldots,n'_d/k]$. Therefore, $\mathbf{x}$ is a periodic point of L_A. □

Corollary 2.3. *Every toral endomorphism $L_A : T^d \to T^d$ of the torus T^d has dense periodic points.*

Proof. Let p be a prime number with $(p,|\det(A)|) = 1$ and let $R = \mathbb{Z}_{(p)}$. Then by Lemma 2.2, the points with coordinates in R are periodic. Moreover, $\mathbb{Z}^d \setminus R^d$, the set of points in T^d with coordinates in R, is a dense subset of the torus T^d. □

3. Nil Endomorphisms

In this section, we first recall from [6–10] some definitions about nilpotent Lie groups and give some basic properties which are necessary for our discussion.

Let G be a connected, simply connected nilpotent Lie group. A discrete cocompact subgroup Γ of G is said to be a *lattice* of G, and in this case, the quotient space $\Gamma \setminus G$ is said to be a *nilmanifold*.

Let Γ be a lattice of G. Then Γ is a finitely generated torsion-free nilpotent group. Recall that the lower central series of Γ is defined inductively by $\gamma_1(\Gamma) = \Gamma$ and $\gamma_{i+1}(\Gamma) = [\gamma_i(\Gamma),\Gamma]$. Suppose that Γ is c-step nilpotent, that is, $\gamma_c(\Gamma) \neq 1$, but $\gamma_{c+1}(\Gamma) = 1$. The isolator of a subgroup H of Γ, denoted by $\sqrt[\Gamma]{H}$, is the set

$$\sqrt[\Gamma]{H} = \left\{x \in \Gamma \mid x^k \in H \text{ for some integer } k > 0\right\}. \tag{3.1}$$

It is well known ([6], [9, page 473] or [10]) that the sequence

$$\Gamma = \Gamma_1 = \sqrt[\Gamma]{\gamma_1(\Gamma)} \supset \Gamma_2 = \sqrt[\Gamma]{\gamma_2(\Gamma)} \supset \cdots \supset \Gamma_c = \sqrt[\Gamma]{\gamma_c(\Gamma)} \supset \Gamma_{c+1} = 1 \tag{3.2}$$

forms a central series with $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}^{k_i}$. It follows that it is possible to choose a generating set

$$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_c \tag{3.3}$$

of Γ in such a way that Γ_i is the group generated by $\mathbf{a}_i = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}$ and Γ_{i+1} for each $i = 1,2,\ldots,c$. We refer to $\mathbf{a} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_c\}$ as a *preferred basis* of Γ.

We use $\mathfrak{G}$ to indicate the Lie algebra of G. This Lie algebra $\mathfrak{G}$ has the same dimension and nilpotency class as G. Moreover, in the case of connected, simply connected nilpotent Lie groups it is known that the exponential map $\exp : \mathfrak{G} \to G$ is a diffeomorphism. We denote its inverse by log. If G' is another connected, simply connected nilpotent Lie group, with Lie algebra $\mathfrak{G}'$, then we have the following properties.

(i) For any homomorphism $\phi : G \to G'$ of Lie groups, there exists a unique homomorphism $d\phi : \mathfrak{G} \to \mathfrak{G}'$ (differential of ϕ) of Lie algebras, making the following diagram commuting:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, shift right, "\log"'] & G' \arrow[d, shift right, "\log"'] \\ \mathfrak{G} \arrow[r, "d\phi"] \arrow[u, shift right, "\exp"'] & \mathfrak{G}' \arrow[u, shift right, "\exp"'] \end{tikzcd} \tag{3.4}$$

(ii) Conversely, for any homomorphism $d\phi : \mathfrak{G} \to \mathfrak{G}'$ of Lie algebras, there exists a unique homomorphism $\phi : G \to G'$ of Lie groups, making the above diagram commuting.

If $\mathbf{a}$ is a preferred basis of Γ, then $\log \mathbf{a} = \{\log \mathbf{a}_1, \log \mathbf{a}_2, \ldots, \log \mathbf{a}_c\} \subset \mathfrak{G}$ can be regarded as a basis for the vector space $\mathfrak{G}$. We call the basis $\log \mathbf{a}$ of $\mathfrak{G}$ *preferred*. In particular, if Γ is a lattice of $\mathbb{R}^d$, then every preferred basis $\mathbf{a}$ of Γ becomes a preferred basis $\log \mathbf{a} = \mathbf{a}$ for the vector space $\mathbb{R}^d$.

We first generalize the concept of toral automorphisms to that of nil endomorphisms and show that every nil endomorphism has eventually dense periodic points.

Let $\Gamma \setminus G$ be a nilmanifold and let $\varphi : G \to G$ be an automorphism satisfying that $\varphi(\Gamma) \subset \Gamma$. Then the automorphism φ induces a surjection φ_Γ on the nilmanifold $\Gamma \setminus G$ and the following diagram is commuting:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & G \arrow[d] \\ \Gamma \setminus G \arrow[r, "\varphi_\Gamma"] & \Gamma \setminus G \end{tikzcd} \tag{3.5}$$

Lemma 3.1. *Let $\varphi : G \to G$ be an automorphism satisfying that $\varphi(\Gamma) \subset \Gamma$. Then $d\varphi$ has a block matrix, with respect to any preferred basis of $\mathfrak{G}$, of the form*

$$d\varphi = \begin{bmatrix} N_1 & 0 & \ldots & 0 \\ * & N_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & N_c \end{bmatrix}, \tag{3.6}$$

where the diagonal blocks N_i's are integral matrices, and $|\det(d\varphi)| = [\Gamma : \varphi(\Gamma)]$. In particular, the automorphism φ on G restricts to an automorphism on a lattice of G if and only if its differential $d\varphi$ has determinant ± 1.

The proof of this lemma is rather straight forward and so we omit the proof. See, for example, [11, Lemma 3.1] and [12, Proposition 3.1].

Definition 3.2. Let $\Gamma \setminus G$ be a nilmanifold and let $\varphi : G \to G$ be an automorphism with $\varphi(\Gamma) \subset \Gamma$. Then φ induces a surjective map φ_Γ on the nilmanifold $\Gamma \setminus G$, which is one of the following two types.

(I) $d\varphi$ has determinant of modulus 1. In this case φ_Γ is called a *nil automorphism*.

(II) $d\varphi$ has determinant of modulus greater than 1. In this case φ_Γ is called a *nil endomorphism*.

If, in addition, φ is hyperbolic (i.e., $d\varphi$ has no eigenvalues of modulus 1), then we say that the nil automorphism or endomorphism φ_Γ is *hyperbolic*.

Example 3.3. Let Nil be the 3-dimensional Heisenberg group with its Lie algebra nil. That is,

$$\text{Nil} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}, \qquad \text{nil} = \left\{ \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}. \tag{3.7}$$

It is easy to show (see [13, Proposition 2.2]) that

$$\text{Aut(nil)} \cong \left\{ \begin{bmatrix} \alpha & \gamma & 0 \\ \beta & \delta & 0 \\ \eta & \mu & \alpha\delta - \beta\gamma \end{bmatrix} \mid \alpha\delta - \beta\gamma \neq 0, \eta, \mu \in \mathbb{R} \right\}. \tag{3.8}$$

Thus we see that the differential of any automorphism φ on Nil has determinant $\det(d\varphi) = (\alpha\delta - \beta\gamma)^2$ and eigenvalues $\alpha\delta - \beta\gamma$ and $(1/2)\{(\alpha + \delta) \pm \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}\}$. Thus if $|\det(d\varphi)| = 1$, then $d\varphi$ has an eigenvalue of modulus 1. Therefore, there are no hyperbolic nil automorphisms on any nilmanifold $\Lambda \setminus \text{Nil}$. (There are examples of hyperbolic nil, nontoral, automorphisms. In fact, we can find such examples from many literatures. For example, we refer to [2, 14–18].)

Via the exponential map

$$\exp : \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \in \text{nil} \longmapsto \begin{bmatrix} 1 & a & c + \dfrac{ab}{2} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \in \text{Nil}, \tag{3.9}$$

we see that every automorphism φ on Nil is given as follows:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & \alpha x + \gamma y & z' \\ 0 & 1 & \beta x + \delta y \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.10}$$

where $z' = (\alpha\delta - \beta\gamma)z + \beta\gamma xy + (\alpha\beta/2)x^2 + \eta x + \mu y + (\gamma\delta/2)y^2$. Consider the subgroups Λ_k, $k \in \mathbb{Z}$, of Nil:

$$\Lambda_k = \left\{ \begin{bmatrix} 1 & m & \dfrac{\ell}{k} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \mid m, n, \ell \in \mathbb{Z} \right\}. \tag{3.11}$$

These are lattices of Nil, and every lattice of Nil is isomorphic to some Λ_k. The following matrices $d\varphi$ give simple examples which induce hyperbolic nil endomorphisms on the nilmanifold $\Lambda_{2k} \setminus \mathrm{Nil}$:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ n & m & 5 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ n & m & 2 \end{bmatrix}. \tag{3.12}$$

Note that the first one has eigenvalues of modulus all greater than 1, and the second one has determinant of modulus greater than 1, and there is at least one eigenvalue with modulus less than 1.

Corollary 3.4. *If φ_Γ is a nil automorphism, then the automorphism $\varphi^{-1} : G \to G$ induces a nil automorphism which is φ_Γ^{-1}. In particular, φ_Γ is a diffeomorphism of $\Gamma \setminus G$.*

By refining the central series of Γ explained in the paragraph above Lemma 3.1, we can find a central series

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots \supset \Gamma_d \supset \Gamma_{d+1} = 1 \tag{3.13}$$

with $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$, for each $i = 1, 2, \ldots, d$. (We are assuming that G is d-dimensional, and using the same symbol for terms of a refinement of the previous central series.) We can choose a generating set

$$\mathbf{a} = \{a_1, a_2, a_3, \ldots, a_d\} \tag{3.14}$$

of Γ in such a way that Γ_i is the group generated by a_i and Γ_{i+1}. Then any element $\gamma \in \Gamma$ is uniquely expressible as a product:

$$\gamma = a_1^{n_1} a_2^{n_2} \cdots a_d^{n_d}, \quad \text{with } \vec{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d, \tag{3.15}$$

and we can regard G as the Mal'cev completion of Γ:

$$G = \left\{ a_1^{r_1} a_2^{r_2} \cdots a_d^{r_d} \mid \vec{r} = (r_1, r_2, \ldots, r_d) \in \mathbb{R}^d \right\}. \tag{3.16}$$

We refer to this preferred basis $\mathbf{a} = \{a_1, a_2, \ldots, a_d\}$ as a *canonical basis* of Γ. Given $\vec{n} \in \mathbb{Z}^d$, we use $\gamma(\vec{n})$ to denote the element of Γ whose canonical coordinate is $\vec{n}$. Thus, we have an identification $\gamma : \mathbb{Z}^d \to \Gamma$ sending $\vec{n}$ to $\gamma(\vec{n})$.

Among interesting properties of this identification, we recall the following ([7, Theorem 2.1.(3)]): for any homomorphism $\kappa : \Gamma \to \Gamma$, there exists a polynomial function with rational coefficients $\psi_\kappa : \mathbb{Z}^d \to \mathbb{Z}^d$ such that $\kappa(\gamma(\vec{n})) = \gamma(\psi_\kappa(\vec{n}))$ for all $\vec{n} \in \mathbb{Z}^d$. Moreover, any homomorphism of Γ extends to a homomorphism of G by using the same polynomial.

Example 3.5. The map $\psi : \mathrm{Nil} \to \mathrm{Nil}$ given by

$$\psi(x, y, z) = \left(3x + y, x + y, 2z + xy + \frac{3}{2}x^2 + nx + my + \frac{1}{2}y^2\right) \tag{3.17}$$

is a polynomial function with rational coefficients, which sends Λ_2 into Λ_2 itself. The polynomial function ψ is associated to the homomorphism

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ n & m & 2 \end{bmatrix} \tag{3.18}$$

on Nil given in Example 3.3.

We recall the famous Campbell-Baker-Hausdorff formula:

$$\log(a \cdot b) = \log a * \log b \quad \forall a, b \in G, \tag{3.19}$$

where

$$A * B = A + B + \frac{1}{2}[A, B] + \sum_{m=3}^{\infty} C_m(A, B). \tag{3.20}$$

Here $C_m(A, B)$ stands for a rational combination of m-fold Lie brackets in A and B. Since our Lie algebra is nilpotent, the sum involved in $A * B$ is always finite. Throughout this paper, we shall use $Q = \mathbb{Z}_{(q_1,\ldots,q_s)}$ whenever $\{q_1, \ldots, q_s\}$ is the set of all prime factors of the denominators of the reduced rational coefficients appearing in the Campbell-Baker-Hausdorff formula. For example, if G is 3-step nilpotent, then

$$\begin{aligned} \log(a \cdot b) = \log a + \log b &+ \frac{1}{2}[\log a, \log b] \\ &+ \frac{1}{12}[[\log a, \log b], \log b] - \frac{1}{12}[[\log a, \log b], \log a], \end{aligned} \tag{3.21}$$

and hence $Q = \mathbb{Z}_{(2,3)} = \{k/2^m 3^n \mid k \in \mathbb{Z}, m, n \in \mathbb{Z}^+\}$.

Lemma 3.6. *For any homomorphism $\kappa : \Gamma \to \Gamma$, the associated polynomial function ψ_κ has coefficients in $Q = \mathbb{Z}_{(q_1,\ldots,q_s)}$.*

Proof. Suppose that G is a d-dimensional connected, simply connected nilpotent Lie group. The first thing to notice is that for any X, Y in the Lie algebra $\mathfrak{G}$ of G, we have that

$$\exp(X+Y) = \exp(X)\exp(Y)\exp\left(\sum_{m=2} D_m(X,Y)\right), \tag{3.22}$$

where D_m denotes a linear combination of m-fold brackets in X and Y with coefficients in the ring Q. To see this, let us make the following computation:

$$\begin{aligned}\exp(X)\exp(Y) &= \exp(X * Y)\\ &= \exp(X+Y)\exp(-X-Y)\exp(X*Y)\\ &\quad (\text{because } \exp(\mathrm{X}+\mathrm{Y})\exp(-\mathrm{X}-\mathrm{Y}) = 1)\\ &= \exp(X+Y)\exp((-X-Y)*(X*Y)).\end{aligned} \tag{3.23}$$

From this it follows that

$$\exp(X+Y) = \exp(X)\exp(Y)\exp(-(-X-Y)*(X*Y)), \tag{3.24}$$

which is of the form (3.22) claimed above.

Now, let $A_1, A_2, \ldots, A_d$ be a canonical basis of $\mathfrak{G}$ (We mean $A_i = \log(a_i)$ where the a_i form a canonical basis of Γ). Since $D_m(Y,X) = -D_m(X,Y)$, we have from (3.22) that

$$\exp(Y)\exp(X) = \exp(X)\exp(Y)\exp\left(2\sum_{m=2} D_m(X,Y)\right). \tag{3.25}$$

By a repeated use of formulas (3.22) and (3.25) it is now easy to see that

$$\exp(\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_d A_d) = \exp\big(p_1(\alpha_1,\ldots,\alpha_d)A_1\big)\cdots\exp\big(p_d(\alpha_1,\ldots,\alpha_d)A_d\big), \tag{3.26}$$

where $p_i(\alpha_1,\ldots,\alpha_d)$ is a polynomial with coefficients in Q. We will use this fact below.

Finally, let κ be the Lie group homomorphism of G which extends uniquely the given $\kappa : \Gamma \to \Gamma$. Let a_i be a term of the canonical basis of Γ, then $\kappa(a_i) = a_1^{\beta_1} a_2^{\beta_2} \cdots a_d^{\beta_d}$ for some $\beta_j = \beta_j(a_i) \in \mathbb{Z}$. Using the Campbell-Baker-Hausdorff formula, it is then easy to see (look also at the computation below) that

$$d\kappa\big(\log(a_i)\big) = \log\kappa(a_i) = \sum_{j=1}^{d} \gamma_j \log\big(a_j\big) \quad \text{for some } \gamma_j \in \mathrm{Q}. \tag{3.27}$$

We now compute

$$\begin{aligned}\kappa\left(a_1^{x_1}a_2^{x_2}\cdots a_d^{x_d}\right) &= \exp\log\left(\kappa\left(a_1^{x_1}a_2^{x_2}\cdots a_d^{x_d}\right)\right)\\ &= \exp d\kappa\left(\log\left(a_1^{x_1}a_2^{x_2}\cdots a_n^{x_n}\right)\right)\\ &= \exp d\kappa\left(x_1\log(a_1) * \log\left(a_2^{x_2}\cdots a_d^{x_d}\right)\right)\\ &= \exp d\kappa\left(x_1\log(a_1) * \left(x_2\log(a_2) * \log\left(a_3^{x_3}\cdots a_d^{x_d}\right)\right)\right)\\ &= \exp d\kappa\left(\sum_{m=1}^{c}E_m(\log(a_1),\log(a_2),\ldots,\log(a_d))\right).\end{aligned} \tag{3.28}$$

Here E_m stands for a term which is a linear combination of m-fold brackets of the $\log(a_i)$ and where the coefficients are polynomials in the variables x_j over the ring Q. By continuing this computation, we see that

$$\kappa\left(a_1^{x_1}a_2^{x_2}\cdots a_d^{x_d}\right) = \exp\left(\sum_{m=1}E_m\left(d\kappa\left(\log(a_1)\right), d\kappa\left(\log(a_2)\right),\ldots, d\kappa\left(\log(a_d)\right)\right)\right). \tag{3.29}$$

Now using (3.27) we derive that

$$\kappa\left(a_1^{x_1}a_2^{x_2}\cdots a_d^{x_d}\right) = \exp\left(q_1(x_1,x_2,\ldots,x_d)\log(a_1)+\cdots+q_d(x_1,x_2,\ldots,x_d)\log(a_d)\right), \tag{3.30}$$

where the q_i are polynomials with coefficients in Q. Therefore, using (3.26), this implies that the polynomial ψ_κ is as required. □

Remark 3.7. Our original proof was longer treating the case where G is a 2-step nilpotent Lie group. This one was provided by one of the referees.

Now we fix a canonical basis $\{a_1,\ldots,a_d\}$ of Γ. A point Γx of the nilmanifold $\Gamma\setminus G$ is said to have *rational coordinates* or simply x has rational coordinates if $x=\gamma(\overrightarrow{q})=a_1^{q_1}a_2^{q_2}\cdots a_d^{q_d}$ for some $\overrightarrow{q}\in\mathbb{Q}^d$. First we show that if $\Gamma x=\Gamma y$ and $x=\gamma(\overrightarrow{q})$ with $\overrightarrow{q}\in\mathbb{Q}^d$, then $y=\gamma(\overrightarrow{p})$ for some $\overrightarrow{p}\in\mathbb{Q}^d$. We recall the following ([7, Theorem 2.1.(1)]): there exists a polynomial function with rational coefficients $\mu:\mathbb{Z}^d\times\mathbb{Z}^d\to\mathbb{Z}^d$ satisfying $\gamma(\overrightarrow{m})\cdot\gamma(\overrightarrow{n})=\gamma(\mu(\overrightarrow{m},\overrightarrow{n}))$ for all $\overrightarrow{m},\overrightarrow{n}\in\mathbb{Z}^d$. The group product on G is defined using this polynomial μ. Now, suppose that $\Gamma x=\Gamma y$ and $x=\gamma(\overrightarrow{q})$ with $\overrightarrow{q}\in\mathbb{Q}^d$. Then $y=zx$ for some $z\in\Gamma$. Since $z\in\Gamma$, $z=\gamma(\overrightarrow{n})$ for some $\overrightarrow{n}\in\mathbb{Z}^d$. Hence we have $y=zx=\gamma(\overrightarrow{n})\cdot\gamma(\overrightarrow{q})=\gamma(\mu(\overrightarrow{n},\overrightarrow{q}))$. Since $\overrightarrow{n}\in\mathbb{Z}^d,\overrightarrow{q}\in\mathbb{Q}^d$, and μ is a polynomial function with rational coordinates, we must have $\mu(\overrightarrow{n},\overrightarrow{q})\in\mathbb{Q}^d$. This proves our assertion. Therefore the points of $\Gamma\setminus G$ with rational coordinates are well defined. Consequently for a subring R of $\mathbb{Q}$ with $\mathbb{Z}\subset R$, the points of $\Gamma\setminus G$ with coordinates in R are well defined.

It is known that every (infra-)nil automorphism has dense periodic points (see the proof of [4, Lemma 3]). Now we will generalize this to the case of (infra-)nil endomorphisms. The proof below is exactly the same as that of Lemma 2.2, except that the coefficients involved are different and hence Lemma 3.6 is essential.

Theorem 3.8. *Let $\varphi_\Gamma : \Gamma \setminus G \to \Gamma \setminus G$ be a nil endomorphism of the nilmanifold $\Gamma \setminus G$. Let R be a ring obtained from Q by adding finitely many primes p_j, that is, $R = Q + \mathbb{Z}_{(p_1,\ldots,p_r)}$. Then every point with coordinates in R is an eventually periodic point of φ_Γ. Moreover, if $(q_i, |\det(d\varphi)|) = (p_j, |\det(d\varphi)|) = 1$ for all i, j, then every point with coordinates in R is a periodic point of φ_Γ.*

Proof. We will show this by induction on the nilpotency class c of G. If $c = 1$, then $\Gamma \setminus G$ is a torus and this case is proved in Lemma 2.2.

Now let $c > 1$ and assume that the assertion is true for any connected, simply connected nilpotent Lie group G' of nilpotency class $\leq c - 1$ and for any ring obtained from Q' by adding finitely many primes.

Consider $\widehat{G} = \gamma_c(G)$ and $\widehat{\Gamma} = \sqrt[\Gamma]{\gamma_c(\Gamma)}$, and the principal fiber bundle $T \to M \to B$ where $M = \Gamma \setminus G$, $T = \widehat{\Gamma} \setminus \widehat{G}$ is a torus and $B = \overline{\Gamma} \setminus \overline{G}$ is a nilmanifold of dimension less than that of M. Since the automorphism $\varphi : G \to G$ maps Γ into itself, its induced map φ_Γ is fiber-preserving. That is, the following diagram is commuting:

$$\begin{tikzcd}
T \arrow[r, "\hat{\varphi}"] \arrow[d] & T \arrow[d] \\
M \arrow[r, "\varphi_\Gamma"] \arrow[d] & M \arrow[d] \\
B \arrow[r, "\bar{\varphi}"] & B
\end{tikzcd} \tag{3.31}$$

Now we note that $\widehat{\Gamma} = \Gamma_i$ for some Γ_i in the refined central series of Γ. Thus $\widehat{\Gamma}$ and $\overline{\Gamma}$ have central series

$$\begin{gathered}
\widehat{\Gamma} = \Gamma_i \supset \Gamma_{i+1} \supset \cdots \supset \Gamma_d \supset \Gamma_{d+1} = 1, \\
\overline{\Gamma} = \Gamma / \widehat{\Gamma}_i = \Gamma_1/\Gamma_i \supset \Gamma_2/\Gamma_i \supset \cdots \supset \Gamma_{i-1}/\Gamma_i \supset \Gamma_i/\Gamma_i = 1.
\end{gathered} \tag{3.32}$$

The canonical basis $\mathbf{a} = \{a_1, a_2, \ldots, a_d\}$ of Γ induces the canonical bases $\widehat{\mathbf{a}} = \{a_i, a_{i+1}, \ldots, a_d\}$ and $\overline{\mathbf{a}} = \{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{i-1}\}$ of $\widehat{\Gamma}$ and $\overline{\Gamma}$, respectively, where $\overline{a}$ stands for the image of $a \in \Gamma$ in $\overline{\Gamma}$ under the natural surjection $\Gamma \to \overline{\Gamma}$. Hence the points in $T = \widehat{\Gamma} \setminus \widehat{G}$ with rational coordinates are well defined. Furthermore the points in $B = \overline{\Gamma} \setminus \overline{G}$ with rational coordinates are also well-defined.

For $\overrightarrow{q} = (q_1, q_2, \ldots, q_d) \in R^d$, write

$$x = a_1^{q_1} a_2^{q_2} \cdots a_d^{q_d}, \quad y = a_1^{q_1} a_2^{q_2} \cdots a_{i-1}^{q_{i-1}}, \quad z = a_i^{q_i} a_{i+1}^{q_{i+1}} \cdots a_d^{q_d}. \tag{3.33}$$

Then $x = yz$ and $z \in \widehat{G}$. Since $\overline{x} = \overline{y} \in \overline{G}$, $\overline{\mathbf{x}} = \overline{\mathbf{y}} = \overline{\Gamma}\overline{y}$ is a point of $\overline{\Gamma} \setminus \overline{G}$ with coordinates in R. (Note that the ring $\overline{Q}$ when working over the group $\overline{G}$ is a subring of Q and so $\overline{Q} \subset R$.) By induction hypothesis $\overline{\varphi}^{t+k}(\overline{\mathbf{y}}) = \overline{\varphi}^{t}(\overline{\mathbf{y}})$ for some $k \geq 1$ and $t \geq 0$. On the other hand, since

$\widehat{\mathbf{z}} = \widehat{\Gamma} z$ is a point of the torus $\widehat{\Gamma} \setminus \widehat{G}$ with coordinates in R, by Lemma 2.2, $\widehat{\varphi}^{\tau+\ell}(\widehat{\mathbf{z}}) = \widehat{\varphi}^{\tau}(\widehat{\mathbf{z}})$ for some $\ell \geq 1$ and $\tau \geq 0$. We may assume that $t = \tau$ and $k = \ell$ so that

$$\begin{aligned} \overline{\varphi}^{t+k}(\overline{\mathbf{y}}) &= \overline{\varphi}^{t}(\overline{\mathbf{y}}), \\ \widehat{\varphi}^{t+k}(\widehat{\mathbf{z}}) &= \widehat{\varphi}^{t}(\widehat{\mathbf{z}}). \end{aligned} \tag{3.34}$$

Then $\overline{\varphi^{t+k}(y)} = \overline{\xi \varphi^{t}(y)}$ in $\overline{G}$ for some $\xi \in \Gamma$; $\varphi^{t+k}(y) = \xi \varphi^{t}(y)\widehat{w}$ for some $\widehat{w} \in \widehat{G}$. By Lemma 3.6, $\widehat{w} \in \widehat{G}$ has coordinates in R. Furthermore, $\varphi^{t+k}(z) = \gamma \varphi^{t}(z) = \varphi^{t}(z)\gamma$ for some $\gamma \in \widehat{\Gamma}$. Let $x_1 = \varphi^{t}(x)$. Then

$$\begin{aligned} \varphi^{k}(x_1) = \varphi^{k+t}(x) &= \varphi^{k+t}(y)\varphi^{k+t}(z) \\ &= \left(\xi \varphi^{t}(y)\widehat{w}\right)\left(\gamma \varphi^{t}(z)\right) = \xi w' \varphi^{t}(x) = \xi w' x_1 \\ &\quad \text{for some } w' \in \widehat{G}. \end{aligned} \tag{3.35}$$

Simply taking $\psi = \varphi^{k}$, we may assume that $\psi(x_1) = \xi w x_1$ where $\xi \in \Gamma$ and $w \in \widehat{G}$ with coordinates in R. Hence Lemma 2.2 can be used to conclude that $\psi^{m+u}(w) = \gamma \psi^{u}(w)$ for some $m \geq 1$, $u \geq 0$, and $\gamma \in \widehat{\Gamma}$. Thus $\psi^{im+u}(w) = \gamma_i \psi^{u}(w)$ for some $\gamma_i \in \widehat{\Gamma}$. We note further that for any $n > 0$,

$$\begin{aligned} \psi^{nm}(x_1) &= \nu' \psi^{nm-1}(w)\psi^{nm-2}(w)\cdots\psi^{2}(w)\psi(w)w x_1 \\ &= \nu' \left\{ \prod_{j=0}^{m-1} \psi^{j}\left(\psi^{(n-1)m}(w)\cdots\psi^{m}(w)w\right) \right\} x_1, \\ \psi^{nm+u}(x_1) &= \psi^{u}(\nu') \left\{ \prod_{j=0}^{m-1} \psi^{j}\left(\psi^{(n-1)m+u}(w)\cdots\psi^{m+u}(w)\psi^{u}(w)\right) \right\} \psi^{u}(x_1) \\ &= \nu \left\{ \prod_{j=0}^{m-1} \psi^{j}\left(\psi^{u}(w)^{n}\right) \right\} \psi^{u}(x_1) = \nu \left\{ \prod_{j=0}^{m-1} \psi^{j}\left(\psi^{u}(w^{n})\right) \right\} \psi^{u}(x_1) \end{aligned} \tag{3.36}$$

for some $\nu', \nu \in \Gamma$. Since $w \in \widehat{G} = \mathbb{R}^{k}$ with coordinates in R, there is $n > 0$ such that $w^{n} \in \widehat{\Gamma} = \mathbb{Z}^{k}$. Since $\psi(\widehat{\Gamma}) \subset \widehat{\Gamma}$, $\psi^{j}(\psi^{u}(w)^{n}) \in \widehat{\Gamma}$ for all $j = 0, 1, \ldots, m-1$. Hence $\psi^{nm+u}(x_1) = \nu \psi^{u}(x_1)$, or $\varphi^{knm+ku+t}(x) = \nu \varphi^{ku+t}(x)$ for some $\nu \in \Gamma$. Therefore

$$\varphi_{\Gamma}^{knm+ku+t}(\mathbf{x}) = \varphi_{\Gamma}^{ku+t}(\mathbf{x}), \tag{3.37}$$

which implies that $\mathbf{x}$ is an eventually periodic point of φ_{Γ}.

Moreover, if $(q_i, |\det(d\varphi)|) = (p_j, |\det(d\varphi)|) = 1$ for all i, j, then by Lemma 2.2 and induction hypothesis, we can choose $t = u = 0$ and so $\varphi_{\Gamma}^{knm}(\mathbf{x}) = \mathbf{x}$. Thus $\mathbf{x}$ is a periodic point of φ_{Γ}. □

Corollary 3.9. *Every nil endomorphism $\varphi_\Gamma : \Gamma \setminus G \to \Gamma \setminus G$ of the nilmanifold $\Gamma \setminus G$ has dense eventually periodic points.*

Proof. Using the fact that the points of $\Gamma \setminus G$ with coordinates in R are dense in $\Gamma \setminus G$, we obtain the result. □

Example 3.10. Let φ_{Λ_2} be the (hyperbolic) nil endomorphism on the nilmanifold $\Lambda_2 \setminus \mathrm{Nil}$ induced by the automorphism on Nil:

$$\varphi = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 3x+y & 2z + \frac{3}{2}x^2 + xy + \frac{1}{2}y^2 \\ 0 & 1 & x+y \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.38}$$

Then

$$\varphi\left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{8} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\mathrm{mod}\,\Lambda_2). \tag{3.39}$$

Thus the point

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{8} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \in \Lambda_2 \setminus \mathrm{Nil} \tag{3.40}$$

is not a periodic point, but an eventually periodic point of φ_{Λ_2} with least period 1 (i.e., an eventually fixed point). Note here that $\det(d\varphi) = 2^2$ and $1/2$ is the coefficient coming from the nilpotent Lie group Nil.

At this moment, we donot know whether Corollary 3.9 is true for periodic points in the general case, that is, the case where $(q_i, \det(d\varphi)) \neq 1$ for some i. We now propose naturally the following problem.

Question 1. Every nil endomorphism has dense periodic points.

Corollary 3.11. *Every nil automorphism $\varphi_\Gamma : \Gamma \setminus G \to \Gamma \setminus G$ of the nilmanifold $\Gamma \setminus G$ has dense periodic points.*

Proof. The proof follows from that $|\det(d\varphi)| = 1$. □

4. Infra-Nil Endomorphisms

Let G be a connected, simply connected nilpotent Lie group and let C be a maximal compact subgroup of $\mathrm{Aut}(G)$. A discrete and cocompact subgroup Π of $G \rtimes C \subset \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ is called an *almost crystallographic group*. Moreover, if Π is torsion-free, then Π is called an *almost Bieberbach group* and the quotient space $\Pi \setminus G$ an *infra-nilmanifold*. In particular, if $\Pi \subset G$, then $\Pi \setminus G$ is a nilmanifold. Recall from [19] that $\Gamma = \Pi \cap G$ is the maximal normal nilpotent subgroup of Π with finite quotient group $\Psi = \Pi/\Gamma$, called the *holonomy group* of $\Pi \setminus G$.

Definition 4.1. Let $\Pi \setminus G$ be an infra-nilmanifold and let $\varphi : G \to G$ be an automorphism which is *weakly* Π*-equivariant*; that is, there is a homomorphism $\theta = \theta_\varphi$ of Π such that

$$\varphi(\alpha x) = \theta(\alpha)\varphi(x), \quad \alpha \in \Pi,\ x \in G. \tag{4.1}$$

Then φ induces a surjection $\varphi_\Pi : \Pi \setminus G \to \Pi \setminus G$, which is one of the following types.

(I) $d\varphi$ has determinant of modulus 1. In this case φ_Π is called an *infra-nil automorphism*.

(II) $d\varphi$ has determinant of modulus greater than 1. In this case φ_Π is called an *infra-nil endomorphism*.

If, in addition, φ is hyperbolic, then we say that the infra-nil automorphism or endomorphism φ_Π is *hyperbolic*.

Let $\Pi \setminus G$ be an infra-nilmanifold with surjection $\varphi_\Pi : \Pi \setminus G \to \Pi \setminus G$. Let $\Gamma = \Pi \cap G$ be the pure translations of Π. Then it is not difficult to see that there exists a fully invariant subgroup $\Lambda \subset \Gamma$ of Π with finite index. For example, one can take $\Pi^{[\Pi:\Gamma]}$ (see also [20, Lemma 3.1]). Thus $\Lambda \setminus G$ is a nilmanifold which is a finite regular covering of $\Pi \setminus G$ and has Π/Λ as the group of covering transformations. The homomorphism $\theta : \Pi \to \Pi$ associated with φ_Π induces a homomorphism $\widehat{\theta} : \Lambda \to \Lambda$ and in turn induces a homomorphism $\overline{\theta} : \Pi/\Lambda \to \Pi/\Lambda$ so that the following diagram is commuting:

$$\begin{tikzcd} 1 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\hat{\theta}"] & \Pi \arrow[r] \arrow[d, "\theta"] & \Pi/\Lambda \arrow[r] \arrow[d, "\bar{\theta}"] & 1 \\ 1 \arrow[r] & \Lambda \arrow[r] & \Pi \arrow[r] & \Pi/\Lambda \arrow[r] & 1 \end{tikzcd} \tag{4.2}$$

Moreover, the automorphism φ on G induces a surjection $\varphi_\Lambda : \Lambda \setminus G \to \Lambda \setminus G$ so that the following diagram is commuting:

$$\begin{tikzcd} \Lambda \setminus G \arrow[r, "\varphi_\Lambda"] \arrow[d] & \Lambda \setminus G \arrow[d] \\ \Pi \setminus G \arrow[r, "\varphi_\Pi"] & \Pi \setminus G \end{tikzcd} \tag{4.3}$$

Since $\varphi(\lambda x) = \widehat{\theta}(\lambda)\varphi(x)$ for all $\lambda \in \Lambda, x \in G$, we have $\varphi(\lambda) = \widehat{\theta}(\lambda)$ for all $\lambda \in \Lambda$. Hence $\varphi : G \to G$ is the unique extension of the homomorphism $\widehat{\theta} : \Lambda \to \Lambda$ of the lattice Λ

of G. If θ is an isomorphism, then $\widehat{\theta}$ is also an isomorphism. Conversely, assume that $\widehat{\theta}$ is an isomorphism. Using the fact that Π is torsion-free, we can show that θ is injective. This fact implies that $\overline{\theta}$ is also injective on the finite group Π/Λ and hence $\overline{\theta}$ must be an isomorphism. Therefore, θ is an isomorphism. (The converse was suggested by a referee.) If φ_{Π} is an infra-nil automorphism, then being $|\det(d\varphi)| = 1$ implies by Lemma 3.1 that $\widehat{\theta}$ is an isomorphism and thus φ_{Λ} is a nil automorphism, and vice versa. Note also that φ_{Π} is an infra-nil endomorphism if and only if φ_{Λ} is a nil endomorphism.

Let $\mathrm{ePer}(f)$ denote the set of eventually periodic points of a self-map $f : X \to X$.

Theorem 4.2. *Every infra-nil endomorphism $\varphi_{\Pi} : \Pi \setminus G \to \Pi \setminus G$ has dense eventually periodic points.*

Proof. Consider the following commuting diagram:

$$\begin{tikzcd} \Lambda \setminus G \arrow[r, "\varphi_{\Lambda}"] \arrow[d, "p"] & \Lambda \setminus G \arrow[d, "p"] \\ \Pi \setminus G \arrow[r, "\varphi_{\Pi}"] & \Pi \setminus G \end{tikzcd} \tag{4.4}$$

where φ_{Π} is an infra-nil endomorphism, and hence φ_{Λ} is a nil endomorphism. First we observe that $\mathrm{ePer}(\varphi_{\Lambda}) = p^{-1}(\mathrm{ePer}(\varphi_{\Pi}))$. The inclusion $\subseteq$ is obvious. For the converse, let $\widehat{\mathbf{x}} \in p^{-1}(\mathrm{ePer}(\varphi_{\Pi}))$ and $p(\widehat{\mathbf{x}}) = \mathbf{x} \in \mathrm{ePer}(\varphi_{\Pi})$. Then $\varphi_{\Pi}^{m+t}(\mathbf{x}) = \varphi_{\Pi}^{t}(\mathbf{x})$ for some $m > 0$ and $t \geq 0$. Clearly $p(\varphi_{\Lambda}^{t}(\widehat{\mathbf{x}})) = \varphi_{\Pi}^{t}(\mathbf{x})$ and $\varphi_{\Lambda}^{m} : p^{-1}(\varphi_{\Pi}^{t}(\mathbf{x})) \to p^{-1}(\varphi_{\Pi}^{t}(\mathbf{x}))$ is a permutation on the finite set $p^{-1}(\varphi_{\Pi}^{t}(\mathbf{x}))$. Hence $(\varphi_{\Lambda}^{m})^{\ell}(\varphi_{\Lambda}^{t}(\widehat{\mathbf{x}})) = \varphi_{\Lambda}^{t}(\widehat{\mathbf{x}})$ for some ℓ. The reverse inclusion $\supseteq$ is proved. Now by the continuity of p and by Corollary 3.9, we have

$$\Pi \setminus G = p(\Lambda \setminus G) = p\left(\overline{\mathrm{ePer}(\varphi_{\Lambda})}\right) \subset \overline{p(\mathrm{ePer}(\varphi_{\Lambda}))} = \overline{pp^{-1}(\mathrm{ePer}(\varphi_{\Pi}))} = \overline{\mathrm{ePer}(\varphi_{\Pi})}. \tag{4.5}$$

This proves that $\mathrm{ePer}(\varphi_{\Pi})$ is dense in $\Pi \setminus G$. □

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